Higher Fano Manifolds

Carolina Araujo - IMPA

Geometria em Lisboa Seminar - June 1st, 2021

Fano Manifolds

X complex projective manifold

Fano Manifolds

X complex projective manifold

T_X tangent bundle of X

Fano Manifolds

X complex projective manifold

T_X tangent bundle of X $\quad \rightsquigarrow \quad -K_X = c_1(T_X) \in H^2(X, \mathbb{Z})$

(the anti-canonical class of X)

Fano Manifolds

X complex projective manifold

T_X tangent bundle of X $\quad\rightsquigarrow\quad$ $-K_X = c_1(T_X) \in H^2(X,\mathbb{Z})$

(the anti-canonical class of X)

Remark

$C \subset X \quad\rightsquigarrow\quad -K_X \cdot C$

Fano Manifolds

X complex projective manifold

T_X tangent bundle of X $\rightsquigarrow$ $-K_X = c_1(T_X) \in H^2(X, \mathbb{Z})$

(the anti-canonical class of X)

Remark

$C \subset X \rightsquigarrow -K_X \cdot C = \frac{1}{2\pi}(\textit{ average of Ricci curvature of } X \textit{ along } C\ )$

Fano Manifolds

X complex projective manifold

T_X tangent bundle of X $\quad\rightsquigarrow\quad -K_X = c_1(T_X) \in H^2(X, \mathbb{Z})$

(the anti-canonical class of X)

Remark

$C \subset X \quad\rightsquigarrow\quad -K_X \cdot C = \frac{1}{2\pi}\big($ *average of Ricci curvature of* X *along* C $\big)$

Definition (The Fano condition)

X is a Fano manifold if $-K_X$ is ample (positive)

Fano Manifolds

X complex projective manifold

T_X tangent bundle of X $\rightsquigarrow$ $-K_X = c_1(T_X) \in H^2(X, \mathbb{Z})$

(the anti-canonical class of X)

Remark

$C \subset X \rightsquigarrow -K_X \cdot C = \frac{1}{2\pi}$(*average of Ricci curvature of* X *along* C)

Definition (The Fano condition)

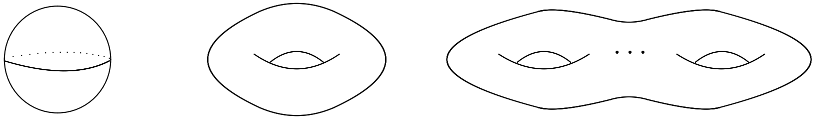
X is a Fano manifold if $-K_X$ is ample (positive)

Remark

$$-K_X \textit{ is ample} \implies -K_X \cdot C > 0 \quad \forall C \subset X$$

Fano Manifolds

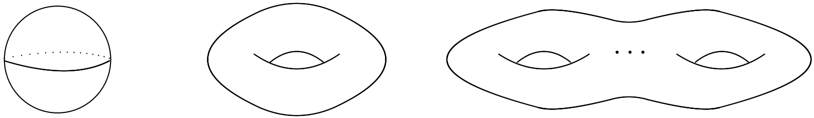
Example (X smooth projective curve)



$-K_C > 0$ $-K_C = 0$ $-K_C < 0$

Fano Manifolds

Example (X smooth projective curve)



$-K_C > 0$ $-K_C = 0$ $-K_C < 0$

Examples

- $\mathbb{CP}^n$
- Hypersurfaces of degree $d \leq n$ in $\mathbb{CP}^n$
- Grassmannians and other rational homogeneous spaces
- Several moduli spaces (of vector bundles)

Fano Manifolds

Remark

Minimal Model Program $\rightsquigarrow$ *every projective manifold is built up from varieties with*

$$-K_X > 0 \;\;,\;\; -K_X = 0 \;\; \textit{or} \;\; -K_X < 0$$

Fano Manifolds

Remark

Minimal Model Program $\rightsquigarrow$ *every projective manifold is built up from varieties with*

$$-K_X > 0 \ , \ -K_X = 0 \ \text{or} \ -K_X < 0$$

Remark

The Calabi Problem - *Which Fano manifolds admit Kähler-Einstein metrics?*

X Fano manifold

Definition (The index of a Fano manifold)

$$i(X) \;:=\; max\{m \in \mathbb{Z} \mid -K_X = mA,\; A \in H^2(X,\mathbb{Z})\}$$

