Loops in the fundamental group of $\mathrm{Symp}(M^4, \omega)$ which are not represented by circle actions

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Joint work with
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Outline

1. Introduction
2. Symplectic forms on rational ruled surfaces
3. Result on loops in $\pi_1(\mathrm{Symp}(\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2, \omega))$
4. Main steps in the proof
5. Further Questions

Symplectic manifolds and associated structures

- (M^{2n}, ω) symplectic manifold: smooth manifold M^{2n} with a non-degenerate closed 2–form ω.
- An almost complex structure J is called ω–tamed if $\omega(v, Jv) > 0$ for any $v \neq 0$.
- An ω-tamed almost complex structure J is called ω-compatible if $\omega(Ju, Jv) = \omega(u, v)$. Compatible (ω, J) is a Kähler structure if J is integrable.
  $\mathcal{J}_\omega$: the **nonempty contractible** space of ω-tamed (or compatible) almost complex structures.
  $c_1(M, \omega) := c_1(M, J)$.
- A symplectic form ω is called **monotone** if its class $[\omega] = \lambda c_1(M, \omega) \in H^2(M, \mathbb{Z}), \lambda > 0$.

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- $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$ is the subgroup generated by vector fields X_t such that $\iota(X_t)\omega = dH_t$ is exact.
- If M is simply connected, $\mathrm{Ham}(M, \omega)$ is the identity component of $\mathrm{Symp}(M, \omega)$. In this case, $\mathrm{Symp}(M, \omega)$ equipped with the C^∞–topology, is a ∞–dimensional Fréchet Lie group.

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Question: to which extent the topology of the symplectomorphism group is determined by compact subgroups arising from Lie group actions?

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- However the topology of $\mathrm{Symp}(M^4, \omega)$ is complicated in general!
- More results: Almost all in dimension 4. By Abreu–Granja–Kitchloo, Seidel, Pinsonnault, Evans, A–Pinsonnault, A–Eden, Li–Li–Wu, Smirnov–Shevchishin, Sheridan–Smith.
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When M is simply connected, $\pi_1(\mathrm{Ham}(M, \omega)) = \pi_1(\mathrm{Symp}(M, \omega))$.

Applications of $\pi_1(\mathrm{Ham}(M, \omega))$:

- Dynamical conjecture: for any compact (M^{2n}, ω), $\mathrm{Ham}(M, \omega)$ has infinite diameter with respect to the Hofer metric. [Polterovich, Lalonde, McDuff].

  Some proofs use a powerful tool:

  Seidel morphism: $\mathcal{S} : \pi_1(\mathrm{Ham}(M, \omega)) \to \mathrm{QH}_*(M)$ is a homomorphism to the degree $2n$ multiplicative units $\mathrm{QH}_{2n}(M)^\times$ of the small quantum homology.
- It may determine the full (rational) homotopy type of $\mathrm{Symp}(M^4, \omega)$ [A–Pinsonnault (2013), A–Eden (2017)].

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Question[McDuff, Karshon]: To what extent are $\pi_1(\mathrm{Ham}(M, \omega))$ and $\pi_1(\mathrm{Symp}(M, \omega))$ generated by symplectic S^1 actions ?

Suppose that $\pi_1(\mathrm{Symp}(M, \omega))$ is nontrivial. Is it true that some nonzero element is represented by a loop $S^1 \mapsto \mathrm{Symp}(M, \omega)$ that is a homomorphism (a circle action on M)?

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An Answer

Theorem (Kędra)

Let (M, ω) be a symplectic blow-up (in a small ball) of a closed simply connected Kähler surface, which is neither a rational nor a ruled surface up to blow-up. Then (M, ω) admits no symplectic circle action and $\pi_1(\mathrm{Symp}(M, \omega))$ is nontrivial.

Example (Kędra)

A concrete example is obtained by taking a K3 surface with any symplectic form.

Example (Buse)

On a ruled surface: there is an element $\gamma \in \pi_1(\mathrm{Ham}(\mathbb{T}^2 \times S^2))$ for which the Samelson product $[\gamma, \gamma]_{\mathbb{Q}}$ does not vanish.

