Bott Canonical basis ?

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complex geometry motivated by repr. theory

following Mike Grossberg

Canonical basis

$K \circlearrowright V$ unirrep

K compact, connected, simply connected

$T \subset K$ maximal torus

$$V = \bigoplus_{\alpha \in t^*_{\mathbb{Z}}} V_\alpha$$

isotypic decomposition into weight spaces

$$T \xrightarrow[a \mapsto a^\alpha]{} S^1 \circlearrowright \mathbb{C}_\alpha$$

$$V_\alpha = \underbrace{\mathbb{C}_\alpha \oplus \ldots \oplus \mathbb{C}_\alpha}_{\text{mult}(\alpha)}$$

goal: decompose "canonically"

We achieve this using a complex analytic big torus action modulo a conjectural cohomology-vanishing condition.

Choices: T, $\underset{\text{Borel}}{B} \subset G := K_{\mathbb{C}}$, reduced expression for longest element of Weyl group

Big torus action

$$(S^1)^{\text{big}} \circlearrowright \begin{array}{c} L \\ \downarrow \\ M \end{array} \quad \begin{array}{l} \text{holomorphic line bundle} \\ \\ \text{connected, } \exists \text{ fixed point} \end{array}$$

big $:= \dim_{\mathbb{C}} M$

Then $\Gamma_{hol}(M,L)$ splits into one dim'l weight spaces.

Geometric models for representations

$$V = \Gamma_{hol}(M,L) \quad \text{for} \quad \begin{array}{c} L \\ \downarrow \\ M \end{array}$$

Borel-Weil : M = flag mfld

Demazure : M = complex Bott-samelson mfld

Bott : use $(S^1)^{\text{big}} \circlearrowright$ Bott-Samelson mfld
to decompose $V = \Gamma_{hol}(M,L)$

problem: action is holomorphic
for different cplx structure
"Bott tower"

Borel-Weil. $\qquad G = K_{\mathbb{C}}$

e.g. $\quad K = SU(n)$, $\quad G = SL(n, \mathbb{C})$

$H = T_{\mathbb{C}}$ Cartan

$\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ positive Weyl chamber

$B \subset G$ Borel e.g. $\begin{bmatrix} * & \ddots & * \\ 0 & & * \end{bmatrix}$

$$\{\text{unirreps } K \circlearrowright V\} \longleftrightarrow \mathfrak{t}^*_{\mathbb{Z}} \cap \mathfrak{t}^*_+$$

$V \longmapsto \lambda \quad$ = highest weight

$V \longleftarrow\!\shortmid \lambda \quad$ Borel-Weil:

$$B = H \cdot U \xrightarrow{\text{project}} H \circlearrowright \mathbb{C}_{-\lambda}$$

$$\begin{array}{ccccc} & K \times_T \mathbb{C}_{-\lambda} & \xrightarrow{\cong} & G \times_B \mathbb{C}_{-\lambda} & =: L \\ K \circlearrowright & \downarrow & & \downarrow & \\ & K/T & \xrightarrow{\cong} & G/B & = M \end{array}$$

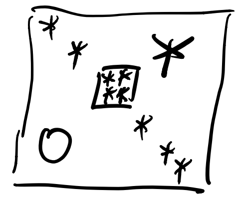
$$V = \Gamma_{hol}(M, L)$$

Bott–Samelson

$\alpha_1, \ldots, \alpha_r \in \mathfrak{t}^*_{\mathbb{Z}}$ simple positive roots

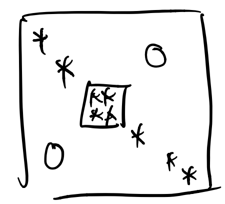
P_i parabolic

e.g.



