Torsion line bundles and branes on the Hitchin system

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work based on a paper joint with:
accepted in Adv. Math

- Peter Gothen
- André Oliveira
- Ana Peon-Nieto

Introducing the objects

X = smooth projective curve of genus $g \geq 2$

(E, φ) = **Higgs bundle** on X

- E = vector bundle
- $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$

the canonical bundle of X

For this talk, we fix $\mathrm{rk} = 2$ and $\deg = 0$

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Ω on $T^*\mathcal{N}^{st}$ extends to the whole $\mathcal{M}^{st}$

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Theorem (Non-abelian Hodge theory)

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$\mathcal{M}_{DR}$: moduli space of flat connections on X

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[Hitchin, Donaldson] (Simpson, Corlette)

$$\mathcal{M} \overset{\text{diffeo.}}{\cong} \mathcal{M}_{DR} \Longrightarrow \text{hyperkähler structure } (\mathcal{M}, I, J, K, \eta)$$

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J: moduli of flat connections $\mathcal{M}_{DR}$; ω_J

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$\mathcal{M} \overset{\text{diffeo.}}{\cong} \mathcal{M}_{DR}$ ← moduli space of flat connections on X

$\Longrightarrow$ hyperkähler structure $(\mathcal{M}, I, J, K, \eta)$, with $K = IJ$ and Kähler forms $\omega_I, \omega_J, \omega_K$

$\Omega_I = \omega_J + i\,\omega_K$

The Hitchin fibration

$$h : \overset{8g-6}{\mathcal{M}} \longrightarrow \overset{4g-3}{B} \overset{=}{=} \overset{g}{H^0(K_X)} \overset{+}{\oplus} \overset{3g-3}{H^0(K_X^2)}$$

$$(E, \varphi) \longmapsto (\mathrm{tr}(\varphi), \det(\varphi))$$

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$$4g-3 = g + 3g-3$$

$$(E, \varphi) \longmapsto (\mathrm{tr}(\varphi), \det(\varphi))$$

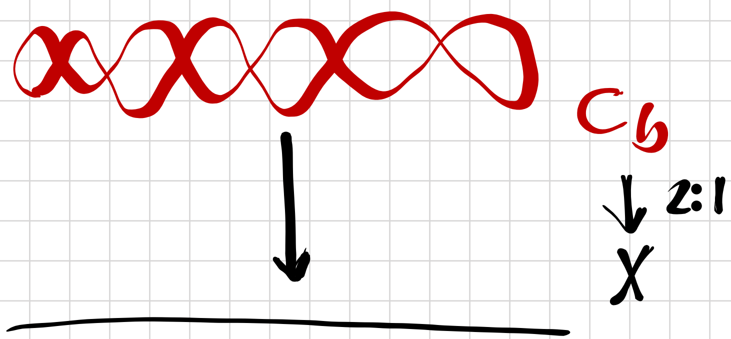
given $b = (b_1, b_2)$ $b_i \in H^0(K_X^i)$ construct <u>Spectral curve</u>

$\pi^* K_X$

λ tautological section

$\mathrm{tot}(K_X) \xrightarrow{\pi} X$

$$\mathrm{tot}(K_X) \supset C_b \equiv \lambda^2 - b_1 \lambda + b_2 = 0$$



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$\pi^* K_X$, tautological section λ, $\mathrm{tot}(K_X) \xrightarrow{\pi} X$

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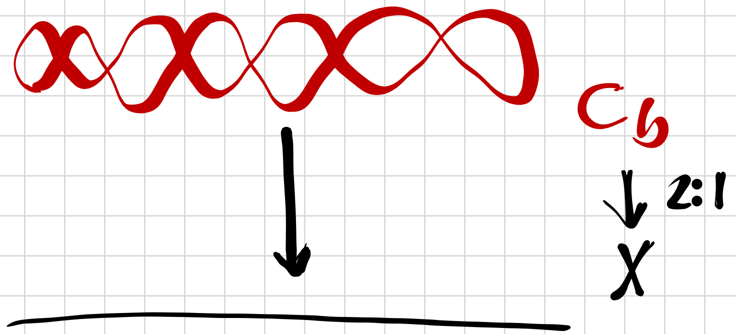

The fibres of h

$$h^{-1}(b) = \overline{\mathrm{Jac}}^{\delta}(C_b) \qquad \delta = 2g-2$$

Compactified Jacobian classifying rk = 1 torsion free sheaves on the spectral curve C_b (for C_b smooth (generic) abelian varieties)

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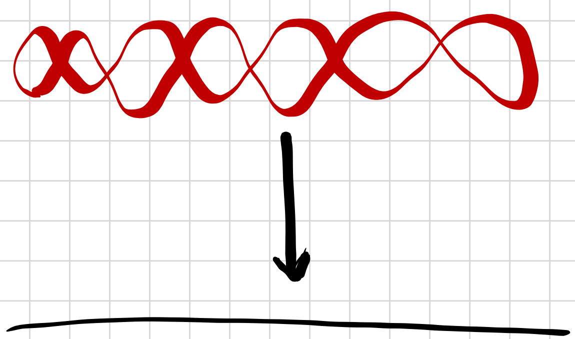

C_b

$\downarrow 2:1$

X

$$\begin{tikzcd} \pi^* K_X \arrow[d] \\ \mathrm{tot}(K_X) \arrow[u, bend left, "\lambda"] \arrow[r, "\pi"] & X \end{tikzcd}$$

tautological section

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$$\dim(h^{-1}(b)) = 4g-3 = \frac{1}{2}\dim(\mathcal{M})$$

Lagrangian for ω_J, ω_K

Mirror Symmetry for Higgs moduli spaces

For any cx. red. Lie group G

Langlands dual of G

$$\mathcal{M}(G) \longrightarrow B \longleftarrow \mathcal{M}({}^LG)$$

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$$\begin{tikzcd}[column sep=huge]
\mathcal{M}(G) \arrow[rr, leftrightarrow, "\text{Fourier-Mukai transform}", "\text{(over the locus of integral spectral curves)}"'] \arrow[dr] & & \mathcal{M}({}^LG) \arrow[dl] \\
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\end{tikzcd}$$