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Reduced form

$\mathbb{X}_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$

Definition (Reduced symplectic form)

Consider $\mathbb{X}_n$ with the standard basis $\{L, V_1, \ldots, V_n\}$ of $H_2(\mathbb{X}_n; \mathbb{Z})$. A symplectic form ω is called **reduced** if it can be normalized to have area $1, \delta_1, \ldots, \delta_n$ on the basis $L, V_1, \ldots, V_n$ such that

$$1 > \delta_1 \geq \ldots \geq \delta_n > 0 \quad \text{and} \quad \nu \geq \delta_i + \delta_j + \delta_k.$$

Fact

- *The diffeomorphism class of ω only depends on its cohomology class $[\omega] = PD(H - \delta_1 V_1 - \ldots - \delta_n V_n)$*
- *Any ω on $\mathbb{X}_n$ is diffeomorphic to a reduced one.*
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Equivalence with the $S^2 \times S^2$ model

$M_{\mu,c_1,\dots,c_{n-1}} := (S^2 \times S^2 \# (n-1)\overline{\mathbb{CP}}^2, \omega_{\mu,c_1,\dots,c_{n-1}})$

is obtained from $(S^2 \times S^2, \mu\sigma \oplus \sigma)$, by performing $n-1$ successive blow-ups of capacities $c_1, \dots, c_{n-1}$, where σ denotes the standard symplectic form on S^2 that gives area 1 to the sphere and $\mu \geq 1$.

This can be naturally identified with $(\mathbb{X}_n, \omega)$. If $\{B, F, E_1, \dots, E_{n-1}\}$ is the natural basis for $H_2(S^2 \times S^2 \# (n-1)\overline{\mathbb{CP}}^2; \mathbb{Z})$ then the transition on the basis is explicitly given by

$$B = L - V_2, \quad F = L - V_1, \quad E_1 = L - V_1 - V_2, \quad E_i = V_{i+1}, \forall i \geq 2.$$

And for parameters satisfying the relations

$$\mu = \frac{1-\delta_2}{1-\delta_1}, \quad c_1 = \frac{1-\delta_1-\delta_2}{1-\delta_1}, \quad \text{and} \quad c_i = \frac{\delta_{i+1}}{1-\delta_1}, \quad 2 \leq i \leq n-1.$$

there exists a symplectomorphism between the two symplectic manifolds encoded by these parameters such that

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This can be naturally identified with $(\mathbb{X}_n, \omega)$. If $\{B, F, E_1, \dots, E_{n-1}\}$ is the natural basis for $H_2(S^2 \times S^2 \# (n-1)\overline{\mathbb{CP}}^2; \mathbb{Z})$ then the transition on the basis is explicitly given by

$$B = L - V_2, \quad F = L - V_1, \quad E_1 = L - V_1 - V_2, \quad E_i = V_{i+1}, \forall i \geq 2.$$

And for parameters satisfying the relations

$$\mu = \frac{1-\delta_2}{1-\delta_1}, \quad c_1 = \frac{1-\delta_1-\delta_2}{1-\delta_1}, \quad \text{and} \quad c_i = \frac{\delta_{i+1}}{1-\delta_1}, \quad 2 \leq i \leq n-1.$$

there exists a symplectomorphism between the two symplectic manifolds encoded by these parameters such that

$$H - \delta_1 V_1 - \dots - \delta_n V_n = \mu B + F - c_1 E_1 - \dots - c_n E_n.$$

Symplectomorphisms of $M_{\mu,c_1,\ldots,c_{n-1}}$

$\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_n})$ is the group of symplectomorphisms of $M_{\mu,c_1,\ldots,c_n}$ acting trivially in homology.

- If $n \leq 3$ then $\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_n})$ is connected [Pinsonnault, Evans, Li-Li-Wu];
- If $n \leq 3$ then $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_n}))$ is generated by S^1 actions [follows from Pinsonnault, A-Pinsonnault, A-Eden, Li-Li-Wu];
- $\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_4}, \omega_{mon}) \simeq \mathrm{Diff}^+(S^2, 5)$, where $\mathrm{Diff}^+(S^2, 5)$ is the group of orientation preserving diffeomorphisms of S^2 fixing 5 points [Evans, Seidel]. $\pi_0(\mathrm{Diff}^+(S^2, 5)) = PB_5(S^2)/\mathbb{Z}^2$ is the pure braid group of 5 strings on S^2 and its fundamental group is trivial.

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Topology of $\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega)$

Consider the edge in P_5, denoted by MA, starting at the monotone point M, where $\mu > 1$ and $c_i = \frac{1}{2}$.

