ON THE GEOMETRIC P=W CONJECTURE

joint work with Mazzon and Stevenson

CHARACTER VARIETIES

C smooth proj curve / $\mathbb{C}$ of genus g

$G = GL_n, SL_n$ (complex reductive group)

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$\mathcal{M}_B(g, G)$ = BETTI MODULI SPACE or CHARACTER VARIETY $= \mathrm{Hom}(\pi_1(C), G)/\!/G$

$$= \left\{ (A_1, B_1, \ldots A_g, B_g) \in G^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = I \right\} /\!/G$$

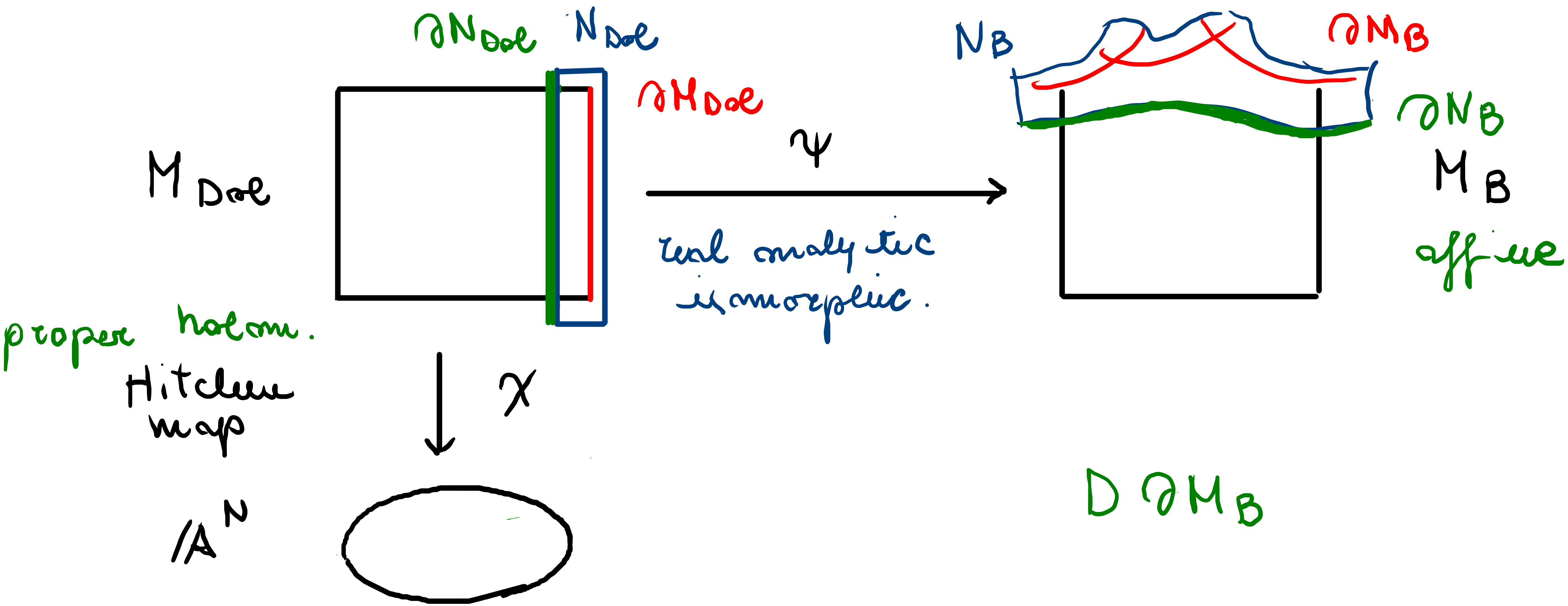
$\mathcal{M}_{Dol}(C, G)$ = DOLBEAULT MODULI SPACE $= \{ G\text{-}\overset{\text{semistable}}{\text{Higgs}}\text{ bundles} \}$

for $G = GL_n$ $\quad \{ (E, \varphi) \mid E$ vector bundle of rank n and degree $0 \}$

