P=W CONJECTURES FOR CHARACTER VARIETIES WITH SYMPLECTIC RESOLUTION

joint work with M. Mauri

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Plan of the talk

1st part:

- Introduce M_{Dol} and M_B and their twists
- NAHT : $M_{Dol}^{tw} \xrightarrow{\simeq} M_B^{tw}$
- $P\,H^\bullet(M_{Dol}^{tw}) \simeq W H^\bullet(M_B^{tw})$

2nd part:

- How to restore the $P=W$ for M_{Dol} and M_B?

$PI = WI$ $P=W$ for resolution

- prove both conjectures for char. var. with symplectic resolution

C smooth projective curve $/\mathbb{C}$ of genus $g \geq 1$

$G = GL_n\mathbb{C}, SL_n\mathbb{C}$

$M_B(g, G) =$ BETTI MODULI SPACE or character variety $= \mathrm{Hom}(\pi_1, G)/$conjugation

$= \left\{ (A_1, \ldots, A_g, B_1, \ldots B_g) \in GL_n^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = Id \right\} /\!\!/ G$

$\pi_1(C, o) = \langle a_1, b_1, \ldots, a_g, b_g \rangle / \langle \prod [a_i, b_i] = id \rangle$

Example GL_1 $n=1$

$$M_B(g, \mathbb{C}^*) = \{ A_1 \dots A_g B_1 \dots B_g \in (\mathbb{C}^*)^{2g} \mid \cancel{\prod [A_i, B_i] = Id} \}$$

$$= (\mathbb{C}^* \times \mathbb{C}^*)^g$$

- $g = 1$

$$M_B(1, GL_n) = \{ (A, B) \in GL_n^2 \mid ABA^{-1}B^{-1} = Id \} / \text{conj}$$

$$= (\mathbb{C}^* \times \mathbb{C}^*)^{(n)}$$

In general: singular affine variety

$$M_B^{sm} = \{ \text{irreducible reps} \}$$

Def (Higgs bundle)

A GL_n- Higgs bundle is a pair (E,ϕ) SL_n

- E holomorphic vector bundle of rank n and degree 0 on C $\det E = \mathcal{O}$
- $\phi \in H^0(\mathrm{End}\,E \otimes K_C)$ $+ \ \mathrm{tr}\,\phi = 0$

$\uparrow$ Higgs field

$$M_{Dol}(C,G) := \text{DOLBEAULT MODULI SPACE} = \left\{ \begin{matrix} \text{semistable} \\ G\text{-Higgs bundles on } C \end{matrix} \right\} \Big/ \sim$$

quasi-projective variety of dim $2n^2(g+1)-2$ (for GL_n)

Def A vector bundle E is called (semi)stable iff $\forall$ proper subbundles $F \subseteq E$

$$\mu(F) = \frac{\deg F}{\operatorname{rank} F} \underset{(=)}{<} \frac{\deg E}{\operatorname{rank} E} = \frac{d}{n}$$

Any semistable bdl admits

$\{0\} \subset E_0 \subset \dots \quad E_n = E$ s.t.

E_i / E_{i-1} are <u>stable</u> bds with the same slope as E

$$E \longrightarrow \operatorname{Gr} E = \bigoplus E_i / E_{i-1}$$

We say that E and E' are S-equivalent iff $\operatorname{Gr} E' \cong \operatorname{Gr} E$

Thm The moduli space of semistable vector bdls is a proj. variety of dim $n^2(g+1) - 1$

<u>Rmk</u> For Higgs bundle the def. of stability is analogous considering just ϕ-invariant proper subbundles

Example

$n=1$ $\qquad L \in \mathrm{Pic}^0(C) + \phi \in H^0(K)$

$$\mathcal{M}_{Dol}(C, \mathbb{C}^*) = \mathrm{Pic}^0(C) \times H^0(K)$$

$g=1$ $\qquad C$ is an elliptic curve $\begin{cases} K_C = \mathcal{O} \\ \underset{\text{for vb}}{\text{semistab}} = \underset{\text{for HB}}{\text{semist.}} \end{cases}$

$$M_{Dol} = \left\{ (L_1, \phi_1) \oplus \dots \oplus (L_n, \phi_n) \;\middle|\; \begin{array}{l} L_i \in \mathrm{Pic}^0(C) \\ \phi_i \in H^0(\mathcal{O}) \end{array} \right\}$$

$$= (\mathrm{Pic}^0(C) \times \mathbb{C})^{(n)} = (C \times \mathbb{A}^1)^{(n)}$$

