The essential minimal volume of manifolds

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Definitions

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$$\text{If } M_{>\delta} := \{x \in M; \quad \mathrm{injrad}_g(x) > \delta\},$$

$$\text{ess-MinVol}(M) := \lim_{\delta\to 0} \quad \inf\{\mathrm{Vol}(M_{>\delta},g); \quad |\sec_g| \le 1\}.$$

$$\le \quad \mathrm{MinVol}(M)$$

Motivations

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Taken literally, the above question is overly optimistic; one needs to allow certain degenerations of the manifold.

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Taken literally, the above question is overly optimistic; one needs to allow certain degenerations of the manifold.

We consider the following problem: realize those invariants by geometric objects, i.e. find natural maps from sets of topological spaces to sets of Riemannian spaces.

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Examples of positive answers: Uniformization of 2-surfaces, Geometrization of 3-manifolds, (other results under additional assumptions like in conformal geometry or Kähler geometry).

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Works of Nabutovsky on recognizability of Einstein metrics, Nabutovsky-Weinberger on local minima of diameter when $|\sec| \leq 1$.

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We will use curvature bound assumptions on the manifold M^n. Denote by Scal and Sec the scalar and sectional curvature and recall that

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By the solution of the Yamabe problem, it is expected that if g_i are metrics on M such that $|\,\mathrm{Scal}_{g_i}\,| \leq n(n-1)$ and

$$\lim_{i\to\infty} \mathrm{Vol}(M, g_i) = \inf\{\mathrm{Vol}(M, g); \quad |\,\mathrm{Scal}_g\,| \leq n(n-1)\},$$

then g_i should subsequentially converge to a generalized Einstein metric space (when non-empty).

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- the "thin part" $M_{\leq\epsilon}$ is well described by the collapsing theory of Cheeger-Fukaya-Gromov.

Here, for $\epsilon > 0$,

$$M_{>\epsilon} := \{x \in M; \quad \text{injectivity radius at } x > \epsilon\},$$

$$M_{\leq\epsilon} := \{x \in M; \quad \text{injectivity radius at } x \leq \epsilon\}.$$

Motivations

$M_{>\epsilon}$ admits a triangulation with number of vertices bounded by $C_\epsilon \operatorname{Vol}(M, g)$ and degree at each vertex bounded by C_ϵ. In fact by Cheeger, if g_i is a sequence of metrics on M with $|\operatorname{Sec}_{g_i}| \leq 1$ and $\operatorname{injrad}_{g_i} \geq \epsilon$, then a subsequence g_i converges to a $C^{1,\alpha}$-metric (see also Peters, Greene-Wu).

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$M_{\leq\epsilon}$ carries an F-structure (Cheeger-Gromov) and an N-structure (Cheeger-Fukaya-Gromov). These structures generalize respectively actions by tori and nilpotent Lie groups.

CG: $\exists$ F-structure $\Longleftrightarrow$ collapsing with $|\sec| \leq C$

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Besson-Courtois-Gallot: If (X, g_X) has negative curvature $\sec_{g_X} \leq -1$, then

$$\mathrm{MinVol}(X) \geq \mathrm{Vol}(X, g_X)$$

with equality if and only if X is hyperbolic.

(if X hyp, MinVol(X) = Hyp Vol(X))

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Our goal is to introduce a natural variant of MinVol which is achieved by the volume of a minimizer, and study those minimizers (geometric interpretation, estimates for negatively curved manifolds, Einstein 4-manifolds and complex surfaces).
It is a "sectional curvature" approach to Hopf-Thom-Yau's question.

Definition of ess-MinVol

Let $\mathcal{M}_{|\,\mathrm{Sec}\,|\leq 1}(M)$ be the set of metrics on M with $|\,\mathrm{Sec}\,| \leq 1$.
Recall that

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Alternative definition for ess-MinVol

We consider a weak closure $\overline{\mathcal{M}}^{w}_{|\,\mathrm{Sec}\,|\leq 1}(M)$ of $\mathcal{M}_{|\,\mathrm{Sec}\,|\leq 1}(M)$, by using a weak notion of convergence:

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a sequence of metrics $g_i \in \mathcal{M}_{|\,\mathrm{Sec}\,|\leq 1}(M)$ weakly converges if for any $\delta > 0$, the δ-thick part of (M, g_i) converges in a multi-pointed Gromov-Haudorff sense.

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By Cheeger/Gromov, it turns out that

$$\text{ess-MinVol}(M) = \inf\{\operatorname{Vol}(Y, h); \quad (Y, h) \in \overline{\mathcal{M}}^{w}_{|\operatorname{Sec}|\leq 1}(M)\},$$

and that there exists a weak minimizer $(M_\infty, g_\infty) \in \overline{\mathcal{M}}^{w}_{|\operatorname{Sec}|\leq 1}(M)$ with volume equal to ess-MinVol(M).

$C^{1,\alpha}$

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(ess-MinVol(M) is thus relevant to the generalized Hopf-Thom-Yau question)

First comparisons with MinVol

Similarly to MinVol,

$$\text{ess-MinVol}(M) \geq C_n \text{simplicial volume}(M),$$ (Gromov)

$$\text{ess-MinVol}(M) \geq C_n \, \mathbf{e}(M).$$

↑ Euler charact

$= \int \cdots |Rm| \cdots$

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By definition

$$\text{ess-MinVol}(M) \leq \text{MinVol}(M).$$

However, ess-MinVol can be arbitrarily smaller than MinVol!