Higgs field $\quad \varphi : E \longrightarrow E \otimes K_C$

for $G = SL_n$ $\quad \{ (E, \varphi) + \det E \simeq \mathcal{O}_C,\ \mathrm{tr}\,\varphi = 0 \}$

NON-ABELIAN HODGE CORRESPONDENCE



DUAL COMPLEX

$\Delta = \sum \Delta_i$ simple normal crossing divisor
(det like ∂M_B)

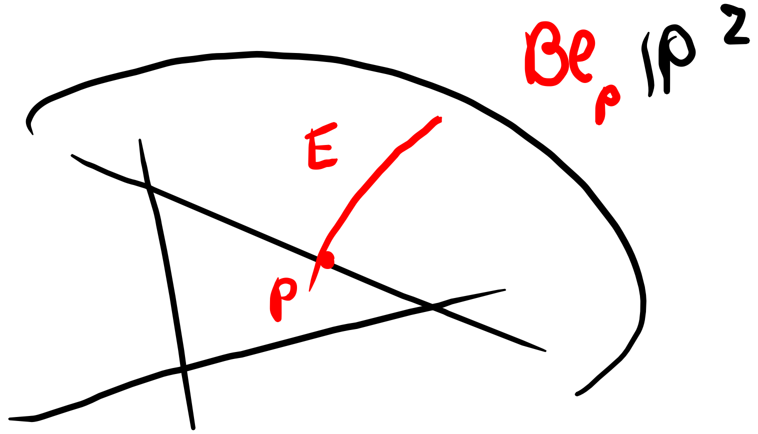
$D(\Delta)$ is the CW-complex

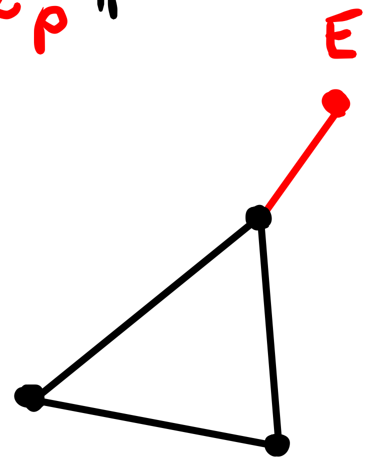
0-cells $\longrightarrow$ Δ_i

$\vdots$

k-cells $\longrightarrow$ STRATA of Δ = irr. comp. of $\Delta_{i_0} \cap \dots \cap \Delta_{i_k}$

Ex. $M_B(g=1, \mathbb{C}^\times) = \mathbb{C}^\times \times \mathbb{C}^\times \subset Bl_p \mathbb{P}^2$





$D\partial \mathbb{C}^\times \times \mathbb{C}^\times \not\simeq S^1$

THE EVALUATION MAP

$$\begin{tikzcd} \partial N_{Dol} \arrow[r, "\Psi"] \arrow[d, "\chi"'] & \partial N_B \arrow[d, dashed, "ev"] \\ S^{2N-1} & D\partial M_B \end{tikzcd}$$

$$\boxed{\begin{array}{rccc} ev: & \partial N_B & \longrightarrow & \mathbb{R}^I \\ & x & \longmapsto & \chi_i(x) \end{array}}$$