Note that vertex representing the monotone case corresponds to $\mu = 1$ and $c_i = \frac{1}{2}$.

Theorem (Li-Li-Wu'18)

- *Along MA, $\pi_0(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega)) = \pi_0(\mathrm{Diff}^+(S^2,4)) = PB_4(S^2)/\mathbb{Z}^2$ and π_0 is trivial for the remaining points in P_5.*
- *$\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega)) = \mathbb{Z}^5$ along the edge MA.*
- *$rank(\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega))) = N_\omega - 5 + rank(\pi_0(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega)))$, where N_ω is the number of symplectic -2 spheres classes and the rank of π_0 means the rank of its abelianization.*

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Main Result

Recall that along the edge MA: $\mu > 1$ and $c_i = \frac{1}{2}$, $i = 1, \ldots, 4$.

Theorem (A-Barata-Pinsonnault-Reis)

- *If* $1 < \mu \leq \frac{3}{2}$ *then along the edge MA there is a loop in* $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_4},\omega))$ *which cannot be represented by a circle action.*
- *If* $\mu > \frac{3}{2}$ *then* $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_4},\omega))$ *is generated by Hamiltonian circle actions.*

Conjecture: There is a neighbourhood of the monotone point M in the reduced cone such that the generators of the fundamental group of $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_4},\omega))$ cannot all be realized by circle actions.

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Karshon's classification

- Hamiltonian S^1-space (M, ω, Φ): symplectic manifold with a Hamiltonian circle action and moment map $\Phi : M \to \mathbb{R}$.
- Critical set of $\Phi = \{\text{fixed points}\}$. $n = 4$: critical set consists of isolated points and 2-dim submanifolds (only at the extrema of Φ).

Decorated graphs:

- each isolated fixed point $p \to$ a vertex $\langle p \rangle$, labeled by $\Phi(p)$.
- each two-dimensional invariant surface $S \to$ a fat vertex $\langle S \rangle$, labeled by $\Phi(S)$, the symplectic area $\omega(S)$, and the genus g of the surface S.
- A $\mathbb{Z}_k$-sphere is a sphere in M on which S^1 acts with isotropy $\mathbb{Z}_k$. Each $\mathbb{Z}_k$-sphere containing two fixed points p and $q \to$ an edge connecting the vertices $\langle p \rangle$ and $\langle q \rangle$ labeled by the integer k.

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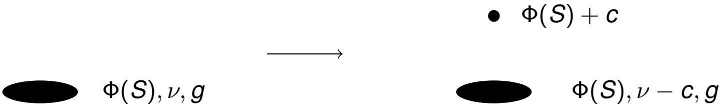
The classification keeps track of symplectic blow-ups.

Blowing-up at a point inside an invariant surface at the minimum value of Φ:



Blowing-up at an interior fixed point:

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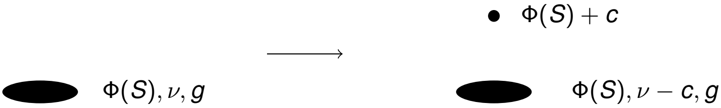

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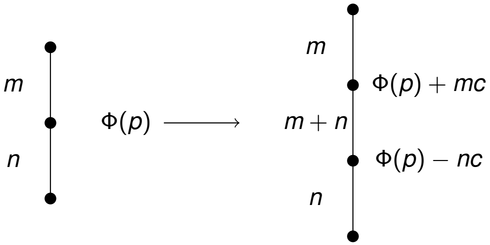

Classification of compact four dimensional Hamiltonian S^1-spaces

Theorem (Karshon)

Every compact four dimensional Hamiltonian S^1-space space is obtained by a sequence of S^1-equivariant symplectic bow-ups from

- *a space with two fixed surfaces and no interior fixed points, or*
- $\mathbb{CP}^2$ *or a Hirzebruch surface, with a symplectic form and a circle action that come from a Kähler form and a toric action.*

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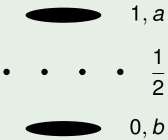