X Fano manifold

Definition (The index of a Fano manifold)

$$i(X) \ := \ max\{ m \in \mathbb{Z} \mid -K_X = mA, \ A \in H^2(X, \mathbb{Z}) \ \}$$

Theorem (Kobayashi-Ochiai 1973)

- $i(X) \leq \dim(X) + 1$

X Fano manifold

Definition (The index of a Fano manifold)

$$i(X) \ := \ max\{ m \in \mathbb{Z} \mid -K_X = mA, \ A \in H^2(X, \mathbb{Z}) \ \}$$

Theorem (Kobayashi-Ochiai 1973)

- $i(X) \leq \dim(X) + 1 \quad \& \quad i(X) = \dim(X) + 1 \iff X \cong \mathbb{CP}^n$

X Fano manifold

Definition (The index of a Fano manifold)

$$i(X) \ := \ max\{m \in \mathbb{Z} \mid -K_X = mA, \ A \in H^2(X,\mathbb{Z}) \}$$

Theorem (Kobayashi-Ochiai 1973)

- $i(X) \leq \dim(X) + 1 \quad \& \quad i(X) = \dim(X) + 1 \iff X \cong \mathbb{CP}^n$
- $i(X) = \dim(X) \iff X \cong Q \subset \mathbb{CP}^n$

X Fano manifold

Definition (The index of a Fano manifold)

$$i(X) \ := \ max\{ m \in \mathbb{Z} \mid -K_X = mA, \ A \in H^2(X, \mathbb{Z}) \ \}$$

Theorem (Kobayashi-Ochiai 1973)

- $i(X) \leq \dim(X) + 1 \quad \& \quad i(X) = \dim(X) + 1 \iff X \cong \mathbb{CP}^n$
- $i(X) = \dim(X) \iff X \cong Q \subset \mathbb{CP}^n$

Theorem (Fujita 1982)

Classification when $i(X) = \dim(X) - 1$ (del Pezzo manifolds)

X Fano manifold

Definition (The index of a Fano manifold)

$$i(X) \;:=\; max\{ m \in \mathbb{Z} \;\big|\; -K_X = mA,\; A \in H^2(X,\mathbb{Z}) \;\}$$

Theorem (Kobayashi-Ochiai 1973)

- $i(X) \leq \dim(X) + 1 \quad \& \quad i(X) = \dim(X) + 1 \iff X \cong \mathbb{CP}^n$
- $i(X) = \dim(X) \iff X \cong Q \subset \mathbb{CP}^n$

Theorem (Fujita 1982)

Classification when $i(X) = \dim(X) - 1$ (del Pezzo manifolds)

Theorem (Mukai 1992)

Classification when $i(X) = \dim(X) - 2$ (Mukai manifolds)

Positivity in the Tangent Bundle

Positivity in the Tangent Bundle

Theorem (Mori 1979 - Frankel's conjecture)

X compact Kähler manifold with positive sectional curvature $\iff$

$$X \cong \mathbb{CP}^n$$

Positivity in the Tangent Bundle

Theorem (Mori 1979 - Frankel's conjecture)

X compact Kähler manifold with positive sectional curvature $\iff$

$$X \cong \mathbb{CP}^n$$

Theorem (Mori 1979 - Hartshorne's conjecture)

X projective manifold with ample tangent bundle $\iff$

$$X \cong \mathbb{CP}^n$$

Positivity in the Tangent Bundle

X complex projective manifold

T_X tangent bundle of X

Positivity in the Tangent Bundle

X complex projective manifold

T_X tangent bundle of X $\quad\rightsquigarrow\quad$ $ch_k(T_X) \in H^{2k}(X,\mathbb{Z})$

Positivity in the Tangent Bundle

X complex projective manifold

T_X tangent bundle of X $\quad\rightsquigarrow\quad$ $ch_k(T_X) \in H^{2k}(X, \mathbb{Z})$

Remark

$$\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r \implies ch_k(\mathcal{E}) = \sum c_1(\mathcal{L}_i)^k$$

Positivity in the Tangent Bundle

X complex projective manifold

T_X tangent bundle of X $\rightsquigarrow$ $ch_k(T_X) \in H^{2k}(X, \mathbb{Z})$

Remark

$$\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r \implies ch_k(\mathcal{E}) = \sum c_1(\mathcal{L}_i)^k$$

