Quasisymmetric Schur Functions

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Outline

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- A *composition* $\alpha = (\alpha_1, \dots, \alpha_l)$ is an ordered sequence of positive integers. Its *size* $|\alpha|$ is $\sum \alpha_i$. If $|\alpha| = n$, we say that α is a composition of n and write $\alpha \vDash n$. The only composition of 0 is $\emptyset$.
- A composition $\lambda = (\lambda_1, \dots, \lambda_l)$ such that $\lambda_1 \geq \dots \geq \lambda_l$ is called a *partition* of $|\lambda|$. We denote it by $\lambda \vdash |\lambda|$.
- Given a composition α, we denote by $\tilde{\alpha}$ the *underlying partition*, i.e. the partition obtained by reordering its parts.
- We identify a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ with its *Young diagram*, a sequence of $|\lambda|$ cells, arranged in l rows, left-justified, such that each row i has λ_i cells, following the matrix notation.
- If λ and μ are partitions such that $\mu \subseteq \lambda$ (as Young diagrams), the *skew shape* λ/μ is the diagram with the cells of λ that are not cells of μ. A partition λ is a skew shape λ/μ, with $\mu = \emptyset$.

- Given a skew shape, the *semistandard Young tableau* (SSYT) is the filling of its diagram with positive integers such that
  1. the entries in each row are weakly increasing, left to right.
  2. the entries in each column are strictly increasing, top to bottom.
- A *standard Young tableau* (SYT) is a SSYT in which the numbers in $\{1, \ldots, |\lambda/\mu|\}$ appear exactly once.
- A *semistandard reverse tableau* (SSRT) is a filling of the diagram with positive integers such that
  1. the entries in each row are weakly decreasing, left to right.
  2. the entries in each column are strictly decreasing, top to bottom.
- A *standard reverse tableau* (SRT) is defined in analogous way.
- If T is a tableau, its *content* is the sequence defined as $cont(T) = (\mu_1, \ldots, \mu_k)$, where μ_i is equal to the number of occurrences of i.

Definition

Let n be a nonnegative integer.

1. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$ we define its associated set by

$$Set(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \ldots + \alpha_{l-1}\} \subseteq [n-1]$$

   If $\alpha = (n)$, then $Set(\alpha) = \emptyset$.

2. Given a nonempty set $A = \{a_1 \leq \ldots \leq a_k\} \subseteq [n-1]$ we define its associated composition by

$$Comp(A) = (a_1, a_2 - a_1, \ldots, a_k - a_{k-1}, n - a_k) \vDash n$$

   We set $Comp(\emptyset)$ to be $\emptyset$ if $n = 0$ and (n) otherwise.

- Given compositions $\alpha = (\alpha_1, \ldots, \alpha_l)$ and $\beta = (\beta_1, \ldots, \beta_k)$, we define the following operations:
  - Concatenation $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k)$
  - Near concatenation $\alpha \odot \beta = (\alpha_1, \ldots, \alpha_l + \beta_1, \ldots, \beta_k)$

Example

$(1, 2, 2, 1, 3) \cdot (2, 3) = (1, 2, 2, 1, 3, 2, 3)$
$(1, 2, 2, 1, 3) \odot (2, 3) = (1, 2, 2, 1, 5, 3)$

- Let $\sigma = \sigma(1) \ldots \sigma(n)$ be a permutation of n. The *descent set* of σ is defined as

$$d(\sigma) = \{i : \sigma(i) > \sigma(i+1)\} \subseteq [n-1]$$

Example

If $\sigma = 4|136|25$, then $d(\sigma) = \{1, 4\}$.

Definition

Let α be a composition.

- A *semistandard reverse composition tableau* (SSRCT) τ of shape α is a filling of the cells of the composition diagram of α with positive integers such that
  1. The entries in the first column are strictly increasing, from top to bottom.
  2. The entries in each row are weakly decreasing, from left to right.
  3. If $i < j$, (i, k) and $(j, k+1)$ are cells in the diagram of α and $\tau(i, k) \geq \tau(j, k+1)$, then
     - $(i, k+1)$ is in the diagram.
     - $\tau(i, k+1) > \tau(j, k+1)$.
- A *standard reverse composition tableau* (SRCT) of shape α is a SSRCT of shape α such that the numbers $1, \ldots, |\alpha|$ appear exactly once.
- Given a SRCT τ, its *column word* $w_{col}(\tau)$ is the sequence obtained by writing the entries in each column, from left to right, in increasing order.