Generic fibres (when C_b smooth) dual abelian varieties [HT] [DP]

for $G = GL(n,\mathbb{C}) = {}^LG$

[Ar] $\overline{Jac}(C_b)$ self-dual when C_b integral

[MRV] $\overline{Jac}(C_b)$ fine $\Rightarrow$ self-dual

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$\boxed{\text{[SYZ] framework for MS}}$

$\boxed{\text{Homological MS [Kontsevich]}}$

$$D^b(\mathcal{M}_{DR}(G)) \cong D(Fuk(\mathcal{M}_{DR}({}^LG)))$$

A-branes $\begin{cases} (F,\nabla) \quad \nabla^2 = 0 \\ \downarrow \\ \text{Lag. subvar.} \end{cases}$

$\Updownarrow$

B-branes (coherent sheaves) $\begin{cases} (E,\bar\partial) \quad \bar\partial^2 = 0 \\ \downarrow \\ \text{hol. subvar.} \end{cases}$

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$\Leftarrow$ **Homological MS** [Kontsevich]

$$D^b(\mathcal{M}^J_{DR}(G)) \simeq D(Fuk(\mathcal{M}^K_{DR}({}^LG)))$$

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Semi-classical limit

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$\Leftarrow$ **Geometric Langlands** [BD]

$$D^b(\mathcal{M}_{DR}(G)) \simeq D^b(Bun({}^LG), \mathcal{D})$$

[KW] $\Leftarrow$ **Dimensional reduction + S-duality**

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for I

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B_A-branes

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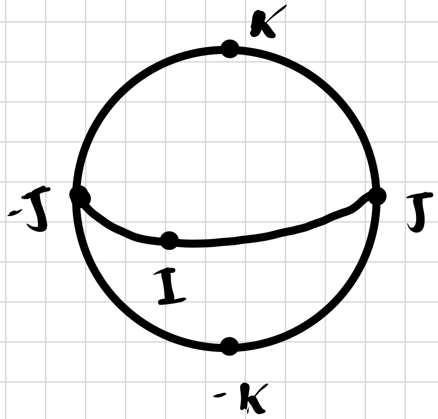

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Topological Mirror Symmetry for Higgs moduli spaces

Semi-classical Limit
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Cohomological shadow

[HT] The (stringy) E-polynomials of $\mathcal{M}(G)$ and $\mathcal{M}({}^LG)$ are conjectured to coincide

- [HT] $G = SL_2(\mathbb{C})$, ${}^LG = PGL(2,\mathbb{C})$ and $G = SL(3,\mathbb{C})$, ${}^LG = PGL(3,\mathbb{C})$
- [GWZ] $G = SL_n(\mathbb{C})$, ${}^LG = PGL(n,\mathbb{C})$

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Strange phenomenon: Equality of terms contributed by two different actions:

- $\mathbb{C}^* \circlearrowright \mathcal{M}$ $\quad (E,\varphi) \longmapsto (E, \lambda\varphi)$
- $\Gamma \circlearrowright \mathcal{M}$ where $\Gamma = \mathrm{Jac}^0(X)[2]$ $\quad (E,\varphi) \longmapsto (L_\gamma \otimes E, \varphi)$

Motivation: Provide a geometric explanation of this

Our project: Transform the fixed loci $\mathcal{M}^\gamma$ under Fourier-Mukai

The fixed locus under tensorization $\mathcal{M}^\gamma$

$\gamma \in \Gamma$
$\quad\ \|$
$L_\gamma \in Jac^0(X)[2]$

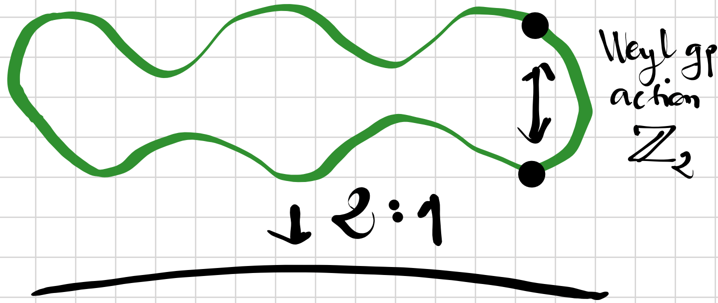
Since $Jac^0(X)[2] \cong H^0(X, \mathbb{Z}_2)$

$$L_\gamma \Longleftrightarrow \Sigma_\gamma \xrightarrow[2:1]{\pi_\gamma} X$$

unramified ($K_\gamma \cong \pi_\gamma^* K$)

$g(\Sigma_\gamma) = 2(g-1)+1$

$$\begin{array}{c} \Sigma_\gamma \\ \pi_\gamma \big\downarrow \\ X \end{array}$$



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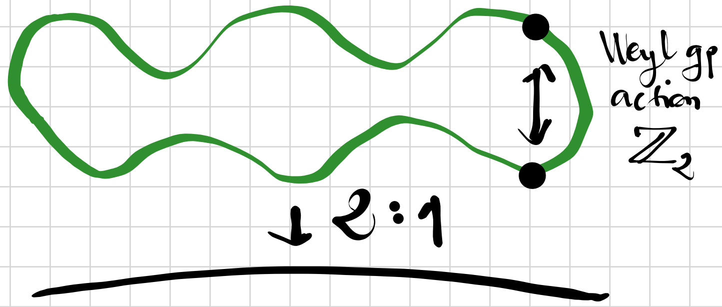

> observe: $\pi_\gamma^* L_\gamma \cong \mathcal{O}_{\Sigma_\gamma}$
>
> So: $(\pi_{\gamma,*}\mathcal{L}) \otimes L_\gamma \cong \pi_{\gamma,*}(\mathcal{L} \otimes \pi_\gamma^* L_\gamma) \cong \pi_{\gamma,*}\mathcal{L}$

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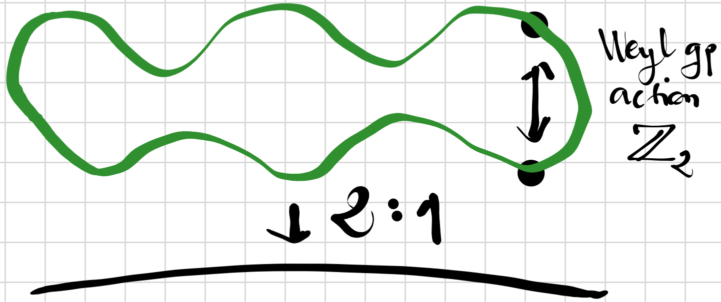