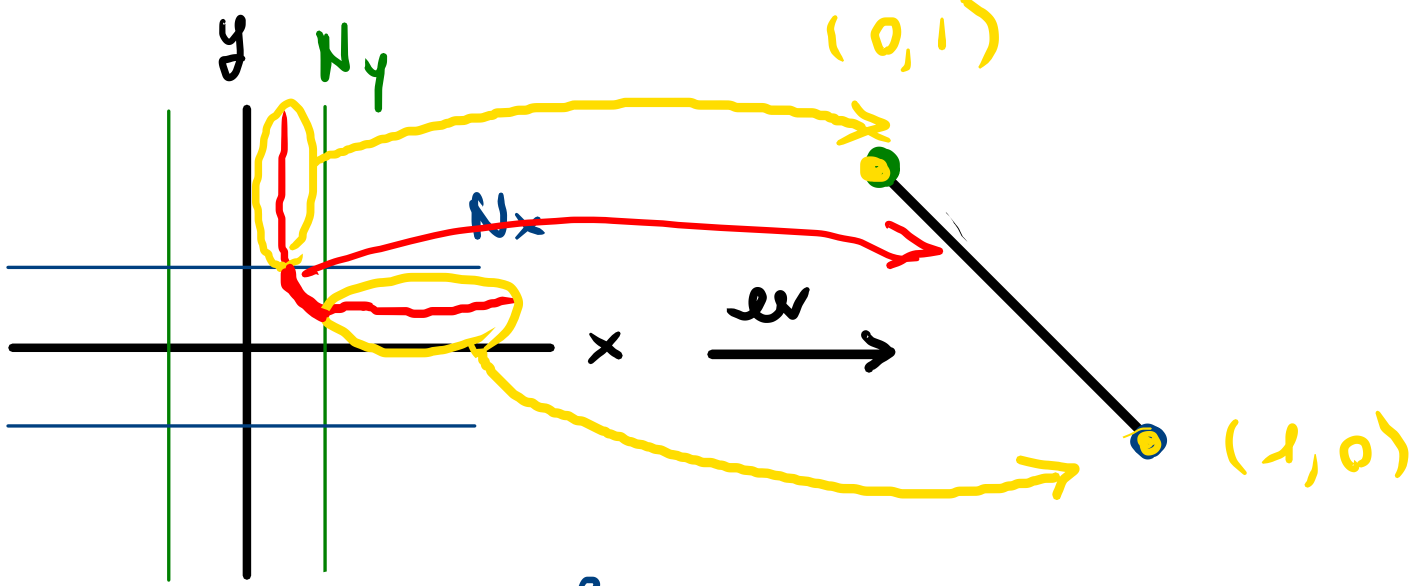
$\Delta = \sum_{i \in I} \Delta_i = \partial M_B$

N_i = tubular neigh of Δ_i

$\partial M_B \subset N_B \subset \bigcup N_i$

$\{\chi_i\}$ partition of unity

Ex. $(\bar{X}, \partial X) = (\mathbb{C}^2, xy = 0)$



$\partial N_B \subset N_x \cup N_y \longrightarrow \mathbb{R}^2, \quad x \longmapsto (\chi_1(x), \chi_2(x))$

THE GEOMETRIC P = W CONJECTURE

Conj. [Katzarkov - Noll - Pandit - Simpson]

(i) There exists a homotopy equivalence

$$\tau : S^{2N-1} \longrightarrow D\partial M_B$$

(ii) The square commutes up to homotopy

$$\begin{tikzcd}
\partial N_{Dol} \arrow[r, "\psi"] \arrow[d, "\chi"'] \arrow[dr, phantom, "\circlearrowright"] & \partial N_B \arrow[d, "ev"] \\
S^{2N-1} \arrow[r, "\tau"'] & D\partial M_B
\end{tikzcd}$$

STATE OF THE ART

- $G = SL_2$, $C = \mathbb{P}^1 \setminus \{p_1, \dots p_k\}$ (fixed tame at punctures)

  $k = 5$ Komyo '13

  $k > 5$ Simpson '15

  $D\partial M_B \sim S^{2(k-3)-1}$

- full geom. $P = W$ conj holds for Painlevé cases [Szabo]

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  $$D\partial M_B \sim S^{2(k-3)-1}$$
- full geom. $P=W$ conj holds for Painlevé cases

<u>Thm</u> [MMS] The geometric $P=W$ conj. holds for

(i) $G = \mathbb{C}^*$ and $g(C)$ arbitrary;

(ii) $G = GL_n, SL_n$ and $g(C) = 1$.