Example

The following are SSRCT fo shape $\alpha = (1,3,2)$

$$\tau_1 = \begin{array}{|c|c|c|} \hline 1 \\ \cline{1-1} 2 & 2 & 1 \\ \hline 3 & 3 \\ \cline{1-2} \end{array} \qquad \tau_2 = \begin{array}{|c|c|c|} \hline 1 \\ \cline{1-1} 3 & 3 & 2 \\ \hline 4 & 2 \\ \cline{1-2} \end{array}$$

Example

The following are SRCT of shape $\alpha = (1,3,2)$ and $\beta = (1,2,1)$

$$\tau_1 = \begin{array}{|c|c|c|} \hline 2 \\ \cline{1-1} 5 & 4 & 1 \\ \hline 6 & 3 \\ \cline{1-2} \end{array} \qquad \tau_2 = \begin{array}{|c|c|} \hline 1 \\ \cline{1-1} 3 & 2 \\ \hline 4 \\ \cline{1-1} \end{array}$$

Moreover, $w_{col}(\tau_1) = 256341$ and $w_{col}(\tau_2) = 1342$

Proposition

There exists a bijection ρ between SSRCT and SSRT.

- Given a SSRCT τ, $\rho(\tau)$ is the SSRT obtained by taking the entries in each column and write them in decreasing order, starting on top.
- Given a SSRT T, $\rho^{-1}(T)$ is the SSRCT obtained by:
  1. writing the entries in the first column in increasing order, top to bottom;
  2. taking the entries in column $k = 2, \ldots$ in decreasing order and place them in the highest row such that the cell in the immediate left is filled and the entries in that row are weakly decreasing, left to right.

Example

Let $\tau = \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-1} 3 & 3 & 2 \\ \hline 5 & 4 \\ \cline{1-2}\end{array}$ be a SSRCT and $T = \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 3 & 1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$ be a SSRT. Then,

$$\rho(\tau) = \begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 3 & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \qquad \rho^{-1}(T) = \begin{array}{|c|c|c|}\cline{1-2} 2 & 1 \\ \cline{1-2} 3 \\ \hline 4 & 4 & 3 \\ \hline\end{array}$$

Definition

Let τ be a SRCT of shape $\alpha \vDash n$. The *descent set of* τ is defined by

$$Des(\tau) = \{i : \ i+1 \text{ appears weakly right of } i\} \subseteq [n-1]$$

Example

Consider the following SRCT

$$\tau_1 = \begin{array}{|c|c|c|} \hline 2 \\ \cline{1-3} 4 & 3 & 1 \\ \hline 6 & 5 \\ \cline{1-2} \end{array} \qquad \tau_2 = \begin{array}{|c|c|c|} \hline 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2} \end{array}$$

We have,

$$Des(\tau_1) = \{2, 4\} \qquad Des(\tau_2) = \{1, 4\}$$

Outline

Introduction and Combinatorial Concepts

Symmetric Functions

Quasisymmetric Functions

Quasisymmetric Schur Functions

Symmetric Functions

Let $\mathbb{Q}[[x_1, x_2, \ldots]]$ be the algebra of formal power series of infinitely many commutative variables over $\mathbb{Q}$. The symmetric group $\mathfrak{S}_\infty = \bigcup_{n \geq 0} \mathfrak{S}_n$ acts naturally on $\mathbb{Q}[[x_1, x_2, \ldots]]$ by

$$\sigma \cdot (x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n}) = (x_{\sigma(i_1)}^{\alpha_1} \ldots x_{\sigma(i_n)}^{\alpha_n})$$

Definition

A *symmetric function* is a function $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ with finite degree that is invariant under the previous action. We denote the set of symmetric functions by **Sym**.

Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. The *monomial symmetric function* m_λ is defined by

$$m_\lambda = \sum x_{i_1}^{\beta_1} \ldots x_{i_k}^{\beta_k}$$

where the sum is over the distinct permutations $(\beta_1, \ldots, \beta_k)$ of the entries of $(\lambda_1, \ldots, \lambda_k)$, and $m_\emptyset = 1$.

We have

$$\mathbf{Sym} = \bigoplus_{n \geq 0} \mathbf{Sym}^n$$

where $\mathbf{Sym}^n = span\{m_\lambda : \lambda \vdash n\}$

Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. The *Schur function* s_λ is defined by

$$s_\lambda = \sum x^T$$

where $x^T = x^{cont(T)}$ and the sum is over all the SSYT of shape λ, and $s_\emptyset = 1$.

The Schur functions are symmetric and are a $\mathbb{Z}$-basis for **Sym**. Moreover, $\mathbf{Sym}^n = span\{s_\lambda : \lambda \vdash n\}$.

Proposition

$$s_\lambda = \sum_{\mu \vdash |\lambda|} K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu}$ is equal to the number of SSYT of shape λ and content μ. The numbers $K_{\lambda\mu}$ are called the *Kostka numbers*.

A skew shape λ/μ is said to be an *horizontal strip* (resp. *vertical strip*) if there are no more than one cell in each column (resp. row).

Theorem (Pieri Rules)

Let λ be a partition. Then,

1. $$s_{(n)}s_\lambda = \sum_{\mu} s_\mu$$

   where μ is such that μ/λ is an horizontal strip of size n.

2. $$s_{(1^n)}s_\lambda = \sum_{\mu} s_\mu$$

   where μ is such that μ/λ is a vertical strip of size n.

Theorem (Littlewood Richardson Rule)

Let λ, μ and ν be partitions. Then,

1. $$s_{\nu/\mu} = \sum_{\lambda} c^{\nu}_{\lambda\mu} s_{\lambda}$$

2. $$s_{\lambda} s_{\mu} = \sum_{\nu} c^{\nu}_{\lambda\mu} s_{\nu}$$

where $c^{\nu}_{\lambda\mu}$ is equal to the number of Littlewood-Richardson tableaux of shape ν/μ and content λ.

Hopf Algebra structure in **Sym**

- Product (LR Rule)

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

- Coproduct

$$\Delta(s_\lambda) = \sum_\mu \sum_\nu c^\nu_{\lambda\mu} s_\nu \otimes s_\mu$$

- Counit

$$\varepsilon(s_\lambda) = \delta_{\emptyset\lambda}$$

- Antipode

$$S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda^*}$$

Proposition

Sym is an Hopf algebra.

Outline

Introduction and Combinatorial Concepts

Symmetric Functions

Quasisymmetric Functions

Quasisymmetric Schur Functions

Quasisymmetric Functions

There is another action of $\mathfrak{S}_\infty$ on $\mathbb{Q}[[x_1, x_2, \ldots]]$, given by

$$\sigma \cdot (x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}) = x_{\sigma \cdot I}^{\alpha}$$

where $I = (i_1, \ldots, i_k)$ and $\sigma \cdot I$ is the k-tuple obtained by rearranging $\sigma(i_1), \ldots, \sigma(i_k)$ in increasing order.

Example

If $\sigma = 132 \in \mathfrak{S}_3$, then

$$\sigma \cdot (x_1 x_2^2 x_3) = x_1 x_2^2 x_3$$

$$\sigma \cdot (x_1 x_3^2 x_2) = x_1 x_2^2 x_3$$

Definition

A *quasisymmetric function* is a function $f \in \mathbb{Q}[[x_1, x_2 \ldots]]$ with finite degree that is invariant under the previous action. The set of quasisymmetric functions is denoted by **QSym**.