Extended graphs

- $\mathrm{Symp}_h(M,\omega) \to \mathrm{Symp}(M,\omega) \to \mathrm{Aut}(H_2(M,\mathbb{Z})) = \mathrm{Aut}_{c_1,[\omega]}$;
- Along MA, $\mathrm{Symp}/\mathrm{Symp}_h \simeq \mathbb{D}_4$, where $\mathbb{D}_4$ is the Weyl group of the Dynkin diagram of type $\mathbb{D}$ with 4 vertices;
- To keep track of the action of Symp on homology $\to$ consider graphs in which can read the homology;
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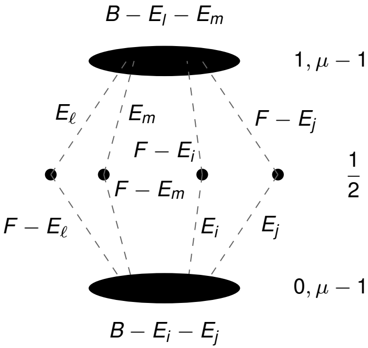
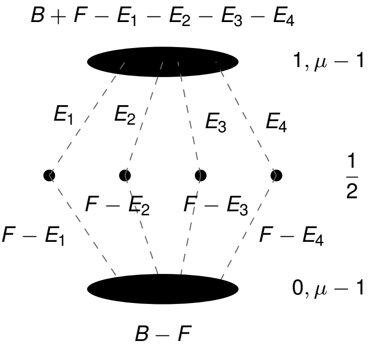
Family of graphs along MA ($\mu > 1$ and $c_i = \frac{1}{2}$)



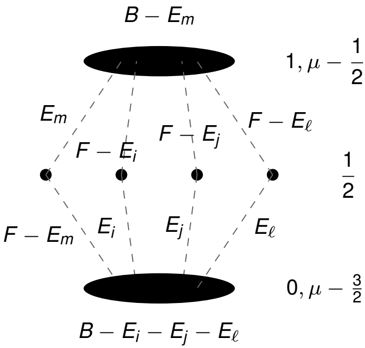
Circle action $z_{0,ij}$



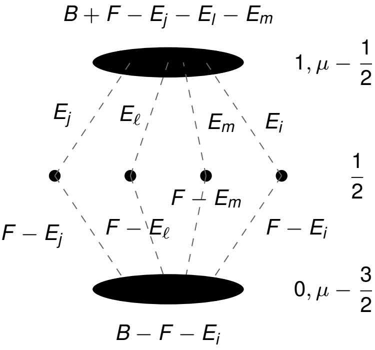
Circle action z_1

$i, j, \ell, m \in \{1, \ldots, 4\}$ are all distinct

New family of graphs if $\mu > \frac{3}{2}$



Circle action $z_{0,ij\ell}$



Circle action $z_{1,i}$

List of Hamiltonian S^1-spaces along MA

Lemma

The Hamiltonian circle actions on the symplectic manifolds encoded by the edge MA are of 5 types:

- z_k, *with fixed spheres in classes* $B - kF$ *and* $B + kF - E_1 - E_2 - E_3 - E_4$ *(exists iff* $\mu > k$ *and* $\mu > 2 - k$*);*
- $z_{k,i}$, *with fixed spheres in classes* $B - kF - E_i$ *and* $B + kF - E_j - E_\ell - E_m$ *(exists iff* $\mu > k + \frac{1}{2}$ *and* $\mu > \frac{3}{2} - k$*);*
- $z_{k,ij}$, *with fixed spheres in classes* $B - kF - E_i - E_j$ *and* $B + kF - E_\ell - E_m$ *(exists iff* $\mu > k + 1$*);*
- $z_{k,ij\ell}$, *with fixed spheres in classes* $B - kF - E_i - E_j - E_\ell$ *and* $B + kF - E_m$ *(exists iff* $\mu > k + \frac{3}{2}$*);*
- $z_{k,1234}$, *with fixed spheres in classes* $B - kF - E_1 - E_2 - E_3 - E_4$ *and* $B + kF$ *(exists iff* $\mu > k + 2$*).*

Remarks

- When $1 < \mu \leq \frac{3}{2}$ there exist only four Hamiltonian circle actions: $z_{0,12}, z_{0,13}, z_{0,14}, z_1$. Not enough to justify $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_4},\omega)) = \mathbb{Z}_5$. The graphs only encode equivariant blow-ups. But there are no "exotic" circle actions by works of Karshon, Kessler and Pinsonnault.
  $\Rightarrow$ there exist a loop in π_1 which is not realized by a circle action.
- The number of Hamiltonian circle actions keeps increasing as the values of μ increase, but the rank of π_1 remains constant along MA [Li-Li-Wu] as μ increases $\Rightarrow$ there can only be at most 5 independent circle actions as elements of the fundamental group.