Definition (de Jong - Starr 2007, A.-Castravet 2012)

X is a k-Fano manifold if $ch_i(T_X) > 0$ for $i \in \{1, \ldots, k\}$

Positivity in the Tangent Bundle

X complex projective manifold

T_X tangent bundle of X $\quad \rightsquigarrow \quad ch_k(T_X) \in H^{2k}(X, \mathbb{Z})$

Remark

$$\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r \quad \Longrightarrow \quad ch_k(\mathcal{E}) = \sum c_1(\mathcal{L}_i)^k$$

Definition (de Jong - Starr 2007, A.-Castravet 2012)

X is a k-Fano manifold if $ch_i(T_X) > 0$ for $i \in \{1, \ldots, k\}$, i.e.,

$$ch_i(T_X) \cdot Z > 0 \quad \forall Z \subset X \text{ with } \dim(Z) = i$$

Special properties of Fano manifolds

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$$X_d \; = \; Z(F_d) \; = \; \left\{ \, (x_0 : \cdots : x_n) \;\middle|\; F_d(x_0, \ldots, x_n) = 0 \, \right\} \; \subset \; \mathbb{CP}^n$$

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$$X_d \;=\; Z(F_d) \;=\; \left\{\, (x_0 : \cdots : x_n) \;\middle|\; F_d(x_0, \ldots, x_n) = 0 \,\right\} \;\subset\; \mathbb{CP}^n$$

$$-K_{X_d} \;=\; (n+1-d)\,\left[H_{|X_d}\right]$$

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$$X_d \;=\; Z(F_d) \;=\; \{\; (x_0 : \cdots : x_n) \;\big|\; F_d(x_0, \ldots, x_n) = 0 \;\} \;\subset\; \mathbb{CP}^n$$

$$-K_{X_d} \;=\; (n+1-d)\,\left[H_{|X_d}\right]$$

X_d is Fano $\iff$ $d \leq n$

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$$X_d \;=\; Z(F_d) \;=\; \{\, (x_0 : \cdots : x_n) \mid F_d(x_0, \ldots, x_n) = 0 \,\} \;\subset\; \mathbb{CP}^n$$

$$-K_{X_d} \;=\; (n+1-d)\,\left[H_{|X_d}\right]$$

X_d is Fano $\iff$ $d \leq n$

$d \leq n-1 \implies X$ is covered by lines

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$X_d \;=\; Z(F_d) \;=\; \{\ (x_0 : \cdots : x_n) \mid F_d(x_0, \ldots, x_n) = 0\ \} \;\subset\; \mathbb{CP}^n$

$$-K_{X_d} \;=\; (n+1-d)\,\left[H_{|X_d}\right]$$

X_d is Fano $\iff$ $d \leq n$

$d \leq n-1 \implies X$ is covered by lines

$d \leq n \implies X$ is covered by conics

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$$X_d = Z(F_d) = \{ (x_0 : \cdots : x_n) \mid F_d(x_0, \ldots, x_n) = 0 \} \subset \mathbb{CP}^n$$

$$-K_{X_d} = (n + 1 - d) \left[H_{|X_d} \right]$$

X_d is Fano $\iff$ $d \leq n$

$d \leq n - 1 \implies$ X is covered by lines

$d \leq n \implies$ X is covered by conics

$d \geq n + 1 \implies$ $\nexists$ rational cruve through a general point of X

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$X_d \ = \ Z(F_d) \ = \ \left\{ \, (x_0 : \cdots : x_n) \ \middle| \ F_d(x_0, \ldots, x_n) = 0 \, \right\} \ \subset \ \mathbb{CP}^n$

$$-K_{X_d} \ = \ (n+1-d) \ \left[H_{|X_d} \right]$$

X_d is Fano $\iff$ $d \leq n$

$d \leq n-1 \implies X$ is covered by lines

$d \leq n \implies X$ is covered by conics

$d \geq n+1 \implies \nexists$ rational cruve through a general point of X

Conclusion

X_d is Fano $\iff$ $d \leq n$ $\iff$ X_d is covered by rational curves

Special properties of Fano manifolds

Theorem (Mori 1979)

Fano manifolds are covered by rational curves

Special properties of Fano manifolds

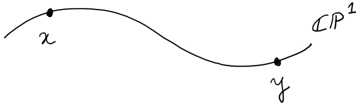
Theorem (Mori 1979)