Geometry of M_{Dol} $\quad G = GL_n \quad SL_n$

- M_{Dol} is a quasi-projective variety, generally singular

$$\{\text{singularities}\} \longleftrightarrow \left\{\begin{array}{l}\text{strictly semistable} \\ \text{Higgs bundles}\end{array}\right\}$$

- $\exists$ a symplectic form on M_{Dol}^{sm}
- $\exists$ a proper holomorphic map Hitchin fibration

$$\chi : M_{Dol}(C,G) \longrightarrow$$

$$A = \bigoplus_{i=1}^{n} H^0(K^{\otimes i})$$

$$(E,\phi) \longmapsto \text{char pol}\,\phi$$

$n=2$

$$(E,\phi) \longmapsto \cancel{\mathrm{tr}\,\phi},\ \det\phi$$

Non Abelian Hodge theorem

$$M_{Dol}(C, G) \xrightarrow[\Psi]{\overset{\mathbb{R}}{\simeq}} M_B(g, G)$$

fibred with cpt subvar.

'affine

<u>Rmk</u>:

- Preserves the local structure of sing
- It is NOT algebraic

Example

$g = 1 \qquad C = \mathbb{C}/_{\langle 1, i\rangle}$

$$\begin{tikzcd} M_{Dol} = (\underset{\overset{\parallel}{S^1\times S^1}}{C} \times \underset{\overset{\parallel}{\mathbb{R}\times\mathbb{R}}}{\mathbb{A}^1})^{(n)} \arrow[r, "\Psi"] \arrow[d, "\chi = pr_L^{(n)}"'] & (\underset{\overset{\parallel}{S^1\times\mathbb{R}}}{\mathbb{C}^*} \times \underset{\overset{\parallel}{S^1\times\mathbb{R}}}{\mathbb{C}^*})^{(n)} = M_B \\ (\mathbb{A}^1)^{(n)} = \mathbb{A}^n & \end{tikzcd}$$

$n = 1$

$$\underset{S^1\times S^1}{(\theta_1, \theta_2}, \underset{\mathbb{R}}{\rho_1} + i\underset{\mathbb{R}}{\rho_2}) \longrightarrow (e^{-2\rho_1}e^{i\theta_1}, e^{+2\rho_2}e^{i\theta_2})$$

$$C \times \mathbb{A}^1 = S^1 \times S^1 \times \mathbb{R} \times \mathbb{R} = \mathbb{C}^* \times \mathbb{C}^*$$

P=W CONJECTURE [de Cataldo - Hausel Migliorini]

$$H^\bullet(M_{Dol}(C,G)) \xleftarrow{\psi^*} H^\bullet(M_B(g,G))$$

$$P_k H^\bullet(M_{Dol}(C,G)) \xleftarrow{\simeq} W_{2k}(M_B(g,G))$$

perverse Leray filtration associated to χ

weight filtration arising from the MHS

Mixed Hodge structures

$$H^k(M)_{/\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q} \qquad H^{p,q} = \overline{H^{q,p}}$$

A mixed Hodge structure on $H^\bullet(M)$ is the datum of

- an increasing filtration $W_\bullet$ on $H^\bullet(M)$ WEIGHT FILTRATION
- a decreasing filtration $F_\bullet$ on $H^\bullet(M) \otimes \mathbb{C}$ Hodge filtration

such that $Gr^k_W := W_k / W_{k-1} \otimes \mathbb{C}$ admit a Hodge decomp. induced by F