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[Narasimhan & Ramanan
Hausel & Thaddeus
García-Prada & Ramanan]

$$\begin{tikzcd} \pi_{\gamma,*} : T^* Jac^0(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[r, "\cong"] & \mathcal{M}^\gamma \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{M} \arrow[d, "h"] \\ & h(\mathcal{M}^\gamma) =: \mathcal{B}^\gamma \arrow[r, phantom, "\subset"] & \mathcal{B} \end{tikzcd}$$

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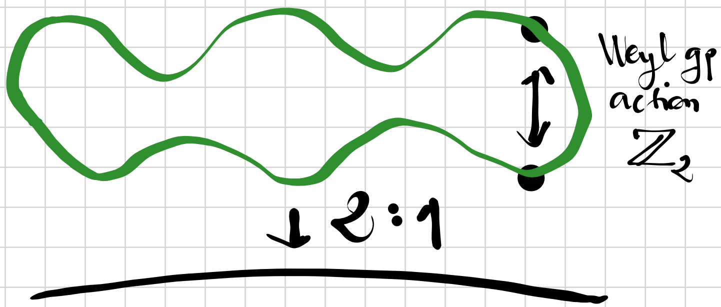


> **observe:** $\pi_\gamma^* L_\gamma \cong \mathcal{O}_{\Sigma_\gamma}$
>
> **So:** $(\pi_{\gamma,*}\mathcal{L}) \otimes L_\gamma \cong \pi_{\gamma,*}(\mathcal{L} \otimes \pi_\gamma^* L_\gamma) \cong \pi_{\gamma,*}\mathcal{L}$

[Narasimhan & Ramanan; Hausel & Thaddeus; García-Prada & Ramanan]

$$\begin{array}{ccccc} \pi_{\gamma,*} : T^*\mathrm{Jac}^0(\Sigma_\gamma)/\mathbb{Z}_2 & \xrightarrow{\cong} & \mathcal{M}^\gamma & \subset & \mathcal{M} \\ & & \downarrow & & \downarrow h \\ & h(\mathcal{M}^\gamma) =: & \mathcal{B}^\gamma & \subset & \mathcal{B} \end{array}$$

We define

$$\boxed{BBB^\gamma := \left\{ \begin{array}{c} (\mathcal{U}, \nabla) \\ \downarrow \\ \mathcal{M}^\gamma \end{array} \right.}$$

The fixed locus under tensorization, $\mathcal{M}^\gamma$

$\gamma \in \Gamma$

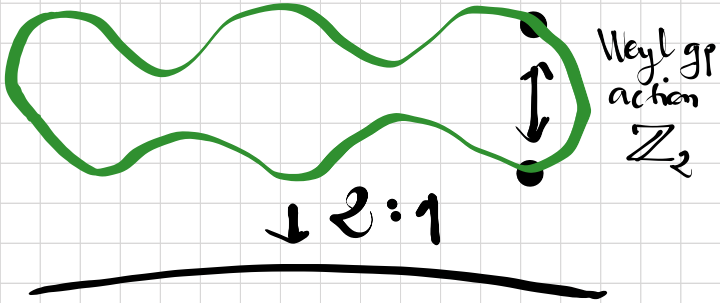
$\|$

$L_\gamma \in \mathrm{Jac}^0(X)[2]$

Since $\mathrm{Jac}^0(X)[2] \cong H^0(X, \mathbb{Z}_2)$

$$L_\gamma \Longleftrightarrow \Sigma_\gamma \xrightarrow[2:1]{\pi_\gamma} X$$

$\Sigma_\gamma \xrightarrow{\pi_\gamma} X$



observe: $\pi_\gamma^* L_\gamma \cong \mathcal{O}_{\Sigma_\gamma}$

So: $(\pi_{\gamma,*}\mathcal{L}) \otimes L_\gamma \cong \pi_{\gamma,*}(\mathcal{L} \otimes \pi_\gamma^* L_\gamma) \cong \pi_{\gamma,*}\mathcal{L}$

unramified ($K_\gamma \cong \pi_\gamma^* K$)

$g(\Sigma_\gamma) = 2(g-1)+1$

proposition: [Narasimhan & Ramanan, Hausel & Thaddeus, García-Prada & Ramanan]

$$\begin{tikzcd} \pi_{\gamma,*} : T^*\mathrm{Jac}^0(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[r, "\cong"] & \mathcal{M}^\gamma \arrow[d] & \subset & \mathcal{M} \arrow[d, "h"] \\ & h(\mathcal{M}^\gamma) =: \mathcal{B}^\gamma & \subset & \mathcal{B} \end{tikzcd}$$

We define

$$\boxed{BBB^\gamma := \left\{ \begin{array}{c} (\mathbb{C}, \nabla) \\ \downarrow \\ \mathcal{M}^\gamma \end{array} \right.}$$

- $\mathcal{B}^\gamma \cong H^0(K) \oplus H^0(L_\gamma K)/\mathbb{Z}_2 \hookrightarrow \mathcal{B} = H^0(K) \oplus H^0(L_\gamma K)$

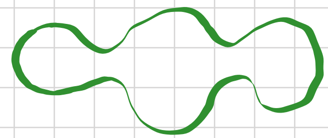
  $(\alpha, \beta) \longmapsto (2\alpha, \alpha^2 - \beta^2)$

- Given $b = (\alpha, \pm\beta) \in \mathcal{B}^\gamma$, if
  - $\beta = 0 \Longrightarrow C_b$ non-reduced
  - $\beta \neq 0 \Longrightarrow C_b$ reduced & irreducible (only the traslated nilpotent cone)

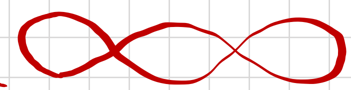
    with normalization $\nu_b : \Sigma_\gamma \longrightarrow C_b$

    $\mathrm{Sing}(C_b) = $ vanishing of $\beta = 2g-2$

    $\mathcal{M}^\gamma \cap h^{-1}(b) = \nu_{b,*}\mathrm{Jac}^0(\Sigma_\gamma) \subset \overline{\mathrm{Jac}}^0(C_b)$



$$\begin{tikzcd} \Sigma_\gamma \arrow[r, "\nu_b"] \arrow[d, "\pi_\gamma"'] & C_b \arrow[dl, "\pi_b"] \\ X & \end{tikzcd}$$