Fano manifolds are covered by rational curves

Theorem (Campana, Kollár-Miyaoka-Mori 1992)

Fano manifolds are rationally connected :

Any 2 points of X can be connected by a rational curve



Special properties of Fano manifolds

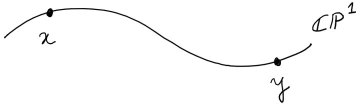
Theorem (Mori 1979)

Fano manifolds are covered by rational curves

Theorem (Campana, Kollár-Miyaoka-Mori 1992)

Fano manifolds are rationally connected :

Any 2 points of X can be connected by a rational curve



- Projective manifolds X with $-K_X \leq 0$ do not contain any rational curve through a general point

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$$X_d = Z(F_d) = \{ (x_0 : \cdots : x_n) \mid F_d(x_0, \ldots, x_n) = 0 \} \subset \mathbb{CP}^n$$

X_d is Fano $\iff$ $d \leq n$ $\iff$ X_d is rationally connected

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$X_d \; = \; Z(F_d) \; = \; \{\ (x_0 : \cdots : x_n) \mid F_d(x_0, \ldots, x_n) = 0\ \} \; \subset \; \mathbb{CP}^n$

X_d is Fano $\iff$ $d \leq n$ $\iff$ X_d is rationally connected

Theorem (Tsen 1936)

B complex algebraic curve

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

$d \leq n \implies \pi$ admits a holomorphic section $s : B \to \mathcal{X}$

Special properties of Fano manifolds

Example (Smooth hypersurfaces)

$X_d \;=\; Z(F_d) \;=\; \{\, (x_0 : \cdots : x_n) \mid F_d(x_0, \ldots, x_n) = 0 \,\} \;\subset\; \mathbb{CP}^n$

X_d is Fano $\iff$ $d \leq n$ $\iff$ X_d is rationally connected

Theorem (Tsen 1936)

B complex algebraic curve

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

$d \leq n \implies \pi$ admits a holomorphic section $s : B \to \mathcal{X}$

Theorem (Graber-Harris-Starr 2003)

B complex algebraic curve

$\pi : \mathcal{X} \to B$ family of rationally connected varieties

$\implies \pi$ admits a holomorphic section $s : B \to \mathcal{X}$

Theorem (Tsen - Lang 1936 - 1952)

B complex algebraic variety of dimension $\dim(B) = k$

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

If $d^k \leq n$, then π admits a meromorphic section $s : B \dashrightarrow \mathcal{X}$

Theorem (Tsen - Lang 1936 - 1952)

B complex algebraic variety of dimension $\dim(B) = k$

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

If $d^k \leq n$, then π admits a meromorphic section $s : B \dashrightarrow \mathcal{X}$

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$

Theorem (Tsen - Lang 1936 - 1952)

B complex algebraic variety of dimension $\dim(B) = k$

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

If $d^k \leq n$, then π admits a meromorphic section $s : B \dashrightarrow \mathcal{X}$

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds

Theorem (Tsen - Lang 1936 - 1952)

B complex algebraic variety of dimension $\dim(B) = k$

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

If $d^k \leq n$, then π admits a meromorphic section $s : B \dashrightarrow \mathcal{X}$

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Theorem (Tsen - Lang 1936 - 1952)

B complex algebraic variety of dimension $\dim(B) = k$

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

If $d^k \leq n$, then π admits a meromorphic section $s : B \dashrightarrow \mathcal{X}$

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$ (modulo Brauer obstruction)

Theorem (Tsen - Lang 1936 - 1952)

B complex algebraic variety of dimension $\dim(B) = k$

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

If $d^k \leq n$, then π admits a meromorphic section $s : B \dashrightarrow \mathcal{X}$

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$ (modulo Brauer obstruction)

- We can take $\mathcal{F}_1 =$ (to be Fano)

Theorem (Tsen - Lang 1936 - 1952)

B complex algebraic variety of dimension $\dim(B) = k$

$\pi : \mathcal{X} \to B$ family of hypersurfaces of degree d in $\mathbb{CP}^n$

If $d^k \leq n$, then π admits a meromorphic section $s : B \dashrightarrow \mathcal{X}$

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$ (modulo Brauer obstruction)