The Monomial Quasisymmetric function

Definition

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition. The monomial quasisymmetric function M_α is defined by

$$M_\alpha = \sum_{i_1 < \ldots < i_k} x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}$$

Example

$M_{(1,2,2)} = x_1 x_2^2 x_3^2 + x_1 x_2^2 x_4^2 + x_1 x_3^2 x_5^2 + x_2 x_3^2 x_4^2 + \ldots$

Proposition

We have $\mathbf{QSym}^n = span\{M_\alpha : \alpha \vDash n\}$ and $\mathbf{QSym} = \bigoplus_{n \geq 0} \mathbf{QSym}^n$.

The Fundamental Quasisymmetric function

Definition

Let $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$. The *fundamental quasisymetric function* F_α is defined by

$$F_\alpha = \sum_{\substack{i_1 \leq \ldots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in Set(\alpha)}} x_{i_1} \ldots x_{i_n}$$

Example

Let $\alpha = (1, 2, 1) \vDash 4$. Then, $Set(\alpha) = \{1, 3\}$ and so

$$\begin{aligned} F_{(1,2,1)} &= \sum_{i_1 < i_2 \leq i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} \\ &= x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + x_2 x_3 x_4 x_5 + x_1 x_2^2 x_4 + x_1 x_3^2 x_5 + \ldots \end{aligned}$$

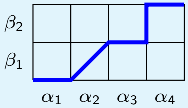
Quasishuffles of compositions

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_l)$ be compositions. Consider a path P from $(0,0)$ to (k,l) that can be obtained with steps $(1,0)$, $(0,1)$ and $(1,1)$. The composition $\gamma_P = (\gamma_1, \ldots, \gamma_m)$ defined by

$$\gamma_i = \begin{cases} \alpha_q \textit{ if } i\textit{-th step is } (1,0) \\ \beta_r \textit{ if } i\textit{-th step is } (0,1) \\ \alpha_q + \beta_r \textit{ if } i\textit{-th step is } (1,1) \end{cases}$$

is a *quasishuffle* of α and β.

Example



$\gamma_P = (\alpha_1, \alpha_2 + \beta_1, \alpha_3, \beta_2, \alpha_4)$

Shuffles of permutations

Definition

Given permutations $\sigma = \sigma(1)\ldots\sigma(n) \in \mathfrak{S}_n$ and $\tau = \tau(1)\ldots\tau(m) \in \mathfrak{S}_m$, a *shuffle* of σ and τ is a permutation of $\mathfrak{S}_{m+n}$ such that, for any $2 \leq i \leq n-1$, the entry $\sigma(i)$ appears to the right of $\sigma(i-1)$ and to the left of $\sigma(i+1)$, and for any $2 \leq j \leq m-1$, the entry $\tau(j)+n$ appears to the right of $\tau(j-1)+n$ and to the left of $\tau(j+1)+n$. We denote the set of shuffles of σ and τ by $\sigma \shuffle \tau$.

Example

Let $\sigma = 132 \in \mathfrak{S}_3$ and $\tau = 21 \in \mathfrak{S}_2$. Then,

$$\sigma \shuffle \tau = \{132\mathbf{54}, 1\mathbf{5}3\mathbf{4}2, 13\mathbf{5}2\mathbf{4}, \mathbf{5}1\mathbf{4}32, \mathbf{5}13\mathbf{4}2, \mathbf{5}132\mathbf{4}, 1\mathbf{5}32\mathbf{4}, 1\mathbf{54}32, 13\mathbf{54}2, \mathbf{54}132\} \subseteq \mathfrak{S}_5$$

Hopf algebra structure on **QSym**

Product of Monomial Quasisymmetric Functions

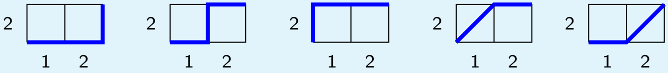
Proposition

$$M_\alpha M_\beta = \sum_P M_{\gamma_P}$$

where the sum is over all quasishuffles of α and β.