FIRST CASE : $G = \mathbb{C}^\times$

$$M_{Dol}(C, \mathbb{C}^\times) = \left\{ (E, \varphi) \;\middle|\; \begin{array}{l} E \text{ rank } 1 \text{ v.b. of degree } 0 \\ \varphi \in H^0(C, \mathrm{Hom}(E, E \otimes K_C)) \end{array} \right\}$$

$$= \mathrm{Pic}^0(C) \times H^0(C, K_C)$$

$$\chi : M_{Dol}(C, \mathbb{C}^\times) \to \mathbb{A}^g \qquad \mathrm{Pic}^0(C) \times H^0(C, K_C) \to H^0(C, K_C)$$

$$M_B(C, \mathbb{C}^\times) \simeq \{ (A_1, B_1, \dots, A_g, B_g) \in (\mathbb{C}^\times)^{2g} \mid \dots \} = (\mathbb{C}^\times)^{2g}$$

$$\overline{M}_B(C, \mathbb{C}^\times) \simeq \mathbb{P}^{2g}$$

$$D\partial M_B(C, \mathbb{C}^\times) \simeq S^{2g-1}$$

$$
\begin{tikzcd}
M_{Dol}(C, \mathbb{C}^\times) \arrow[r, "\psi"] \arrow[d, "\chi"'] & M_B(C, \mathbb{C}^\times) \simeq (\mathbb{C}^\times \times \mathbb{C}^\times)^g \arrow[r, hook] & Y \\
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\mathbb{C}^g \simeq \mathbb{R}^{2g} & &
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$(z_1, \dots z_{2g})$

$$
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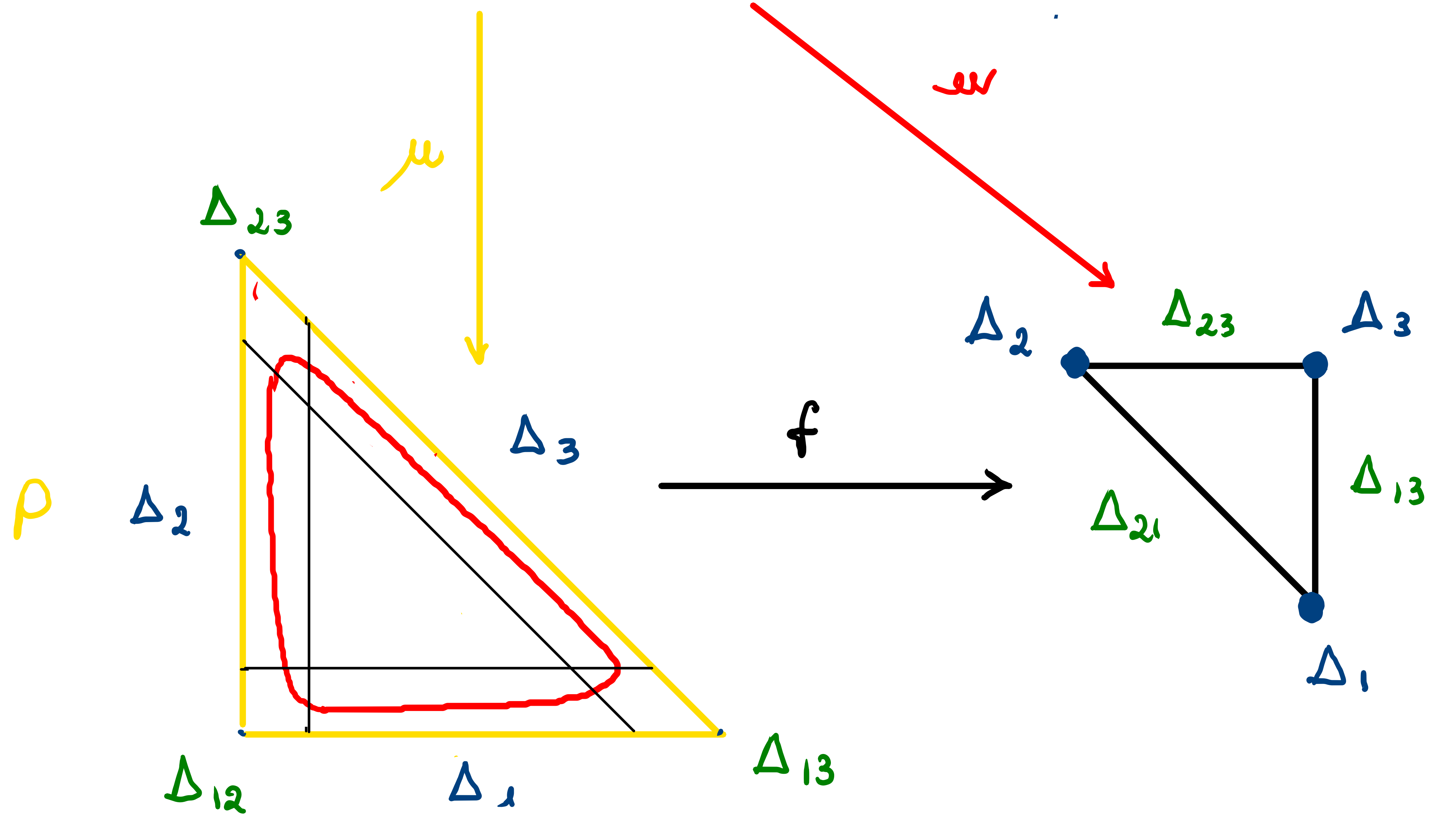
\mathbb{C}^g \simeq \mathbb{R}^{2g} \arrow[rr, hook, "\text{diffeo}"'] & & P
\end{tikzcd}
$$