- We can take $\mathcal{F}_1 =$ (to be Fano) or (to be Rationally Connected)

Higher Fano conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Higher Fano conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Definition

X is a k-Fano manifold if $ch_i(T_X) > 0$ for $i \in \{1, \ldots, k\}$, i.e.,

$$ch_i(T_X) \cdot Z > 0 \quad \forall Z \subset X \text{ with } \dim(Z) = i$$

Higher Fano conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Definition

X is a k-Fano manifold if $ch_i(T_X) > 0$ for $i \in \{1, \ldots, k\}$, i.e.,

$$ch_i(T_X) \cdot Z > 0 \quad \forall Z \subset X \text{ with } \dim(Z) = i$$

Example (Smooth hypersurfaces)

$$X_d = Z(F_d) = \{ (x_0 : \cdots : x_n) \mid F_d(x_0, \ldots, x_n) = 0 \} \subset \mathbb{CP}^n$$

Higher Fano conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Definition

X is a k-Fano manifold if $ch_i(T_X) > 0$ for $i \in \{1, \dots, k\}$, i.e.,

$$ch_i(T_X) \cdot Z > 0 \quad \forall Z \subset X \text{ with } \dim(Z) = i$$

Example (Smooth hypersurfaces)

$$X_d = Z(F_d) = \{ (x_0 : \cdots : x_n) \mid F_d(x_0, \dots, x_n) = 0 \} \subset \mathbb{CP}^n$$

$$X_d \text{ is } k\text{-Fano} \iff d^k \leq n$$

Inspiration from Topology

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \big\downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \pi \\ & & B \end{array}$$

If F is $(k-1)$-connected , then π admits a section $s : B \to X$

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \big\downarrow \pi \\ & & B \end{array}$$

If F is $(k-1)$-connected , then π admits a section $s : B \to X$

- F topological space, $p \in F$

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \big\downarrow \pi \\ & & B \end{array}$$

If F is $(k-1)$-connected , then π admits a section $s : B \to X$

- F topological space, $p \in F \ \rightsquigarrow\ $ loop space $\Omega_p F$

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \pi \\ & & B \end{array}$$

If F is $(k-1)$-connected , then π admits a section $s : B \to X$

- F topological space, $p \in F$ $\rightsquigarrow$ loop space $\Omega_p F$
- $\pi_k(F) \; = \; \pi_{k-1}(\Omega_p F)$

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \big\downarrow \pi \\ & & B \end{array}$$

If F is $(k-1)$-connected , then π admits a section $s : B \to X$

- F topological space, $p \in F \;\rightsquigarrow\;$ loop space $\Omega_p F$
- $\pi_k(F) \;=\; \pi_{k-1}(\Omega_p F)$
- F is k-connected $\leftrightarrow$

$$\pi_0(F) \;=\; \pi_1(F) \;=\; \cdots \;=\; \pi_k(F) \;=\; 0$$

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \pi \\ & & B \end{array}$$

If F is $(k-1)$-connected , then π admits a section $s : B \to X$

- F topological space, $p \in F \;\rightsquigarrow\;$ loop space $\Omega_p F$
- $\pi_k(F) \;=\; \pi_{k-1}(\Omega_p F)$
- F is k-connected $\leftrightarrow$

$$\pi_0(F) \;=\; \pi_1(F) \;=\; \cdots \;=\; \pi_k(F) \;=\; 0$$

- F is 0-connected $\leftrightarrow$ F is path-connected

Inspiration from Topology

$\pi : X \to B$ fibration of CW complexes with typical fiber F, $\dim B = k$

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \pi \\ & & B \end{array}$$

If F is $(k-1)$-connected , then π admits a section $s : B \to X$

- F topological space, $p \in F \rightsquigarrow$ loop space $\Omega_p F$
- $\pi_k(F) = \pi_{k-1}(\Omega_p F)$
- F is k-connected $\leftrightarrow$

$$\pi_0(F) = \pi_1(F) = \cdots = \pi_k(F) = 0$$

- F is 0-connected $\leftrightarrow$ F is path-connected
- F is 1-connected $\leftrightarrow$ F is simply connected

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0,1]$ $\leftrightarrow$ $\mathbb{CP}^1$

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0, 1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0,1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected
- (k=2) F is simply connected (F path-connected + $\Omega_p F$ path-connected) $\leftrightarrow$

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0,1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected
- (k=2) F is simply connected (F path-connected + $\Omega_p F$ path-connected) $\leftrightarrow$ X is rationally connected + ???