Construction of the dual BAA-branes

picking a spin structure

$K^{1/2} \rightsquigarrow \pi_\gamma^* K^{1/2} \rightsquigarrow \sigma_{\Sigma_\gamma}$

Higgs bdles with deg = g-1

$$\begin{tikzcd}
T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma) \arrow[r] \arrow[d] & T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[d] \arrow[r, phantom, "\cong"] & \mathcal{M}^\gamma(g-1) \arrow[r, hook] \arrow[d] & \mathcal{M}(g-1)|_{B^\gamma} \arrow[d] \\
H^0(K_\gamma) \arrow[r] \arrow[u, bend left] & H^0(K_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] & B^\gamma \arrow[u, bend right] & B^\gamma \arrow[u, bend left, "\sigma_\gamma"]
\end{tikzcd}$$

Hitchin section restricted to B^γ

for $b \in B^\gamma$

$$\sigma_\gamma(b) = \left(K^{1/2} \oplus K^{1/2} \otimes L_\gamma, \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right)$$

Construction of the dual BAA-branes

picking a spin structure

$K^{1/2} \rightsquigarrow \pi_\gamma^* K^{1/2} \rightsquigarrow \sigma_{\Sigma_\gamma}$

Higgs bdles with deg = g-1

$$\begin{tikzcd}
T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma) \arrow[r] \arrow[d] & T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[d] \arrow[r, phantom, "\cong"] & \mathcal{M}^\gamma(g-1) \arrow[r, hook] \arrow[d] & \mathcal{M}(g-1)|_{B^\gamma} \arrow[d, shift left] \\
H^0(K_\gamma) \arrow[r] \arrow[u, bend left] & H^0(K_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] & B^\gamma \arrow[u, bend right] & B^\gamma \arrow[u, bend left, "\sigma_\gamma"]
\end{tikzcd}$$

Hitchin section restricted to B^γ

for $b \in B^\gamma$

$$\sigma_\gamma(b) = \left(\begin{matrix} K^{1/2} \\ \oplus \\ K^{1/2} \otimes L_\gamma \end{matrix}, \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right)$$

Hecke modification associated

to $\left\{ \begin{matrix} (E_0, \Phi_0) \\ x_0 \in X \\ v \in E_0^\vee|_{x_0} \end{matrix} \right\}$ is (E, Φ) s.t.

- $0 \longrightarrow E \longrightarrow E_0 \xrightarrow{v} \mathcal{O}_{x_0} \longrightarrow 0$
- $\Phi_0(E) \subset E \otimes K$ $\quad$ ($\Rightarrow$ $\Phi := \Phi_0|_E$)

Construction of the dual BAA-branes

picking a spin structure

$K^{1/2} \rightsquigarrow \pi_\gamma^* K^{1/2} \rightsquigarrow \sigma_{\Sigma_\gamma}$

Higgs bdles with deg = g-1

$$\begin{tikzcd}
T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma) \arrow[r] \arrow[d] & T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{M}^\gamma(g-1) \arrow[r, hook] \arrow[d] & \mathcal{M}(g-1)|_{B^\gamma} \arrow[d] \\
H^0(K_\gamma) \arrow[r] \arrow[u, bend left] & H^0(K_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] & B^\gamma \arrow[u, bend right] & B^\gamma \arrow[u, bend left, "\sigma_\gamma"]
\end{tikzcd}$$

Hitchin section restricted to B^γ

for $b \in B^\gamma$

$$\sigma_\gamma(b) = \left(K^{1/2} \oplus K^{1/2} \otimes L_\gamma, \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right)$$

Hecke modification associated

to $\left\{ \begin{array}{l} (E_0, \Phi_0) \\ x \in X \\ v \in E_0^*|_{x_0} \end{array} \right\}$ is (E, Φ) s.t.

- $0 \to E \to E_0 \xrightarrow{v} \mathcal{O}_{x_0} \to 0$
- $\Phi_0(E) \subset E \otimes K$ (so $\Phi := \Phi_0|_E$)

We define

$$\boxed{BAA^\gamma_{red} := \bigcup_{b \in B^\gamma_{red}} \left\{ \begin{array}{l} \text{Hecke modifications of } (E_{0,b}, \Phi_{0,b}) = \sigma_\gamma(b) \\ \text{at (the image in } X \text{ of) } \mathrm{sing}(C_b) = Z(\beta) \end{array} \right\}}$$

Construction of the dual BAA-branes

picking a spin structure

$K^{1/2} \rightsquigarrow \pi_\gamma^* K^{1/2} \rightsquigarrow \sigma_{\Sigma_\gamma}$

Higgs bdles with $\deg = g-1$

$$\begin{tikzcd}
T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma) \arrow[r] \arrow[d] & T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[d] & \cong \mathcal{M}^\gamma(g-1) \arrow[r, hook] \arrow[d] & \mathcal{M}(g-1)|_{B^\gamma} \arrow[d] \\
H^0(K_\gamma) \arrow[r] \arrow[u, bend left] & H^0(K_\gamma)/\mathbb{Z}_2 & \cong B^\gamma \arrow[u, bend right] & B^\gamma \arrow[u, bend left, "\sigma_\gamma"]
\end{tikzcd}$$

Hitchin section restricted to B^γ

for $b \in B^\gamma$

$$\sigma_\gamma(b) = \left(K^{1/2} \oplus K^{1/2} \otimes L_\gamma, \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\right)$$

Hecke modification associated

to $\left\{\begin{array}{l} (E_0, \Phi_0) \\ x \in X \\ v \in E_0^\vee|_{x_0} \end{array}\right\}$ is (E, Φ) s.t.