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A generating set for $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_4},\omega))$ (if $\mu > \frac{3}{2}$)

Claim: $z_{0,12}, z_{0,13}, z_{0,14}, z_1$ and $z_{1,4}$, seen as elements of the fundamental group, form a basis of $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\ldots,c_4},\omega))$ along MA, if $\mu > \frac{3}{2}$.

Steps in the proof:

- Obtain relations between the loops $z_k, z_{k,i}, z_{k,ij}, z_{k,ij\ell}$ and $z_{k,1234}$, that come from embedding pairs of loops inside torus actions. Show, in particular, all loops are linear combinations of these 5 actions. Uses Delzant's classification of toric actions and Karshon's classification.
- Compute the Seidel elements of $z_{0,12}, z_{0,13}, z_{0,14}, z_1$ and $z_{1,4}$, i.e., the image of these 5 loops in $\mathrm{QH}_4(M_{\mu,c_1,\ldots,c_4})$ by the Seidel morphism $\mathcal{S} : \pi_1(\mathrm{Ham}(M,\omega)) \to \mathrm{QH}_*(M)$;
- Show that the 5 Seidel elements generate a free subgroup of dimension 5 of the group of invertible elements in $\mathrm{QH}_*(M_{\mu,c_1,\ldots,c_4})$.

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- Compute the Seidel elements of $z_{0,12}, z_{0,13}, z_{0,14}, z_1$ and $z_{1,4}$, i.e., the image of these 5 loops in $\mathrm{QH}_4(M_{\mu,c_1,\ldots,c_4})$ by the Seidel morphism $\mathcal{S} : \pi_1(\mathrm{Ham}(M,\omega)) \to \mathrm{QH}_*(M)$;
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A generating set for $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega))$ (if $\mu > \frac{3}{2}$)

Claim: $z_{0,12}, z_{0,13}, z_{0,14}, z_1$ and $z_{1,4}$, seen as elements of the fundamental group, form a basis of $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega))$ along MA, if $\mu > \frac{3}{2}$.

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Delzant classification

Definition

A **Delzant polytope** in $\mathbb{R}^n$ is a convex polytope such that the n edges meeting at each vertex are given by a basis of $\mathbb{Z}^n$.

Definition

A **symplectic toric manifold** is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of a torus $\mathbb{T}^n$ and with a choice of a moment map $\phi : M \to \mathbb{R}^n$.

Theorem (Delzant)

Symplectic toric manifolds up to equivariant symplectomorphisms are classified by Delzant polytopes up to transformations of $GL(2, \mathbb{Z})$.

$$\{\textit{toric manifolds}\} \to \{\textit{Delzant polytopes}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \phi) \mapsto \phi(M).$$

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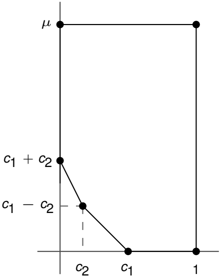
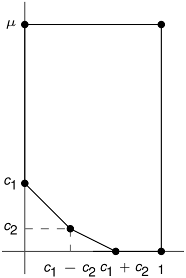
Example

Consider the two toric actions on $\mathrm{Symp}_h(M_{\mu,c_1,c_2})$ $\mathbb{T}^2_0$ and $\mathbb{T}^2_1$



$\mathbf{x}_0$, $\mathbf{y}_0$



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Performing the $SL(2,\mathbb{Z})$ transformation given by the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

in the toric manifolds equipped with $\mathbb{T}^2_0$ ($\mathbf{x}_0$, $\mathbf{y_0}$) and $\mathbb{T}^2_1$($\mathbf{x}_1$,$\mathbf{y_1}$) one obtains

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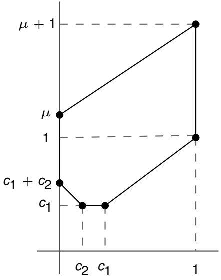

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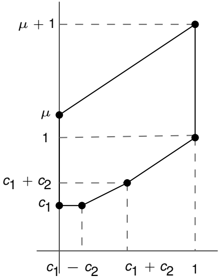