$K_i \subset P_i$ maximal cpct

e.g



$$T_i = Z(K_i)$$

choose sequence of simple positive roots

$$\alpha_{i_1}, \ldots, \alpha_{i_n}$$

Bott–Samelson mfld:

$$K_{i_1} \times_T K_{i_2} \times_T \cdots \times_T K_{i_n} / T$$

$$[k_1, k_2, \ldots, k_n] = [k_1 a_1, a_1^{-1} k_1 a_2, \ldots, a_{n-1}^{-1} k_n a_n]$$

$$(S^1)^{big} = T \times_{T_{i_1}} T \times_{T_{i_2}} \cdots \times_{T_{i_{n-1}}} T / T_{i_n}$$

Complex Bott-Samelson

Hansen, Demazure
mid 1970s

$$\begin{array}{ccc}
K_{i_1} \times_T K_{i_2} \times_T \cdots \times_T K_{i_n} / T & \xrightarrow{\cong} & P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_n} / B \\
\downarrow \text{mult} & & \downarrow \text{mult} \\
K/T & \xrightarrow{\cong} & G/B
\end{array}$$

+ associated line bundles

$$\Gamma_{hol}(G \times_B \mathbb{C}_{-\lambda}) \xrightarrow{\text{pullback}} \Gamma_{hol}(P_{i_1} \times_B \cdots \times_B P_{i_n} \times_B \mathbb{C}_{-\lambda})$$

is an isomorphism

if $\alpha_{i_1}, \ldots, \alpha_{i_n}$ come from a reduced expression for longest element of the Weyl gp.

Demazure
= complex Bott-Samelson:

$$[p_1, \cdots, p_n] = [p_1 b_1, b_1^{-1} p_2 b_2, \cdots, b_{n-1}^{-1} p_n b_n]$$

Bott tower: $[p_1, \cdots, p_n] = [p_1 b_1, h_1^{-1} p_2 b_2, \cdots, h_{n-1}^{-1} p_n b_n]$

$$b_j \longmapsto h_j$$

$$B \longrightarrow H$$

Mike Grossberg: family of cplx strs $\{J_t\}_{t\in[0,1]}$

on Bott-Samelson

s.t. $t=1$: Demazure

$t=0$: Bott tower

Joseph Bernstein, ≈ 1993:

try to extend toric action on special fibre

to action on the family

"Test configuration" $\mathbb{C}^\times \circlearrowright \begin{matrix} \mathcal{X} \\ \downarrow \\ \mathbb{C} \end{matrix}$ algebro geometric

base $= \mathbb{C}$

Donaldson 2002 ; David Witt Nyström 2012

our setup: cplx analytic

base $= \mathbb{C}^n$

Bott Samelson family

$$s : \mathbb{C}^\times \longrightarrow \underset{\text{Cartan}}{H} \qquad \text{s.t.}$$

$\forall$ simple root α_i

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "s"] \arrow[rr, bend right, "t \mapsto t^q"'] & H \arrow[r, "a \mapsto a^{\alpha_i}"] & \mathbb{C}^\times \end{tikzcd}$$

smallest q.

$\Longrightarrow \quad \psi_t : B \to B \,, \quad t \in \mathbb{C}$

$t \neq 0 \;:\quad b \mapsto s(t)\, b\, s(t)^{-1}$

$t = 0 \;:\quad b = hu \longmapsto h$

$B \to H$

e.g. $SU(3)$ $\qquad s(t) = \begin{bmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{bmatrix}$

$$\psi_t : \begin{pmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a & td & t^2 f \\ 0 & b & te \\ 0 & 0 & c \end{pmatrix}$$

$$\begin{array}{c} \mathcal{X} \\ \pi \downarrow \\ \mathbb{C}^n \ni t \end{array} \qquad X_t \ni [t; p_1, \cdots, p_n] = [t; p_1 b_1, \psi_{t_2}(b_1)^{-1} p_2 b_2, \cdots, \psi_{t_n}(b_{n-1})^{-1} p_n b_n]$$

Left action of $H \times (\mathbb{C}^\times)^n$:

$$(h, \tau_1, \cdots, \tau_n) : [t_1, \cdots, t_n; p_1, \cdots, p_n] \mapsto [\tau_1 t_1, \cdots, \tau_n t_n; h\, s(\tau_1) p_1, s(\tau_2) p_2, \cdots, s(\tau_n) p_n]$$