- $0 \to E \to E_0 \xrightarrow{v} \mathcal{O}_{x_0} \to 0$
- $\Phi_0(E) \subset E \otimes K$ (so $\Phi := \Phi_0|_E$)

We define

$$\boxed{BAA^\gamma_{red} := \bigcup_{b \in B^\gamma_{red}} \left\{ \begin{array}{l} \text{Hecke modifications of } (E_{0,b}, \Phi_{0,b}) = \sigma_\gamma(b) \\ \text{at (the image in } X \text{ of) } \mathrm{Sing}(C_b) = Z(\beta) \end{array} \right\}}$$

prop (- GOP) BAA^γ_{red} is a Lagrangian subvariety of $\mathcal{M}$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K^{1/2} \arrow[d, "\beta"] & E \arrow[d] & & \\
0 \arrow[r] & K^{1/2} L_\gamma \arrow[r] \arrow[d] & K^{1/2} \oplus K^{1/2} L_\gamma \arrow[r] \arrow[d] & K^{1/2} \arrow[r] & 0 \\
 & \mathcal{O}_{Z(\beta)} \arrow[d] & \mathcal{O}_{Z(\beta)} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Construction of the dual BAA-branes

picking a spin structure

$K^{1/2} \rightsquigarrow \pi_\gamma^* K^{1/2} \rightsquigarrow \sigma_{\Sigma_\gamma}$

Higgs bdles with deg = g-1

$$\begin{tikzcd}
T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma) \arrow[r] \arrow[d] & T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[d] \arrow[r, phantom, "\simeq"] & \mathcal{M}^\gamma(g-1) \arrow[r, hook] \arrow[d] & \mathcal{M}(g-1)|_{B^\gamma} \arrow[d] \\
H^0(K_\gamma) \arrow[r] \arrow[u, bend left] & H^0(K_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\simeq"] & B^\gamma \arrow[u, bend right] & B^\gamma \arrow[u, bend left, "\sigma_\gamma"]
\end{tikzcd}$$

Hitchin section restricted to B^γ

for $b \in B^\gamma$

$$\sigma_\gamma(b) = \left(K^{1/2} \oplus K^{1/2} \otimes L_\gamma, \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right)$$

Hecke modification associated

to $\begin{cases} (E_0, \Phi_0) \\ x_0 \in X \\ v \in E_0^\vee|_{x_0} \end{cases}$ is (E, Φ) s.t.

- $0 \to E \to E_0 \xrightarrow{v} \mathcal{O}_{x_0} \to 0$
- $\Phi_0(E) \subset E \otimes K$ $\quad (\Rightarrow \Phi := \Phi_0|_E)$

We define

$$\boxed{BAA^\gamma_{red} := \bigcup_{b \in B^\gamma_{red}} \left\{ \begin{array}{l} \text{Hecke modifications of } (E_{0,b}, \Phi_{0,b}) = \sigma_\gamma(b) \\ \text{at (the image in } X \text{ of) } \mathrm{sing}(C_b) = Z(\beta) \end{array} \right\}}$$

prop (– GOP) BAA^γ_{red} is a Lagrangian subvariety of $\mathcal{M}$

exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K^{1/2} \arrow[r] \arrow[d, "\beta"] & E \arrow[r] \arrow[d] & K^{1/2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K^{1/2} L_\gamma \arrow[r] \arrow[d] & K^{1/2} \oplus K^{1/2} L_\gamma \arrow[r] \arrow[d] & K^{1/2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{Z(\beta)} \arrow[r, equal] \arrow[d] & \mathcal{O}_{Z(\beta)} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$BAA^\gamma_{red} \cap h^{-1}(b) \overset{\text{open}}{\supset} \ker\left(H^1(K^{-1}) \xrightarrow{\beta} H^1(L_\gamma)\right) = T_{(E,\Phi)}\left(BAA^\gamma \cap h^{-1}(b)\right)$$

$$0 \to T_{(E,\Phi)}\left(BAA^\gamma \cap h^{-1}(b)\right) \to T_{(E,\Phi)} BAA^\gamma \to T_b B^\gamma \to 0$$

$$T_b B^\gamma = \mathrm{image}\left(\begin{array}{ccc} H^0(K) \oplus H^0(L_\gamma K) & \longrightarrow & H^0(K) \oplus H^0(K^2) \\ (\dot{x}, \dot{\mathfrak{z}}) & \longmapsto & (2\dot{x}, 2\mathfrak{z}\dot{\mathfrak{z}}) \end{array} \right) \cong \mathrm{coker}\left(H^0(L_\gamma K) \xrightarrow{\mathfrak{z}} H^0(K^2)\right)$$

$$\ker\left(H^1(K^{-1}) \xrightarrow{\beta} H^1(L_\gamma)\right) \perp \mathrm{coker}\left(H^0(L_\gamma K) \xrightarrow{\mathfrak{z}} H^0(K^2)\right) \Rightarrow \text{isotropy of } BAA^\gamma$$

Construction of the dual BAA-branes

6

picking a spin structure

$K^{1/2} \rightsquigarrow \pi_\gamma^* K^{1/2} \rightsquigarrow \mathcal{J}_{\Sigma_\gamma}$

Higgs bdles with deg = g-1

$$\begin{tikzcd}
T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma) \arrow[r] \arrow[d] & T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[d] \arrow[r, phantom, "\cong"] & \mathcal{M}^\gamma(g-1) \arrow[r, hook] \arrow[d] & \mathcal{M}(g-1)|_{B^\gamma} \arrow[d] \\
H^0(K_\gamma) \arrow[r] \arrow[u, bend left] & H^0(K_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] & B^\gamma \arrow[u, bend right] & B^\gamma \arrow[u, bend left, "\mathcal{J}_\gamma"]
\end{tikzcd}$$

Hitchin section restricted to B^γ

for $b \in B^\gamma$

$$\mathcal{J}_\gamma(b) = \left(K^{1/2} \oplus K^{1/2}\otimes L_\gamma, \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\right)$$

Hecke modification associated

to $\begin{Bmatrix} (E_0, \Phi_0) \\ x \in X \\ v \in E_0^\vee|_{x_0} \end{Bmatrix}$ is (E, Φ) s.t.