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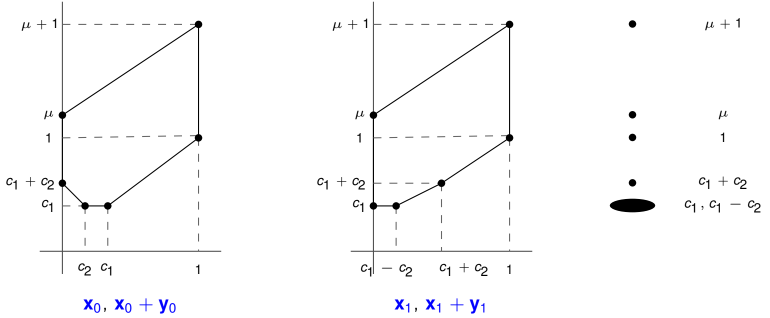

Projecting in the direction perpendicular to the y-axis we obtain the same graph for the two polytopes: .

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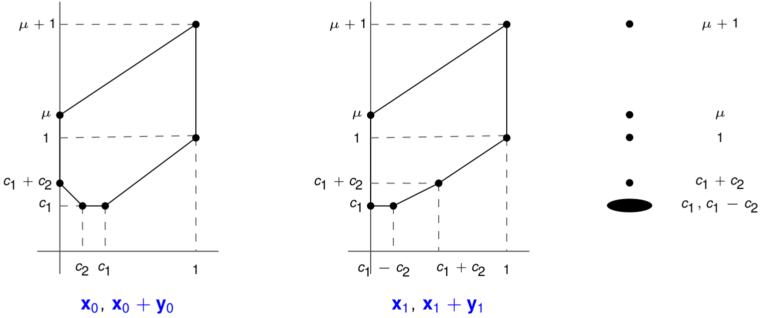

Projecting in the direction perpendicular to the y-axis we obtain the same graph for the two polytopes: $\mathbf{x}_0 + \mathbf{y}_0 = \mathbf{x}_1 + \mathbf{y}_1$.

(small) Quantum homology

$$\mathrm{QH}_*(M;\Pi) = H_*(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \Pi^{\mathrm{univ}}[q,q^{-1}]$$

where q is a polynomial variable of degree 2 and Π^{univ} (Novikov ring) is a generalised Laurent series ring in a variable of degree 0:

$$\Pi^{\mathrm{univ}} := \left\{ \sum_{\kappa\in\mathbb{R}} r_\kappa t^\kappa \,\middle|\, r_\kappa \in \mathbb{Q},\ \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}.$$

$\mathrm{QH}_*(M;\Pi)$ is $\mathbb{Z}$–graded: $\deg(a \otimes q^d t^\kappa) = \deg(a) + 2d$ with $a \in H_*(M)$.

Quantum intersection product: $a * b \in \mathrm{QH}_{i+j-\dim M}(M;\Pi)$, where $a \in H_i(M)$ and $b \in H_j(M)$ depends on some Gromov-Witten invariants.

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$\mathrm{QH}_*(M_{\mu,c_1,\ldots,c_4})$

Let $b_{ij} = (B - E_i - E_j) \otimes q^{\frac{t^{\frac{1}{2}}}{1-t^{1-\mu}}}$, $f_{ij} = (F - E_i - E_j) \otimes q^{\frac{t^{\frac{1}{2}}}{1-t^{1-\mu}}}$ and $e_i = E_i \otimes q^{\frac{t^{\frac{1}{2}}}{1-t^{1-\mu}}}$.

Theorem (A-Barata-Pinsonnault-Reis)

$$\mathrm{QH}_*(M_{\mu,c_1,\ldots,c_4}) \simeq \Pi^{\mathrm{univ}}[b_{ij}, f_{ij}, e_i]/(relations),$$

where the relations are the following:

$b_{ij}^2 = 2b_{ij}f_{ij} + f_{ij} + f_{k\ell} + 1$, $\quad b_{ij}b_{ik} = b_{ij}f_{ij} + f_{j\ell} + 1$, $\quad b_{ij}b_{k\ell} = 1$, $\quad f_{ij}f_{k\ell} = 0$,
$f_{ij}f_{ik} = f_{ij}(b_{ij}+1)$, $\quad f_{ij}^2 = 2f_{ij}(b_{ij}+1)$, $\quad f_{ik}(b_{ij}+1) = 0$, $\quad (f_{ij}+f_{k\ell})(b_{ij}+1) = 0$,
$f_{ij}(e_k + \frac{t^{1-\mu}}{1-t^{1-\mu}}) = 0$, $\quad b_{ij}(f_{ij} + e_i + \frac{t^{1-\mu}}{1-t^{1-\mu}}) = e_j + \frac{t^{1-\mu}}{1-t^{1-\mu}}$, $\quad f_{ij}(b_{ij} + e_i + \frac{1}{1-t^{1-\mu}}) = 0$,
$b_{ij}(e_k + \frac{t^{1-\mu}}{1-t^{1-\mu}}) = f_{k\ell} + e_\ell + \frac{t^{1-\mu}}{1-t^{1-\mu}}$, $\quad e_i^2 = e_ie_j + f_{ij} + b_{ij}f_{ij} + (e_j - e_i)\frac{t^{1-\mu}}{1-t^{1-\mu}}$.

It follows from the formulas for the quantum product on a rational surface obtained by Crauder-Miranda'95.

Seidel morphism

$\mathcal{S} : \pi_1(\mathrm{Ham}(M,\omega)) \to \mathrm{QH}_*(M)$ "counts" pseudo-holomorphic sections of a bundle $M_\Lambda \to S^2$ associated to a loop $\Lambda \subset \mathrm{Ham}(M,\omega)$. M_Λ is the total space of the fibration over S^2 with fiber M which consists of two trivial fibrations over 2–discs, glued along their boundary via Λ.

Theorem (McDuff-Tolman'06)

Let Λ be an Hamiltonian circle action on (M,ω) with moment map Φ_Λ. If the maximal fixed point component $F_{\max}$ is semifree and $[F_{\max}] = A \in H_*(M)$ then there are classes $a_B \in H_*(M)$ s. t.
$\mathcal{S}(\Lambda) = A \otimes qt^{\Phi_{\max}} + \sum_{B \in H_2^S(M;\mathbb{Z})>0} a_B \otimes q^{1-c_1(B)} t^{\Phi_{\max} - \omega(B)}$.

- if there exists an almost complex structure J on M so that (M,J) is Fano (all J–pseudo-holomorphic spheres in M have positive first Chern number) and the codimension of $F_{\max}$ is 2 then $\mathcal{S}(\Lambda) = A \otimes qt^{\Phi_{\max}}$. [McDuff-Tolman'06]
- if (M,J) is NEF ($c_1(B) \geq 0$ for every class $B \in H_2(M)$ with a J-holomorphic sphere representative) $\to$ there are infinitely many contributions to the Seidel elements, but can be expressed by closed formulas [A-Leclercq'18].

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Seidel elements

Seidel elements of a generating set of $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4}))$ if $\mu > \frac{3}{2}$:

- $\mathcal{S}(z_{0,1i}) = b_{j\ell}, \quad i = 2, 3, 4$ and i, j, ℓ all distinct;
- $\mathcal{S}(z_1) = b_{12} + f_{34}$;
- $\mathcal{S}(z_{1,4}) = (b_{12} + f_{34} + e_4 + \frac{t^{1-\mu}}{1-t^{1-\mu}})t^{\alpha}(1 - t^{1-\mu})$ where $\alpha = \frac{1}{6(1-2\mu)}$.

Further questions

Question 1: Are there other points in the reduced cone P_5 for which not all the generators of $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4}))$ can be represented by Hamiltonian circle actions ?

Conjecture: Yes. There is a neighbourhood of the monotone point M in the reduced cone such that the generators of the fundamental group of $\pi_1(\mathrm{Symp}_h(M_{\mu,c_1,\dots,c_4},\omega))$ cannot all be realized by circle actions.

Main reason: it appears that at least one circle action that has a fixed sphere with positive area in class $B-E_j-E_k-E_\ell$ or $B-F-E_i$ ($\mu-c_j-c_k-c_\ell>0$ or $\mu-1-c_i>0$) has to be included in the set of generators. However, this condition does not necessarily hold for all points in the symplectic cone, in particular, for points close to the monotone point M ($\mu=1$ and $c_i=\frac{1}{2}$).

Question 2: Does the loop which cannot be represented by an Hamiltonian circle action give rise to new elements in higher homotopy groups (via Samelson products) as in Buse's example?

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Thank you!