Systolic questions in metric and symplectic geometry

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What is the supremum of the systolic ratio

$$\rho_{\text{sys}}(M, g) := \frac{\min\{\text{length}(\gamma, g)^n \mid \gamma \text{ closed geodesic on } (M, g)\}}{\text{vol}(M, g)}$$

of the n-dimensional closed Riemannian manifold (M, g) over the space of all Riemannian metrics g?

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Gromov, 1983: If the n-dimensional closed manifold M is essential (i.e. $[M] \neq 0$ in $K(\pi_1(M), 1)$), then $\rho_{\mathrm{sys}}(M, g) \leq C_n$.

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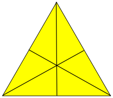

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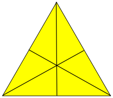

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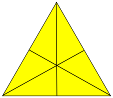

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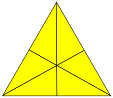

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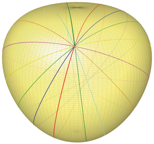

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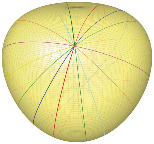

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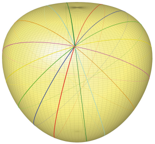

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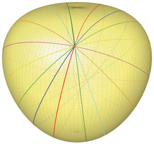

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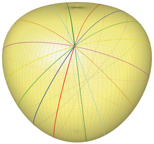

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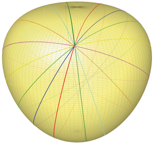

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Remark 1: Two Riemannian metrics on M having conjugated geodesic flows have the same systolic ratio.

Remark 2: Any two Zoll metrics on S^2 have conjugate geodesic flows (up to rescaling).

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Example 2: W closed hypersurface in $\mathbb{R}^{2n} \setminus \{0\}$ transverse to the radial direction, contact form $\alpha = \frac{1}{2} \sum_{j=1}^n (x_j\, dy_j - y_j\, dx_j)\Big|_W$.

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The volume $\mathrm{vol}(W, \alpha)$ is the Euclidean volume of the region bounded by W times $n!$.

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By Example 1, the contact systolic ratio generalizes the metric one: $\rho_{\mathrm{sys}}(S_g^* M, \alpha) = \frac{1}{n! \omega_n} \rho_{\mathrm{sys}}(M, g)$.

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Main example: S^{2n-1} with standard contact form α_0, whose Reeb orbits are the fibers of the Hopf fibration $\pi : S^{2n-1} \to \mathbb{CP}^{n-1}$, and $\rho_{\mathrm{sys}}(S^{2n-1}, \alpha_0) = 1$.

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- Any contact form that is a local maximizer of ρ_{sys} must be Zoll.
- Is the converse true?

Local systolic maximality of Zoll contact forms

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Theorem 1 (A.-Benedetti, 2020). Let α_0 be a Zoll contact form on the closed manifold W. Then α_0 has a C^3-neighborhood $\mathcal{U}$ in the space of contact forms on W such that

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Corollary 1. Zoll Riemannian metrics are local maximizers of the metric systolic ratio in the C^3-topology (answering question of Berger, 1970).

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Conjecture (Viterbo, 2000 - reformulation). For every smooth convex body $K \subset \mathbb{R}^{2n}$ containing the origin in its interior we have

$$\rho_{\mathrm{sys}}(\partial K, \alpha) \leq 1$$

with equality if and only K is symplectomorphic to a ball.

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Corollary 2. There exists a C^3-neighborhood $\mathcal{U}$ of the ball in the space of smooth convex bodies in $\mathbb{R}^{2n}$ such that

$$\rho_{\text{sys}}(\partial K, \alpha) \leq 1 \qquad \forall K \in \mathcal{U},$$

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See Bonus slides.

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There exist $W \subset T^*M$ fiberwise transverse to the radial direction such that $\rho_{\mathrm{sys}}(W, p\,dq|_W)$ is arbitrarily large (whereas when W bounds a fiberwise convex set and M is essential, then $\rho_{\mathrm{sys}}(W, p\,dq|_W)$ has a uniform upper bound, by Gromov).

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Key fact: Any orbit γ of R_{α_0} consisting of critical points of S is a closed orbit of $R_{u^*\alpha}$ of period $S(\gamma)T_{\min}(\alpha_0)$.

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$$\mathrm{vol}(M, \beta) = \int_M p(x, S(x))\, \alpha_0 \wedge d\alpha_0^{n-1},$$

where $p : M \times \mathbb{R} \to \mathbb{R}$ is polynomial in its second variable,

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Moreover, p_j is C^0-small when η and F are small in suitable norms.

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and we conclude as in the simple case treated before. □

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Gromov's non-squeezing theorem (1985): V symplectic 2-plane in $(\mathbb{R}^{2n}, \omega_0)$, P_V symplectic projector onto V, B unit ball in $\mathbb{R}^{2n}$. Then

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A.-Matveyev, 2013: If V is a symplectic $2k$-plane with $1 < k < n$ and $\epsilon > 0$, then there exists a symplectomorphism $\varphi : B \hookrightarrow \mathbb{R}^{2n}$ such that

$$\mathrm{vol}(P_V \varphi(B), \omega_0^k|_V) < \epsilon.$$

(building on results of Guth, 2008).

Shadows of symplectic balls, II

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Linear symplectomorphisms: If $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear symplectomorphism, then

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Corollary 3. There exists a C^3_{loc}-neighborhood $\mathcal{U}$ of the set of linear mappings in the space of all smooth symplectomorphisms of $\mathbb{R}^{2n}$ such that for every symplectic $2k$-plane $V \subset \mathrm{R}^{2n}$ we have

$$\mathrm{vol}(P_V \varphi(B), \omega_0^k|_V) \geq \pi^k,$$

for every $\varphi \in \mathcal{U}$.