+ associated line bundles

Equivariant families over $\mathbb{C}^n$

$$H \times (\mathbb{C}^\times)^n \circlearrowright \quad \begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathcal{X} \\ \downarrow \\ \mathbb{C}^n \end{array} \quad \begin{array}{l} \text{holomorphic line bundle} \\ \\ \text{complex, connected} \\ \pi = (t_1, \cdots, t_n) \\ \text{holomorphic proper submersion} \end{array}$$

$$X_0 := \pi^{-1}(0, \cdots, 0) \qquad X_1 := \pi^{-1}(1, \cdots, 1)$$

$$L_0 := \mathcal{L}|_{X_0} \qquad L_1 := \mathcal{L}|_{X_1}$$

Assume

- $\forall j \quad (\mathbb{C}^\times)_{j\text{th}} \circlearrowright X_0$ has a fixed point w/ isotropy wts ≤ 0
- $H^{\geq 1}(\mathcal{X}, \mathcal{O}_{\mathcal{L}}) = \{0\}$

$\Rightarrow$ "holomorphic Hadamard lemma":

$\forall \; s \in \Gamma_{hol}(\mathcal{X}, \mathcal{L})$, if $s|_{X_0} = 0$, then

$s = t_1 s_1 + \ldots + t_n s_n$ for some $s_1, \cdots, s_n \in \Gamma_{hol}(\mathcal{X}, \mathcal{L})$

$$\begin{tikzcd} & \mathcal{L} \arrow[d] \\ (\mathbb{C}^\times)^n \arrow[r, phantom, "\circlearrowright"] & \mathcal{X} \arrow[d] \\ & \mathbb{C}^n \end{tikzcd}$$

rough idea:

$$\begin{tikzcd} & \Gamma_{hol}(\mathcal{X}, \mathcal{L}) \arrow[dl] \arrow[dr] & \\ \Gamma_{hol}(X_1, L_1) \arrow[rr, dashed, "?"] & & \Gamma_{hol}(X_0, L_0) \end{tikzcd}$$

<u>"sweep"</u>: $\quad s \in \Gamma_{hol}(X_1, L_1) \longmapsto \tilde{s} \in \Gamma_{hol}(\mathcal{X}_{reg}, \mathcal{L})^{(\mathbb{C}^\times)^n}$

$$\mathcal{X}_{reg} := \bigcap_j \{t_j \neq 0\} \subset \mathcal{X}$$

<u>Lemma</u> singularities of $\tilde{s}$ at each $\{t_j = 0\}$ are poles of bounded order.

<u>"twist"</u>: $\quad \tilde{S} \longmapsto \underbrace{t^{-\vec{\ell}}}_{t_1^{-\ell_1} \cdots t_n^{-\ell_n}} \tilde{S}$ for $\vec{\ell} = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n$

for $t = \pi(x)$

Filtration of $V := \Gamma_{hol}(X_1, L_1)$:

$$F_{\vec{\ell}} := \left\{ s \in \Gamma_{hol}(X_1, L_1) \;\middle|\; t^{-\vec{\ell}}\tilde{s} \text{ extends to } \mathcal{X} \right\}$$

$$\xrightarrow[\text{sweep, twist, extend to } \mathcal{X}, \text{ restrict to } X_0]{} \Gamma_{hol}(X_0, L_0)$$

partial ordering:

$$\vec{\ell}' \geq \vec{\ell} \quad :\Leftrightarrow \quad \forall j \;\; \ell_j' \geq \ell_j$$

Lemma. $\mathrm{Kernel}\left(F_{\vec{\ell}} \to \Gamma_{hol}(X_0, L_0) \right) = F_{>\vec{\ell}}$

$$F_{\vec{\ell}} / F_{>\vec{\ell}} \longleftrightarrow \underbrace{\Gamma_{hol}(X_0, L_0)}_{\text{one dim'l if } X_0 \text{ is toric}}$$

$\vec{\ell}$th weight space in $\Gamma_{hol}(X_0, L_0)$

Thm. $V_{graded} := \bigoplus_{\vec{\ell}} F_{\vec{\ell}} / F_{>\vec{\ell}} \longrightarrow V = \Gamma_{hol}(X, L_1)$

is 1-1 & onto.

THANK YOU!