Example

Let $\alpha = (1,2)$ and $\beta = (2)$. We have



$\gamma_1 = (1,2,2)$ $\gamma_2 = (1,2,2)$ $\gamma_3 = (2,1,2)$ $\gamma_4 = (3,2)$ $\gamma_5 = (1,4)$

and so

$$M_{(1,2)} M_{(2)} = 2M_{(1,2,2)} + M_{(2,1,2)} + M_{(3,2)} + M_{(1,4)}$$

Hopf algebra structure on QSym

Product of Fundamental Quasisymmetric Functions

Proposition

Let $\sigma \in \mathfrak{S}_{|\alpha|}$ and $\tau \in \mathfrak{S}_{|\beta|}$ be such that $d(\sigma) = Set(\alpha)$ and $d(\tau) = Set(\beta)$. Then,

$$F_\alpha F_\beta = \sum_{\pi \in \sigma \sqcup\!\sqcup \tau} F_{Comp(d(\pi))}$$

Example

Let $\alpha = (1,1,2) \vDash 4$ and $\beta = (1) \vDash 1$. Then, $Set(\alpha) = \{1,2\}$ and $Set(\beta) = \emptyset$, and $4|2|13 \in \mathfrak{S}_4$ and $1 \in \mathfrak{S}_1$ are permutations with these descent sets.

$$\begin{array}{cccccc} 4213 \sqcup\!\sqcup 1 = & \{4|2|13\mathbf{5}, & 4|2|1\mathbf{5}|3, & 4|2\mathbf{5}|13, & 4\mathbf{5}|2|13, & \mathbf{5}|4|2|13\} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Comp(d(\pi)) & (1,1,3) & (1,1,2,1) & (1,2,2) & (2,1,2) & (1,1,1,2) \end{array}$$

Consequentely,

$$F_{(1,1,2)}F_{(1)} = F_{(1,1,3)} + F_{(1,1,2,1)} + F_{(1,2,2)} + F_{(2,1,2)} + F_{(1,1,1,2)}$$

Hopf algebra structure on **QSym**

- Coproduct of Monomial Quasisymmetric Functions

$$\Delta(M_\alpha) = \sum_{\alpha=\beta\cdot\gamma} M_\beta \otimes M_\gamma$$

Example

$$\Delta(M_{(1,3,2)}) = 1 \otimes M_{(1,3,2)} + M_{(1)} \otimes M_{(3,2)} + M_{(1,3)} \otimes M_{(2)} + M_{(1,3,2)} \otimes 1$$

- Coproduct Fundamental Quasisymmetric Functions

$$\Delta(F_\alpha) = \sum_{\substack{\alpha=\beta\cdot\gamma \\ \textit{or }\alpha=\beta\odot\gamma}} F_\alpha \otimes F_\gamma$$

Example

$$\Delta(F_{1,3,2}) = 1 \otimes F_{(1,3,2)} + F_{(1)} \otimes F_{(3,2)} + F_{(1,3)} \otimes F_{(2)} + F_{(1,3,2)} \otimes 1$$
$$+ F_{(1,2)} \otimes F_{(1,2)} + F_{(1,1)} \otimes F_{(22)} + F_{(1,3,1)} \otimes F_{(1)}$$

Hopf algebra structure on **QSym**

- Counit

$$\varepsilon(M_\alpha) = \delta_{\emptyset\alpha}$$

- Antipode

$$S(F_\alpha) = (-1)^{|\alpha|} F_{\alpha^t}$$

where $\alpha^t = Comp(\overline{Set(\alpha^r)})$ and α^r is obtained from α by writing its parts in reverse order.

Proposition

QSym is an Hopf algebra.

Moreover, **Sym** is an Hopf subalgebra of **QSym**. In particular, we have

$$m_\lambda = \sum_{\tilde{\alpha}=\lambda} M_\alpha$$

Outline

Quasisymmetric Schur Functions

Definition (Haglund, Luoto, Mason, van Willigenburg, 2011)

Let α be a composition. The *quasisymmetric Schur function* $\mathcal{S}_\alpha$ is given by

$$\mathcal{S}_\alpha = \sum_{\tau} x^\tau$$

where the sum is over all SSRCT of shape α, and $\mathcal{S}_\emptyset = 1$.