- $0 \to E \to E_0 \xrightarrow{v} \mathcal{O}_{x_0} \to 0$
- $\Phi_0(E) \subset E \otimes K \quad (\text{so } \Phi := \Phi_0|_E)$

We define

$$\boxed{\mathrm{BAA}^\gamma_{red} := \bigcup_{b \in B^\gamma_{red}} \left\{ \begin{array}{l} \text{Hecke modifications of } (E_{0,b}, \Phi_{0,b}) = \mathcal{J}_\gamma(b) \\ \text{at (the image in } X \text{ of) } \mathrm{sing}(C_b) = Z(\beta) \end{array} \right\}}$$

prop (− GOP) $\mathrm{BAA}^\gamma_{red}$ is a Lagrangian subvariety of $\mathcal{M}$

exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K^{1/2} \arrow[r] \arrow[d, "\beta"] & E \arrow[r] \arrow[d] & K^{1/2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K^{1/2}L_\gamma \arrow[r] \arrow[d] & K^{1/2} \oplus K^{1/2}L_\gamma \arrow[r] \arrow[d] & K^{1/2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{Z(\beta)} \arrow[r, equal] \arrow[d] & \mathcal{O}_{Z(\beta)} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\mathrm{BAA}^\gamma_{red} \cap h^{-1}(b) \overset{\text{open}}{\supset} \ker\left(H^1(K^{-1}) \xrightarrow{\beta} H^1(L_\gamma)\right) = T_{(E,\Phi)}\left(\mathrm{BAA}^\gamma \cap h^{-1}(b)\right)$$

$$0 \to T_{(E,\Phi)}\left(\mathrm{BAA}^\gamma \cap h^{-1}(b)\right) \to T_{(E,\Phi)}\mathrm{BAA}^\gamma \to T_b B^\gamma \to 0$$

$$T_b B^\gamma = \mathrm{image}\left(\begin{array}{ccc} H^0(K) \oplus H^0(L_\gamma K) & \longrightarrow & H^0(K) \oplus H^0(K^2) \\ (\dot\alpha, \dot\beta) & \longmapsto & (2\alpha\dot\alpha, 2\beta\dot\beta) \end{array}\right) \cong \mathrm{coker}\left(H^0(L_\gamma K) \xrightarrow{\beta} H^0(K^2)\right)$$

$$\ker\left(H^1(K^{-1}) \xrightarrow{\beta} H^1(L_\gamma)\right) \perp \mathrm{coker}\left(H^0(L_\gamma K) \xrightarrow{\beta} H^0(K^2)\right) \Rightarrow \text{isotropy of } \mathrm{BAA}^\gamma$$

$$\dim(\mathrm{BAA}^\gamma) = \dim(B^\gamma) + \dim(\mathrm{BAA}^\gamma \cap h^{-1}(b)) = \left(\dim H^0(K) + \dim H^0(L_\gamma K)\right) + \left(\dim H^1(K^{-1}) - \dim H^1(L_\gamma)\right)$$

Construction of the dual BAA-branes

picking a spin structure

$K^{1/2} \rightsquigarrow \pi_\gamma^* K^{1/2} \rightsquigarrow \sigma_{\Sigma_\gamma}$

Higgs bdles with deg = g-1

$$\begin{tikzcd}
T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma) \arrow[r] \arrow[d] & T^*\mathrm{Jac}^{g-1}(\Sigma_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{M}^\gamma(g-1) \arrow[r, hook] \arrow[d] & \mathcal{M}(g-1)|_{B^\gamma} \arrow[d] \\
H^0(K_\gamma) \arrow[r] \arrow[u, bend left] & H^0(K_\gamma)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] & B^\gamma \arrow[u, bend right] & B^\gamma \arrow[u, bend left, "\sigma_\gamma"]
\end{tikzcd}$$

Hitchin section restricted to B^γ

for $b \in B^\gamma$

$$\sigma_\gamma(b) = \left(K^{1/2} \oplus K^{1/2} \otimes L_\gamma, \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right)$$

Hecke modification associated

to $\left\{ \begin{array}{l} (E_0, \Phi_0) \\ x_0 \in X \\ v \in E_0^*|_{x_0} \end{array} \right\}$ is (E, Φ) s.t.

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- $\Phi_0(E) \subset E \otimes K$ (so $\Phi := \Phi_0|_E$)

We define

$$\boxed{BAA^\gamma_{red} := \bigcup_{b \in B^\gamma_{red}} \left\{ \begin{array}{l} \text{Hecke modifications of } (E_{0,b}, \Phi_{0,b}) = \sigma_\gamma(b) \\ \text{at (the image in } X \text{ of) } \mathrm{sing}(C_b) = Z(\beta) \end{array} \right\}}$$

prop (– GOP) BAA^γ_{red} is a Lagrangian subvariety of $\mathcal{M}$

exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K^{1/2} \arrow[r] \arrow[d, "\beta"] & E \arrow[r] \arrow[d] & K^{1/2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K^{1/2} L_\gamma \arrow[r] \arrow[d] & K^{1/2} \oplus K^{1/2} L_\gamma \arrow[r] \arrow[d] & K^{1/2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{Z(\beta)} \arrow[r, equal] \arrow[d] & \mathcal{O}_{Z(\beta)} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$BAA^\gamma_{red} \cap h^{-1}(b) \overset{\text{open}}{\supset} \ker\left(H^1(K^{-1}) \xrightarrow{\beta} H^1(L_\gamma)\right) = T_{(E,\Phi)}\left(BAA^\gamma \cap h^{-1}(b)\right)$$

$$0 \to T_{(E,\Phi)}\left(BAA^\gamma \cap h^{-1}(b)\right) \to T_{(E,\Phi)} BAA^\gamma \to T_b B^\gamma \to 0$$

$$T_b B^\gamma = \mathrm{image}\left(\begin{array}{ccc} H^0(K) \oplus H^0(L_\gamma K) & \longrightarrow & H^0(K) \oplus H^0(K^2) \\ (\dot\alpha, \dot\beta) & \longmapsto & (2\alpha\dot\alpha, 2\beta\dot\beta) \end{array} \right) \cong \mathrm{coker}\left(H^0(L_\gamma K) \xrightarrow{\beta} H^0(K^2)\right)$$

$$\ker\left(H^1(K^{-1}) \xrightarrow{\beta} H^1(L_\gamma)\right) \perp \mathrm{coker}\left(H^0(L_\gamma K) \xrightarrow{\beta} H^0(K^2)\right) \Rightarrow \text{isotropy of } BAA^\gamma$$

$$\dim(BAA^\gamma) = \dim(B^\gamma) + \dim(BAA^\gamma \cap h^{-1}(b)) = \left(g + (g-1) \right) + \left((3g-3) - (g-1) \right) = \tfrac{1}{2}\dim(\mathcal{M})$$