$(z_1, \dots z_{2g})$

moment map

moment polytope

Example [Evans - M.] $\mathbb{P}^2 \supset \partial N_B$



$$\begin{tikzcd}
\partial N_{Dol} \arrow[r, "\psi"] \arrow[d, "\chi"] & \partial N_B \arrow[d, "\mu"] \\
\mathbb{C}^g \setminus \{0\} \arrow[r, "\text{diffeo}"] & \mathcal{P} \setminus \{0\}
\end{tikzcd}$$

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\partial N_{Dol} \arrow[r, "\psi"] \arrow[d, "\chi"'] & \partial N_B \arrow[d, "\mu"] \\
\mathbb{C}^g \setminus \{0\} \arrow[r, "\text{diffeo}"] \arrow[d] & P \setminus \{0\} \arrow[d] \\
S^{2n-1} \arrow[r, "\text{diffeo}"] & S^{2n-1}
\end{tikzcd}$$

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\partial N_{Dol} \arrow[r, "\psi"] \arrow[d, "\chi"'] \arrow[dr, phantom, "\circlearrowright"] & \partial N_B \arrow[d, "\mu"] \arrow[ddr, "w"] & \\
\mathbb{C}^n \setminus \{0\} \arrow[r, "\text{diffeo}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & P \setminus \{0\} \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \\
S^{2n-1} \arrow[r, "\text{diffeo}"] \arrow[rr, bend right, "\tau"'] & S^{2n-1} \arrow[r, "f"] & D\partial M_B
\end{tikzcd}
$$

homotopic to the identity

A MORE TRICKY CASE: $g(C) = 1$, $G = GL_n$

$$M_{Dol}(C, GL_n) = \{ (E_1, \phi_1) \oplus \dots \oplus (E_n, \phi_n) \} \simeq \Big(\underbrace{Pic^0(C)}_{= C} \times \underbrace{H^0(K_C)}_{= \mathbb{A}^1} \Big)^{(n)}$$

Hitchin map $\quad (C \times \mathbb{A}^1)^{(n)} \longrightarrow (\mathbb{A}^1)^{(n)} \simeq \mathbb{A}^n$

$$M_B(C, GL_n) = \{ (A, B) \in GL_n^2 \mid [A, B] = 1 \} /\!/ GL_n \simeq (\mathbb{C}^\times \times \mathbb{C}^\times)^{(n)}$$

$$\overline{M}_B(C, GL_n) = (\mathbb{P}^2)^{(n)}$$

$\big((\mathbb{P}^2)^{(n)}, \Delta^{(n)} \big)$ is not snc,

log CY, i.e. $K_{(\mathbb{P}^2)^{(n)}} + \Delta^{(n)} \sim_{\mathbb{Q}} 0$

Thm [MMS] Let $g(C) = 1$ and $G = GL_n, SL_n$.