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0, 1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected
- (k=2) F is simply connected (F path-connected + $\Omega_p F$ path-connected) $\leftrightarrow$ X is rationally connected + ???
- $\Omega_p F$ $\leftrightarrow$???

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0,1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected
- (k=2) F is simply connected (F path-connected + $\Omega_p F$ path-connected) $\leftrightarrow$ X is rationally connected + ???
- $\Omega_p F$ $\leftrightarrow$ $H_x = \{$rational curves of minimal degree through $x\}$

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0,1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected
- (k=2) F is simply connected (F path-connected + $\Omega_p F$ path-connected) $\leftrightarrow$ X rationally connected + H_x are rationally connected ???
- $\Omega_p F$ $\leftrightarrow$ $H_x = \{$rational curves of minimal degree through $x\}$

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0,1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected
- (k=2) F is simply connected (F path-connected + $\Omega_p F$ path-connected) $\leftrightarrow$ X rationally connected + H_x are rationally connected ???
- $\Omega_p F$ $\leftrightarrow$ $H_x = \{$rational curves of minimal degree through $x\}$
- Problem: $\Omega_p F \sim \Omega_{p,q} F$

Inspiration from Topology

TOPOLOGY $\leftrightarrow$ ALGEBRAIC GEOMETRY

- F topological space $\leftrightarrow$ X projective variety
- $[0,1]$ $\leftrightarrow$ $\mathbb{CP}^1$
- (k=1) F is path connected $\leftrightarrow$ X is rationally connected
- (k=2) F is simply connected (F path-connected + $\Omega_p F$ path-connected) $\leftrightarrow$ X rationally connected + H_x are rationally connected ???
- $\Omega_p F$ $\leftrightarrow$ $H_x = \{$rational curves of minimal degree through $x\}$
- Problem: $\Omega_p F \sim \Omega_{p,q} F \leftrightarrow H_x \not\simeq H_{x,y}$

Definition (De Jong-Starr)

A complex projective variety X is rationally simply connected if

- X and H_x are rationally connected
- Some version of $H_{x,y}$ is rationally connected
- $+$ some technical conditions

Definition (de Jong-Starr)

A complex projective variety X is rationally simply connected if

- X and H_x are rationally connected
- Some version of $H_{x,y}$ is rationally connected
- $+$ some technical conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Definition (De Jong-Starr)

A complex projective variety X is rationally simply connected if

- X and H_x are rationally connected
- Some version of $H_{x,y}$ is rationally connected
- $+$ some technical conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Question (De Jong-Starr)

Can one take $\mathcal{F}_2$ to be "X is rationally simply connected" ?

Higher Fano conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Higher Fano conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Definition

X is a k-Fano manifold if $ch_i(T_X) > 0$ for $i \in \{1, \ldots, k\}$, i.e.,

$$ch_i(T_X) \cdot Z > 0 \quad \forall Z \subset X \text{ with } \dim(Z) = i$$

Higher Fano conditions

Problem

To find intrinsic (geometric) conditions $\mathcal{F}_k$ such that

- For hypersurfaces of degree d in $\mathbb{CP}^n$, $\quad \mathcal{F}_k \iff d^k \leq n$
- Projective manifolds satisfying $\mathcal{F}_k$ are covered by rational k-folds
- The Tsen-Lang Theorem holds if the fibers of π satisfy $\mathcal{F}_k$

Definition

X is a k-Fano manifold if $ch_i(T_X) > 0$ for $i \in \{1, \ldots, k\}$, i.e.,

$$ch_i(T_X) \cdot Z > 0 \quad \forall Z \subset X \text{ with } \dim(Z) = i$$

Question (A.-Castravet 2012)

Can one take $\mathcal{F}_k$ to be "X is k-Fano" ?