Spectral data of BAA^γ

Recall that $h^{-1}(b) \cap_{\iota} M^\gamma = \nu_{b,*}\, \mathrm{Jac}^0(\Sigma_\gamma) \subset \overline{\mathrm{Jac}}^{\delta}(C_b)$

the "dual" of $\nu_{b,*} : \mathrm{Jac}^0(\Sigma_\gamma) \longrightarrow \overline{\mathrm{Jac}}^{\delta}(C_b)$ is $\nu_b^* : \mathrm{Jac}^{\delta}(C_b) \longrightarrow \mathrm{Jac}^{\delta}(\Sigma_\gamma)$

problem: ν_b^* don't extend to $\overline{\mathrm{Jac}}(C_b)$ $\Longrightarrow$ use other compactification where ν_b^* extends

Spectral data of BAAγ

Recall that $h^{-1}(b) \cap \mathcal{M}^\gamma = \nu_{b,*} \mathrm{Jac}^0(\Sigma_\gamma) \subset \overline{\mathrm{Jac}}^\delta(C_b)$

the "dual" of $\nu_{b,*}: \mathrm{Jac}^0(\Sigma_\gamma) \longrightarrow \overline{\mathrm{Jac}}^\delta(C_b)$ is $\nu_b^*: \mathrm{Jac}^\delta(C_b) \longrightarrow \mathrm{Jac}^{\delta}(\Sigma_\gamma)$

problem: ν_b^* don't extend to $\overline{\mathrm{Jac}}(C_b)$ $\Longrightarrow$ use other compactification where ν_b^* extends

Def [Rego, see also Cook and Bhosle] $\nu_b: \Sigma_\gamma \to C_b$ normalization $D_b = \nu_b^{-1}(\mathrm{sing}(C_b))$

A $\mathrm{rk}=1$ parabolic module for (Σ_γ, D_b) is a pair (M, V) where $M \in \mathrm{Jac}^0(\Sigma_\gamma)$ and V is a subspace of $M \otimes \mathcal{O}_{D_b}$ s.t. $\nu_{b,*}\left(\frac{\mathcal{O}_{D_b} \otimes M}{V}\right) = \mathrm{sing}(C_b)$

Spectral data of BAAγ

Recall that $h^{-1}(b) \cap \mathcal{M}^\gamma = \nu_{b,*} \mathrm{Jac}^0(\Sigma_\gamma) \subset \overline{\mathrm{Jac}}^\delta(C_b)$

the "dual" of $\nu_{b,*} : \mathrm{Jac}^0(\Sigma_\gamma) \longrightarrow \overline{\mathrm{Jac}}^\delta(C_b)$ is $\nu_b^* : \mathrm{Jac}^\delta(C_b) \longrightarrow \mathrm{Jac}^\delta(\Sigma_\gamma)$

problem: ν_b^* don't extend to $\overline{\mathrm{Jac}}(C_b)$ $\Longrightarrow$ use other compactification where ν_b^* extends

Def [Rego, see also Cook and Bhosle] $\nu_b : \Sigma_\gamma \to C_b$ normalization $D_b = \nu_b^{-1}(\mathrm{sing}(C_b))$

A $\mathrm{rk}=1$ parabolic module for (Σ_γ, D_b) is a pair (M,V) where $M \in \mathrm{Jac}^\circ(\Sigma_\gamma)$ and V is a subspace of $M \otimes \mathcal{O}_{D_b}$ s.t. $\nu_{b,*}\left(\mathcal{O}_{D_b} \otimes M / V\right) = \mathrm{sing}(C_b)$

$\mathrm{PMod}^{2g-2}(\Sigma_\gamma, D_b) =$ moduli space

- $\tilde{\nu}_b^* : \mathrm{PMod}^{2g-2}(\Sigma_\gamma, D_b) \longrightarrow \mathrm{Jac}^{2g-2}(\Sigma_\gamma)$ extends ν_b^*
  $(M,V) \longmapsto M$
- $\tau : \mathrm{PMod}^{2g-2}(\Sigma_\gamma, D_b) \longrightarrow \overline{\mathrm{Jac}}^{2g-2}(C_b)$
  $(M,V) \longmapsto \mathrm{Ker}\left(\nu_* M \twoheadrightarrow \nu_{b,*}\left(M \otimes \mathcal{O}_{D_b} / V\right)\right)$
- τ is surjective, PMod reduced, irred and projective.
- $\tau\big|_{\text{fibres of } \tilde{\nu}^*} : (\tilde{\nu}^*)^{-1}(M) \hookrightarrow \overline{\mathrm{Jac}}(C_b)$

Spectral data of BAA$^\gamma$

Recall that $h^{-1}(b) \cap {}_1\mathcal{M}^\gamma = \nu_{b,*}\,\mathrm{Jac}^0(\Sigma_\gamma) \subset \overline{\mathrm{Jac}}^\delta(C_b)$

the "dual" of $\nu_{b,*} : \mathrm{Jac}^0(\Sigma_\gamma) \longrightarrow \overline{\mathrm{Jac}}^\delta(C_b)$ is $\nu_b^* : \overline{\mathrm{Jac}}^\delta(C_b) \longrightarrow \mathrm{Jac}^\delta(\Sigma_\gamma)$

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prop: (—GOP)

$\mathrm{BAA}^\gamma \cap h^{-1}(b) = \tau\left((\overline{\nu}^*)^{-1}(\pi_\gamma^* K^{1/2})\right)$

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prop: (-GOP)

$$\mathrm{BAA}^\gamma \cap h^{-1}(b) = \tau\left((\overline{\nu}^*)^{-1}(\pi_\gamma^* K^{1/2})\right)$$

idea: the righ hand side is the set of spectral data

$$0 \to \mathcal{L} \to \nu_{b,*}(\pi_\gamma^* K^{1/2}) \longrightarrow \pi_* \mathcal{O}_{\mathrm{sing}(C_b)} \to 0$$

$$0 \to \pi_* \mathcal{L} \to \underbrace{\pi_{\gamma,*}(\pi_\gamma^* K^{1/2})}_{K^{1/2} \oplus K^{1/2} L_\gamma} \longrightarrow \mathcal{O}_{\pi(\mathrm{sing}(C_b))} \to 0$$

$=$ Hecke transform of $\mathcal{J}_\gamma(b)$.