Example

We have $\mathcal{S}_{(1,2)} = x_1x_2x_3 + x_1x_3x_4 + x_1x_2^2 + x_1x_3^2 + \dots$

Proposition

Let α be a composition. The quasisymmetric Schur function $\mathcal{S}_\alpha$ is given by

$$\mathcal{S}_\alpha = \sum_{\beta \vDash |\alpha|} d_{\alpha,\beta} F_\beta$$

where $d_{\alpha,\beta}$ is equal to the number of SRCT τ of shape α such that $Des(\tau) = Set(\beta)$.

Example

Consider $\alpha = (1, 3, 2)$. The only SRCT of shape α are

τ	$\begin{array}{lll} 1 & & \\ 4 & 3 & 2 \\ 6 & 5 & \end{array}$	$\begin{array}{lll} 1 & & \\ 5 & 4 & 2 \\ 6 & 3 & \end{array}$	$\begin{array}{lll} 1 & & \\ 5 & 4 & 3 \\ 6 & 2 & \end{array}$	$\begin{array}{lll} 2 & & \\ 4 & 3 & 1 \\ 6 & 5 & \end{array}$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$Des(\tau)$	$\{1, 4\}$	$\{1, 3, 5\}$	$\{1, 2, 5\}$	$\{2, 4\}$
β	$(1, 3, 2)$	$(1, 2, 2, 1)$	$(1, 1, 3, 1)$	$(2, 2, 2)$

So, we have

$$\mathcal{S}_{(1,3,2)} = F_{(1,3,2)} + F_{(1,2,2,1)} + F_{(1,1,3,1)} + F_{(2,2,2)}$$

Proposition (Analogous of Kostka numbers)

Let α be a composition. Then,

$$\mathcal{S}_\alpha = \sum_{\beta \vDash |\alpha|} K_{\alpha,\beta} M_\beta$$

where $\mathcal{S}_\emptyset = 1$ and $K_{\alpha,\beta}$ is equal to the number of SSRCT of shape α and content β.

Example

We have

$$\mathcal{S}_{(1,2)} = M_{(1,1,1)} + M_{(1,2)}$$

corresponding to the SSRCTs

$$\begin{array}{|c|c} \hline 1 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 1 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Theorem (Haglund, Luoto, Mason, van Willigenburg, 2011)

The set of quasisymmetric Schur functions is a $\mathbb{Z}$-basis of **QSym**. Furthermore, we have

$$\mathbf{QSym}^n = span\{\mathcal{S}_\alpha : \alpha \vDash n\}$$

Proposition

Given a partition λ,

$$s_\lambda = \sum_{\tilde{\alpha}=\lambda} \mathcal{S}_\alpha$$

Quasisymmetric Pieri Rules

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition and let $m = max\{\alpha_i\}$. For $s \in [m]$, define the operators:

- *Decrementing operator*:
  $\mathfrak{d}_s(\alpha)$ is equal to $(\alpha_1, \ldots, \alpha_{i-1}, s-1, \alpha_{i+1}, \ldots, \alpha_k)$, if exists $1 \leq i \leq k$ such that $s = \alpha_i$ and $s \neq \alpha_j$ for $j > i$. Otherwise, $\mathfrak{d}_s(\alpha) = \emptyset$. (subtract 1 from the rightmost part of size s, if it exists)
- *Horizontal operator*:
  $\mathfrak{h}_{\{s_1 < \ldots < s_j\}} = \mathfrak{d}_{s_1} \circ \ldots \circ \mathfrak{d}_{s_j}$
- *Vertical operator*:
  $\mathfrak{v}_{\{m_1 \leq \ldots \leq m_j\}} = \mathfrak{d}_{m_j} \circ \ldots \circ \mathfrak{d}_{m_1}$

Zeros are removed if necessary.

Theorem (Quasisymmetric Pieri Rules)

Let α be a composition. Then,

1.
$$\mathcal{S}_{(n)}\mathcal{S}_{\alpha} = \sum_{\beta} \mathcal{S}_{\beta}$$

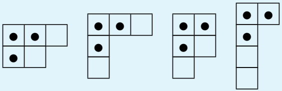
   where β is such that $\tilde{\beta}/\tilde{\alpha}$ is an horizontal strip of size n and $\mathfrak{h}_{\{s_1<\ldots<s_j\}}(\beta) = \alpha$ where $s_1, \ldots, s_j$ are the indices of the columns of $\tilde{\beta}/\tilde{\alpha}$.