Theorem (A.-Castravet 2012)

X Fano manifold

$H_x = \{$rational curves of minimal degree through $x\}$

Theorem (A.-Castravet 2012)

X Fano manifold

$H_x = \{$rational curves of minimal degree through $x\}$

$$ch_k(H_x) = \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L)^j \pi_* e^* \big(ch_{k+1-j}(X)\big) - \frac{1}{k!} c_1(L)^k$$

Theorem (A.-Castravet 2012)

X Fano manifold

$H_x = \{$rational curves of minimal degree through $x\}$

$$ch_k(H_x) = \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L)^j \pi_* e^* \big(ch_{k+1-j}(X)\big) - \frac{1}{k!} c_1(L)^k$$

- X is 2-Fano + $\dim(H_x) \geq 1 \Rightarrow H_x$ is Fano

Theorem (A.-Castravet 2012)

X Fano manifold

$H_x = \{$rational curves of minimal degree through $x\}$

$$ch_k(H_x) = \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L)^j \pi_* e^* \left(ch_{k+1-j}(X)\right) - \frac{1}{k!} c_1(L)^k$$

- X is 2-Fano + $\dim(H_x) \geq 1 \Rightarrow H_x$ is Fano
- X is 3-Fano + $\dim(H_x) \geq 2 \Rightarrow H_x$ is 2-Fano

Theorem (A.-Castravet 2012)

X Fano manifold

$H_x \ = \ \{\text{rational curves of minimal degree through } x\}$

$$ch_k(H_x) \ = \ \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L)^j \pi_* e^* \big(ch_{k+1-j}(X)\big) - \frac{1}{k!} c_1(L)^k$$

- X is 2-Fano + $\dim(H_x) \geq 1 \ \Rightarrow \ H_x$ is Fano
- X is 3-Fano + $\dim(H_x) \geq 2 \ \Rightarrow \ H_x$ is 2-Fano
- X is 2-Fano + $(\cdots) \ \Rightarrow \ X$ is covered by rational surfaces

Theorem (A.-Castravet 2012)

X Fano manifold

$H_x = \{\text{rational curves of minimal degree through } x\}$

$$ch_k(H_x) = \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L)^j \pi_* e^* \left(ch_{k+1-j}(X)\right) - \frac{1}{k!} c_1(L)^k$$

- X is 2-Fano + $\dim(H_x) \geq 1 \Rightarrow H_x$ is Fano
- X is 3-Fano + $\dim(H_x) \geq 2 \Rightarrow H_x$ is 2-Fano
- X is 2-Fano + $(\cdots) \Rightarrow X$ is covered by rational surfaces
- X is 3-Fano + $(\cdots) \Rightarrow X$ is covered by rational 3-folds

Theorem (A.-Castravet 2012)

X Fano manifold

$H_x \ = \ \{$rational curves of minimal degree through $x\}$

$$ch_k(H_x) \ = \ \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L)^j \pi_* e^* \left(ch_{k+1-j}(X)\right) - \frac{1}{k!} c_1(L)^k$$

- X is 2-Fano + $\dim(H_x) \geq 1 \ \Rightarrow \ H_x$ is Fano
- X is 3-Fano + $\dim(H_x) \geq 2 \ \Rightarrow \ H_x$ is 2-Fano
- X is 2-Fano + $(\cdots) \ \Rightarrow \ X$ is covered by rational surfaces
- X is 3-Fano + $(\cdots) \ \Rightarrow \ X$ is covered by rational 3-folds

Conjecture (A.-Castravet 2012)

X is k-Fano $\ \Rightarrow \ H_x$ is $(k-1)$-Fano

X Fano manifold

$H_x = \{$rational curves of minimal degree through $x\}$

Conjecture (A.-Castravet 2012)

X is k-Fano $\Rightarrow$ H_x is $(k-1)$-Fano

X Fano manifold

$H_x = \{\text{rational curves of minimal degree through } x\}$

Conjecture (A.-Castravet 2012)

X is k-Fano $\Rightarrow$ H_x is $(k-1)$-Fano

Theorem (A., Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan 2021)

X is 4-Fano $+ (\cdots) \Rightarrow$ H_x is 3-Fano

X Fano manifold

$H_x = \{\text{rational curves of minimal degree through } x\}$

Conjecture (A.-Castravet 2012)

X is k-Fano $\Rightarrow$ H_x is $(k-1)$-Fano

Theorem (A., Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan 2021)

X is 4-Fano + $(\cdots)$ $\Rightarrow$ H_x is 3-Fano

Theorem (Suzuki 2020, Nagaoka 2019)

- X is k-Fano + $\dim(H_x) \geq N(k)$ $\Rightarrow$ $\underbrace{H_x(H_x(\dots(H_x)}_{k \text{ times}}\dots)$ is Fano