Duality between BAA$^\sigma$ and BBB$^\sigma$

<u>Theorem [Mukai]</u> Σ_γ smooth, $\mathrm{Jac}^\circ(\Sigma_\gamma)$ self-dual abelian variety.
The integral functor associated to the Poincaré bundle is

$$\Psi_\gamma : D^b(\mathrm{Jac}(\Sigma_\gamma)) \xrightarrow{\cong} D^b(\mathrm{Jac}(\Sigma_\gamma))$$

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Theorem [Arinkin] C_b integral planar curve, then $\exists$ Poincaré sheaf on $\overline{\mathrm{Jac}}(C_b) \times \overline{\mathrm{Jac}}(C_b)$ and the associated integral functor is

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Duality between BAA^τ and BBB^τ

Theorem [Mukai] Σ_γ smooth, $Jac^\circ(\Sigma_\gamma)$ self-dual abelian variety. The integral functor associated to the Poincaré bundle is

$$\Psi_\gamma : D^b(Jac(\Sigma_\gamma)) \xrightarrow{\cong} D^b(Jac(\Sigma_\gamma))$$

Theorem [Arinkin] C_b integral planar curve, then $\exists$ Poincaré sheaf on $\overline{Jac}(C_b) \times \overline{Jac}(C_b)$ and the associated integral functor is

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$$\boxed{\begin{array}{l}\textbf{Theorem (– GOP)} \quad \nu_b : \Sigma_\gamma \longrightarrow C_b \text{ normalization, then} \\ \qquad \overline{\Psi}_b(\nu_{b,*}\mathcal{E}^\bullet) \otimes \mathfrak{f}_1 \cong R\tau_* \bar{\nu}^* \Psi_\gamma(\mathcal{E}^\bullet) \otimes \mathfrak{f}_2 .\end{array}}$$

Duality between BAA^T and BBB^T

Theorem [Mukai] Σ_γ smooth, $\mathrm{Jac}^\circ(\Sigma_\gamma)$ self-dual abelian variety. The integral functor associated to the Poincaré bundle is

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Theorem [Arinkin] C_b integral planar curve, then $\exists$ Poincaré sheaf on $\overline{\mathrm{Jac}}(C_b) \times \overline{\mathrm{Jac}}(C_b)$ and the associated integral functor is

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Theorem (– GOP) $\nu_b : \Sigma_\gamma \longrightarrow C_b$ normalization, then

$$\boxed{\overline{\Psi}_b(\nu_{b,*}\mathcal{E}^\bullet) \otimes f_1 \cong R\tau_* \hat{\nu}^* \Psi_\gamma(\mathcal{E}^\bullet) \otimes f_2 .}$$

Corollary $\mathrm{supp}(\overline{\Psi}_b(\nu_{b,*}\mathcal{E}^\bullet)) \cong \tau((\hat{\nu}^*)^{-1}\mathrm{supp}(\Psi_\gamma(\mathcal{E}^\bullet)))$

Duality between BAA^γ and BBB^γ

Theorem [Mukai] Σ_γ smooth, $\mathrm{Jac}^0(\Sigma_\gamma)$ self-dual abelian variety.
The integral functor associated to the Poincaré bundle is

$$\Phi_\gamma : D^b(\mathrm{Jac}(\Sigma_\gamma)) \xrightarrow{\cong} D^b(\mathrm{Jac}(\Sigma_\gamma))$$

Theorem [Arinkin] C_b integral planar curve, then $\exists$ Poincaré sheaf on $\overline{\mathrm{Jac}}(C_b)\times\overline{\mathrm{Jac}}(C_b)$ and the associated integral functor is

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the Hitchin section (BAA)

Theorem (– GOP) $\nu_b : \Sigma_\gamma \longrightarrow C_b$ normalization, then

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Corollary $\mathrm{supp}\left(\overline{\Phi}_b\left(BBB^\gamma \cap h^{-1}(b)\right)\right) \cong \tau\left((\hat{\nu}^*)^{-1}\, \underbrace{\mathcal{J}_{\Sigma_\gamma}(b)}_{=\ \mathrm{supp}(\Phi_\gamma(\mathrm{Jac}^0(\Sigma_\gamma)))}\right) \cong BAA^\gamma \cap h^{-1}(b)$

Hitchin section

An example of transfer

Given a hyperholomorphic morphism of moduli spaces

$$\begin{tikzcd} BBB_{{}^LH} \supset \mathcal{M}_\Sigma(H) \arrow[d, mapsto] \arrow[d] \\ BBB_{{}^LG} \supset \mathcal{M}_X(G) \end{tikzcd}$$

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Given a hyperholomorphic morphism of moduli spaces

$$
\begin{tikzcd}
BBB_{^LH} \arrow[r, phantom, "\supset"] \arrow[d] & \mathcal{M}_\Sigma(H) \arrow[d] \arrow[dr] & & \mathcal{M}_\Sigma(^LH) \arrow[dl] \arrow[r, phantom, "\subset"] & BAA_H \arrow[d, leftrightarrow, "\text{transfer}"] \\
BBB_{^LG} \arrow[r, phantom, "\supset"] & \mathcal{M}_X(G) \arrow[dr] & B_\Sigma(H) \arrow[d] & \mathcal{M}_X(^LG) \arrow[dl] \arrow[r, phantom, "\subset"] & BAA_G \\
& & B_X(G) & &
\end{tikzcd}
$$

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Given a hyperholomorphic morphism of moduli spaces

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\mathrm{BBB}_{{}^LH} \supset \mathcal{M}_\Sigma(H) \arrow[d, mapsto] \arrow[dr] & & \mathcal{M}_\Sigma({}^LH) \subset \mathrm{BAA}_H \arrow[dl] \arrow[d, Mapsto, "\text{transfer}"] \\
\mathrm{BBB}_{{}^LG} \supset \mathcal{M}_X(G) \arrow[dr] & B_\Sigma(H) \arrow[d] & \mathcal{M}_X({}^LG) \subset \mathrm{BAA}_G \arrow[dl] \\
& B_X(G) &
\end{tikzcd}$$

In our case

$$\begin{tikzcd}
\mathcal{M}_\Sigma(1,0) \supset \mathcal{M}_\Sigma(1,0) \arrow[d, mapsto] & \mathcal{M}_\Sigma(1,6) \subset \mathrm{im}(\sigma_\Sigma) \arrow[d, Mapsto, "\text{Hecke modifications at } \mathrm{sing}(C_s)"] \\
\mathcal{M}_X^\gamma \supset \mathcal{M}_X(2,0) & \mathcal{M}_X(2,0) \subset \mathrm{BAA}^\gamma
\end{tikzcd}$$

thanks for your attention