- M_B admits a dlt log CY compactification
- $D\partial M_B(C, GL_n) \simeq_{PL} S^{2n-1}$
- $D\partial M_B(C, SL_n) \simeq_{PL} S^{2n-3}$

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Idea $M_B(C, GL_n) \simeq (\mathbb{C}^\times \times \mathbb{C}^\times) \times \dots \times (\mathbb{C}^\times \times \mathbb{C}^\times) / Sym_n$

$$D\partial M_B(C, GL_n) \simeq_{PL} \underbrace{S^1 * \dots * S^1}_{n\text{-times}} / Sym_n \simeq S^{2n-1}$$

homotopy commutativity of square diagram

$$\begin{tikzcd}
(\mathbb{C} \times \mathbb{C})^{(n)} \arrow[rr] \arrow[d, "\chi = (pr_2)^{(n)}"'] & & (\mathbb{C}^\times \times \mathbb{C}^\times)^{(n)} \\
(\mathbb{C})^{(n)} \simeq \mathbb{C}^n & &
\end{tikzcd}$$

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(\mathbb{C})^{(n)} \simeq \mathbb{C}^n &
\end{tikzcd}
$$

$\overline{M}_B$ is not toric

$$\Psi : (\theta_1, \theta_2, r_1 + i r_2) \longmapsto (e^{-2r_1} e^{i\theta_1}, e^{2r_2} e^{i\theta_2})$$

$$
\begin{array}{ccccc}
C \times \mathbb{A}^1 & \underset{\text{diffeo}}{\simeq} & S^1 \times S^1 \times \mathbb{R} \times \mathbb{R} & \underset{\text{diffeo}}{\simeq} & \mathbb{C}^{\times}_{z_1} \times \mathbb{C}^{\times}_{z_2} \\
\cup & & & & \cup \\
\chi^{-1}(x) = C \times \{0\} & & & & \mathbb{T} := \{ |z_1| = |z_2| = 1 \}
\end{array}
$$

However, moment map exists locally...

(Z, Δ) toric surface with at least n torus-fixed pts $p_1, \dots p_n, \dots$

$p = (p_1, \dots p_n) \in Z^{(n)} \supset (\mathbb{C}^\times \times \mathbb{C}^\times)^{(n)}$

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$(Z^{(n)}, \Delta^{(n)}) \simeq_{\text{loc at } p} (\mathbb{C}^{2n},$ coordinate hyperplane$)$

$$\begin{tikzcd} \mathbb{C}^{2n} \arrow[d, "\mu"] \\ \mathbb{R}^{2n} \end{tikzcd}$$

moment map

with fiber $T^n = ev^{-1}(x)$

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ψ $\quad T$

$$\boxed{\chi^{-1}(x) \sim ev^{-1}(x) \implies \pi N_{\Delta, e} \overset{ev \circ \psi}{\underset{\chi}{\rightrightarrows}} S^{2n-1} \text{ hom. equiv.}}$$

GENERAL TOPOLOGICAL STATEMENTS

<u>Thm</u> [MMS] Let $G = Gl_n$ or Sl_n.

(i) M_B admits a dlt compactification

(ii) $H^*(D\partial M_B, \mathbb{Q}) \simeq H^*(S^{\dim M_B - 1})$ for $n = 2$

<u>Proof</u> $Gr^W_{2N} IH^N(M_B) \simeq \mathbb{Q}$ by [M.].