Januskiewicz: $\exists$ M^4, ess-MinVol $= 0$

minVol $\gg 1$

This M carries an F-structure $\Rightarrow$ ess-MinVol $= 0$

CG

First comparisons with MinVol

But $|B(M)| > 0$

$\Rightarrow$ MinVol $\neq 0$

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However, ess-MinVol can be arbitrarily smaller than MinVol! Moreover ess-MinVol does satisfy the gap property: if ess-MinVol$(M) \leq \epsilon_n$ then actually ess-MinVol$(M) = 0$.

Thickness of minimizing metrics

Theorem 1 (S.)

There is a $\delta_n > 0$, *for a smooth closed manifold* M, *and a weak minimizer* $(M_\infty, g_\infty) \in \overline{\mathcal{M}}^{w}_{|\operatorname{Sec}|\leq 1}(M)$ *with*

$$\operatorname{Vol}(M_\infty, g_\infty) = \text{ess-MinVol}(M),$$

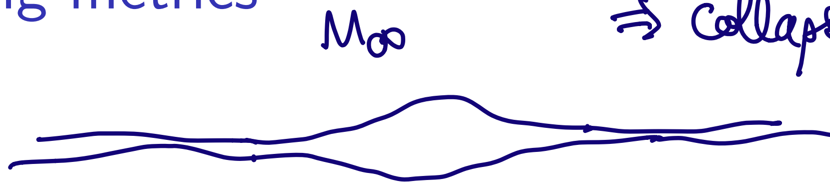
any connected component of M_∞ *contains a point* p *such that*

$$\operatorname{Vol}(B_{g_\infty}(p,1), g_\infty) > \delta_n.$$

($\Rightarrow$ # conn components of M_∞ $< \infty$)

Thickness of minimizing metrics

by CG, ∃F structure ⇒ collapse



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any connected component of M_∞ *contains a point* p *such that*

$$\mathrm{Vol}(B_{g_\infty}(p,1), g_\infty) > \delta_n.$$

It implies that M_∞ has finitely many components, so actually M_∞ lives in a strong closure of $\overline{\mathcal{M}}^{s}_{|\,\mathrm{Sec}\,|\leq 1}(M)$ of $\mathcal{M}_{|\,\mathrm{Sec}\,|\leq 1}(M)$.

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if (M, g) is ϵ_n-collapsed then there is a 1-parameter family of metrics g_t with $g_0 = g$, g_t becomes arbitrarily collapsed as $t \to \infty$, and $|\operatorname{Sec}_{g_t}| \leq C(n, ||g||_3)$.

CG showed it with $C = C(n, g)$.

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We need the N-structures introduced by Cheeger-Fukaya-Gromov.

Minimizing metrics generalize hyperbolic metrics

In dimension at least 3, we have:

Theorem 2 (S.)

If (X, g_X) *has negative curvature* $\mathrm{Sec}_{g_X} \leq -1$*, then*

$$\text{ess-MinVol}(X) \geq \mathrm{Vol}(X, g_X)$$

with equality if and only if X *is hyperbolic.*

[Actually: $\forall$ minimizer (M_∞, g_∞) $\exists$ conn comp $N \subset M_\infty$, $\mathrm{Vol}(N, g_\infty) \geq \mathrm{Vol}(X, g_X)$]

$\mathrm{Vol}(M_\infty, g_\infty) = \text{ess MinVol}$

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This shows that in some sense, ess-MinVol/minimizers generalize hyperbolic volume/metrics.

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We combine Cheeger-Fukaya-Gromov / Paternain-Petean and Besson-Courtois-Gallot.

Examples

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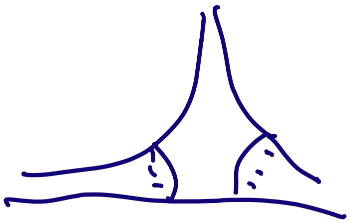
and if $\gamma > 1$, the minimizers are exactly the metrics in the closure of the hyperbolic metrics on Σ_γ.



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In dimension 3: if M is a closed oriented prime 3-manifold, then

$$\text{ess-MinVol}(M) = \text{volume of hyperbolic part of } M.$$

Proof uses the fact that the Yamabe invariant of M is known:

$$\begin{aligned}\sigma(M) &:= \sup_{[g]} \inf_{g' \in [g]} \frac{\int_M \mathrm{Scal}_{g'}}{\mathrm{Vol}(M, g')^{1/3}} \\ &= -6(\text{volume of hyperbolic part of } M)^{2/3}.\end{aligned}$$

In dim 4: $\text{ex-MinVol}(S^4) = ?$

Rmk: $T_1, T_2, \dots$ 2-tori $\hookrightarrow S^4$

(Ivanšić) such that $S^4 \setminus (T_1 \cup T_2 \cup \dots)$ is hyperbolic with finite volume $=$ $\text{vol}(S^4, g_{\text{round}})$

Estimates for Einstein 4-manifolds and complex surfaces

Theorem 3 (S.)

there is a constant C such that if a closed 4-manifold M admits an Einstein metric, or is a complex surface of nonnegative Kodaira dimension, then

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For an Einstein 4-manifold M, the proof treats the thicker part of M using Cheeger-Naber, then the thinner part using Cheeger-Fukaya-Gromov.

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For complex surfaces, we use the Enriques-Kodaira classification and Aubin-Yau's theorem.

Geometric interpretation in dimension 4:
ess-MinVol(M) $\leq C$ if and only if there is a bounded curvature metric on M divided into a part covered by F-structures, and a part with bounded geometry and volume $\lesssim C$.

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ess-MinVol$(M) \leq C$ if and only if there is a bounded curvature metric on M divided into a part covered by F-structures, and a part with bounded geometry and volume $\lesssim C$.

Conjecture: For any closed N^4 and any genus $\gamma > 1$,

$$\text{ess-MinVol}\left(N\sharp(S^2 \times \Sigma_\gamma)\right) \geq C\gamma.$$