X Fano manifold

$H_x = \{\text{rational curves of minimal degree through } x\}$

Conjecture (A.-Castravet 2012)

X is k-Fano $\Rightarrow$ H_x is $(k-1)$-Fano

Theorem (A., Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan 2021)

X is 4-Fano + $(\cdots)$ $\Rightarrow$ H_x is 3-Fano

Theorem (Suzuki 2020, Nagaoka 2019)

- X is k-Fano + $\dim(H_x) \geq N(k)$ $\Rightarrow$ $\underbrace{H_x(H_x(\ldots(H_x)}_{k \text{ times}}\ldots)$ is Fano
- X is k-Fano + $(\cdots)$ $\Rightarrow$ X is covered by rational k-folds

Examples of 2-Fano Manifolds

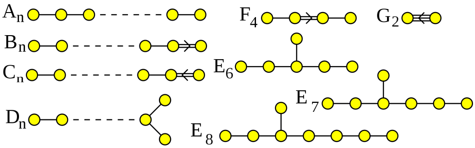
- Complete intersections of low degree in (weighted) projective spaces ($d^2 \leq n$)

[1]Dynkin - By Tomruen - Created by me by copying File:Connected Dynkin Diagrams.svg, , commons.wikimedia.org

Examples of 2-Fano Manifolds

- Complete intersections of low degree in (weighted) projective spaces ($d^2 \leq n$)

- Rational homogeneous spaces of Picard rank 1

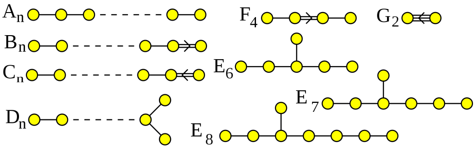


[1]

[1]Dynkin - By Tomruen - Created by me by copying File:Connected Dynkin Diagrams.svg, CC BY-SA 3.0, commons.wikimedia.org

Examples of 2-Fano Manifolds

- Complete intersections of low degree in (weighted) projective spaces ($d^2 \leq n$)
- Rational homogeneous spaces of Picard rank 1



[1]

- Some 2-orbit varieties

Theorem (A.-Castravet 2012)

Classification of 2-Fano Manifolds of index $i(X) \geq \dim(X) - 2$.

[1] Dynkin - By Tomruen - Created by me by copying File:Connected Dynkin Diagrams.svg, CC BY-SA 3.0, commons.wikimedia.org

Examples of 3-Fano Manifolds (2021)

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

Examples of 3-Fano Manifolds (2021)

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

- Complete intersections of low degree in (weighted) projective spaces ($d^3 \leq n$)

Examples of 3-Fano Manifolds (2021)

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

- Complete intersections of low degree in (weighted) projective spaces ($d^3 \leq n$)
- Rational homogeneous spaces of Picard rank 1: $\mathbb{CP}^n$ and $Q \subset \mathbb{CP}^n$

Examples of 3-Fano Manifolds (2021)

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

- Complete intersections of low degree in (weighted) projective spaces ($d^3 \leq n$)
- Rational homogeneous spaces of Picard rank 1: $\mathbb{CP}^n$ and $Q \subset \mathbb{CP}^n$

Theorem

Classification of 3-Fano Manifolds of index $i(X) \geq \dim(X) - 2$: only complete intersections of low degree in (weighted) projective spaces

Higher Fano Manifolds

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

Higher Fano Manifolds

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

Problem

Find examples of 3-Fano manifolds other than complete intersections in weighted projective spaces

Higher Fano Manifolds

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

Problem

Find examples of 3-Fano manifolds other than complete intersections in weighted projective spaces

Conjecture

X k-Fano and $\dim(X) = n$, with $k = \lceil \log_2(n+1) \rceil \implies X \cong \mathbb{CP}^n$

Higher Fano Manifolds

Joint with Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon, Libby Taylor, Nivedita Viswanathan

Problem

Find examples of 3-Fano manifolds other than complete intersections in weighted projective spaces

Conjecture

X k-Fano and $\dim(X) = n$, with $k = \lceil \log_2(n+1) \rceil \implies X \cong \mathbb{CP}^n$

Problem

For fixed n, find the smallest integer $k = k(n)$ such that:

X k-Fano and $\dim(X) = n \implies X$ is a complete intersections in weighted projective space

Obrigada!