(iii) $\pi_1(D\partial M_B) = 1$ (or $\mathbb{Z}$ if $\dim M_B = 2$)

COHOMOLOGICAL VS GEOMETRIC P = W CONJECTURE

Conj. [de Cataldo - Hausel - Migliorini, de Cataldo - Maulik]

$$H^\bullet(M_{Dol}(C,G)) \xleftarrow{\ \Psi^*\ } H^\bullet(M_B(C,G))$$

$$\begin{array}{ccc} \cup\!\! \mid & & \cup\!\! \mid \\ P_k H^\bullet(M_{Dol}(C,G)) & \xleftarrow{\ \simeq\ } & W_{2k} H^\bullet(M_B(C,G)) \end{array}$$

perverse Leray filtration associated to χ — weight filtration

Thm [MMS] Up to a mild hypothesis ...

geometric $P=W$ $\Rightarrow$ cohomological $P=W$ at the highest weight

$$\mathrm{Gr}^{P}_{k} IH^{k}(M_{Dol}) \simeq \mathrm{Gr}^{W}_{2k} IH^{k}(M_{B})$$

$$P_{k-1} IH^{k}(M_{Dol}) \simeq W_{2k-1} IH^{k}(M_{B})$$

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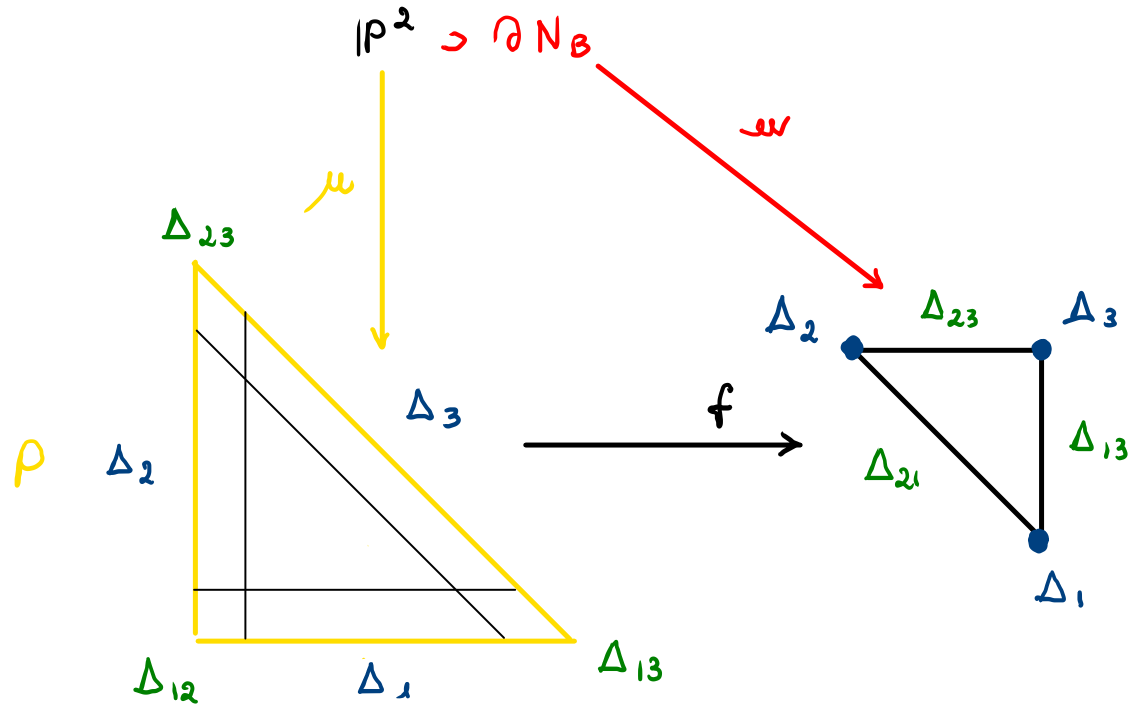
$$P_{k-1} IH^k(M_{Dol}) = \ker\{IH^k(M_{Dol}) \to IH^k(\chi^{-1}(x))\}$$

[de Cataldo - Migliorini]

$$W_{2k-1} IH^k(M_B) = \ker\{IH^k(M_{Dol}) \to IH^k(ev^{-1}(x))\}$$

Thank you!



$\mathbb{P}^2 \supset \partial N_B$
μ
ω
f
ρ
Δ_{23}
Δ_2
Δ_3
Δ_{12}
Δ_1
Δ_{13}
Δ_2
Δ_{23}
Δ_3
Δ_{21}
Δ_{13}
Δ_1