2.
$$\mathcal{S}_{(1^n)}\mathcal{S}_{\alpha} = \sum_{\beta} \mathcal{S}_{\beta}$$

   where β is such that $\tilde{\beta}/\tilde{\alpha}$ is an vertical strip of size n and $\mathfrak{v}_{\{m_1\leq\ldots\leq m_j\}}(\beta) = \alpha$ where $\{m_1, \ldots, m_j\}$ is the multiset of the indices of columns of $\tilde{\beta}/\tilde{\alpha}$ with multiplicities given by the number of cells in each column.

Example

We will compute $\mathcal{S}_{(1,1)}\mathcal{S}_{(1,2)}$. If $\alpha = (1,2)$, then $\tilde{\alpha} = (2,1)$ and we have the following vertical strips of size 2



Then, we have

$$\begin{aligned}
\mathfrak{v}_{\{2,3\}}(2,3) &= (1,2)\\
\mathfrak{v}_{\{1,3\}}(1,3,1) &= \mathfrak{v}_{\{1,3\}}(1,1,3) = (1,2)\\
\mathfrak{v}_{\{1,2\}}(1,2,2) &= (1,2)\\
\mathfrak{v}_{\{1,1\}}(1,2,1,1) &= \mathfrak{v}_{\{1,1\}}(1,1,2,1) = \mathfrak{v}_{\{1,1\}}(1,1,1,2) = (1,2)
\end{aligned}$$

and so

$$\mathcal{S}_{(1,1)}\mathcal{S}_{(1,2)} = \mathcal{S}_{(2,3)} + \mathcal{S}_{(1,3,1)} + \mathcal{S}_{(1,1,3)} + \mathcal{S}_{(1,2,2)} + \mathcal{S}_{(1,2,1,1)} + \mathcal{S}_{(1,1,2,1)} + \mathcal{S}_{(1,1,1,2)}$$

Skew quasisymmetric functions

- Given compositions α and β, we define the *skew reverse composition shape* $\alpha /\!/ \beta$ as the diagram with the cells of α that are not cells in β, with β being placed at the bottom left corner of α.
- The *skew quasisymmetric Schur function* $\mathcal{S}_{\alpha/\!/\beta}$ is defined implicitely by

$$\Delta \mathcal{S}_{\alpha} = \sum_{\beta \text{ composition}} \mathcal{S}_{\alpha/\!/\beta} \otimes \mathcal{S}_{\beta}$$

Proposition (Bessenrodt, Luoto, van Willigenburg, 2011)

$$\mathcal{S}_{\alpha/\!/\beta} = \sum_{\substack{\tau \text{ is a SSRCT} \\ \text{of shape } \alpha/\!/\beta}} x^{\tau} = \sum_{\delta \vDash |\alpha/\!/\beta|} d_{D\delta} F_{\delta}$$

where $d_{D\delta}$ is equal to the number of SRCT τ of shape $D = \alpha /\!/ \beta$ such that $Des(\tau) = Set(\delta)$.

Theorem (Bessenrodt, Luoto, van Willigenburg, 2011)

Let α and β be compositions. Then,

$$\mathcal{S}_{\alpha/\!/\beta} = \sum_{\gamma} \mathcal{C}^{\alpha}_{\gamma\beta} \mathcal{S}_{\gamma}$$

where $\mathcal{C}^{\alpha}_{\gamma\beta}$ is equal to the number of SRCT τ of shape $\alpha /\!/ \beta$ such that, when using Schensted insertion for reverse tableaux, we have

$$\rho^{-1}(P(w_{col}(\tau))) = U_{\gamma}$$

with U_{γ} being the unique SRCT of shape γ and $Comp(Des(U_{\gamma})) = \gamma$.

However, this does *not* apply to the product of any quasisymmetric Schur functions $\mathcal{S}_{\alpha}\mathcal{S}_{\beta}$.

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