YOGA of WEIGHTS

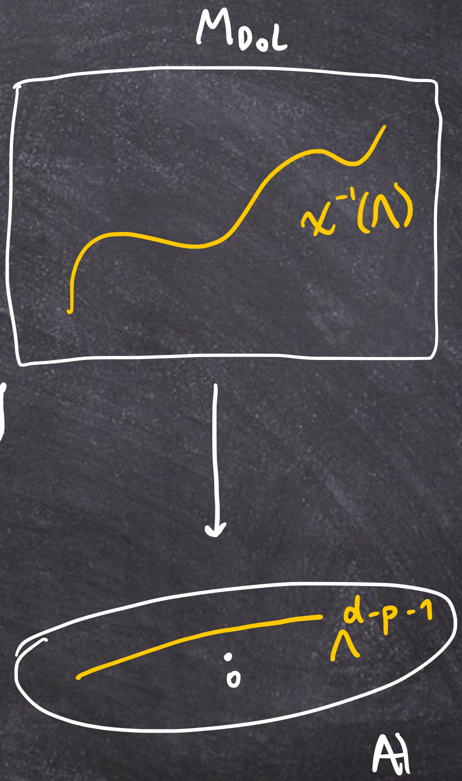
M smooth $\Rightarrow$ $W_k H^i = 0 \quad \forall\, k < i$ high weights

M compact $\Rightarrow$ $W_k H^i = W_i H^i \quad \forall\, k \geq i$ low weights

Perverse filtration

$$\chi : M_{Dol}(C,G) \longrightarrow \mathbb{A} = \bigoplus_{i=1}^{n} H^0(K^{\otimes i})$$

$$P_p H^d = \operatorname{Ker}\left\{ H^\bullet(M_{Dol}) \longrightarrow H^\bullet(\chi^{-1}(\Lambda^{d-p-1})) \right\}$$



Motivation : mixed Hodge structure on twisted moduli

$d \in \mathbb{Z}$, $\underbrace{(n,d)}$ $= 1$ $L \in Pic^d(C)$

$$M_B^{tw}(g,G) = \left\{ (A_1 \dots A_g\, B_1 \dots B_g \in GL_n^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = \widetilde{e^{\frac{2\pi i}{d}} I} \right\} /\!/ G$$

$$\|\wr$$

$M_{Dol}^{tw}(C,G) =$ for $G = GL_n$ $\left\{ (E,\phi) \;\middle|\; \begin{array}{l} E \text{ holo vector bdl} \\ \text{of rk } n \text{ and deg } d \\ \phi \in H^0(\mathrm{End}\, E \otimes K_C) \end{array} \right\}$

for $G = SL_n$ $\left\{ (E,\phi) + \begin{array}{c} \det E \cong L \\ \mathrm{tr}\, \phi = 0 \end{array} \right\}$

Thm (CURIOUS HARD LEFSCHETZ) [Hausel - Rodriguez Villegas, Mellit]

$\exists\, \alpha \in H^2(M_B^{tw}(g, G))$

$$\cup \alpha^\ell : \quad Gr^W_{2d-2\ell} H^\bullet(M_B^{tw}) \xrightarrow{\simeq} Gr^W_{2d+2\ell} H^{\bullet+2\ell}(M_B^{tw})$$

Thm (RELATIVE HARD LEFSCHETZ) for the Hitchin map

$\exists\, \alpha \in H^2(M_{Dol}^{tw}(C, G))$ χ- ample

$$\cup \alpha^\ell \quad Gr^P_{d-\ell} H^\bullet(M_{Dol}^{tw}) \xrightarrow{\simeq} Gr^P_{d+\ell} H^{\bullet+2\ell}(M_{Dol}^{tw})$$

In the untwisted case
RHL and CHL may fail for $H^\bullet(M)$!

<u>Example</u>: $E(M_B^{g=2}, SL_2) = \sum_{k,d} (\dim Gr^W_{2k} H^d_c)\, q^k$

$$= 1 + q^2 + 17q^4 + q^6$$

CHL $\Rightarrow$ E palindromic!

How to restore symmetries?

take intersection cohomology

$\downarrow$

$PI = WI$

resolve singularities

$\downarrow$

$P = W$ for resolutions

P = W CONJECTURE
[de Cataldo - Maulik]

$$\begin{array}{ccc} IH^*(M_{Dol}(C,G)) & \xleftarrow{\Psi^*} & IH^\bullet(M_B(C,G)) \\ \cup\!\mid & & \cup\!\mid \\ P_k IH^*(M_{Dol}(C,G)) & \xleftarrow{\simeq} & W_{2k} IH^*(M_B(C,G)) \end{array}$$

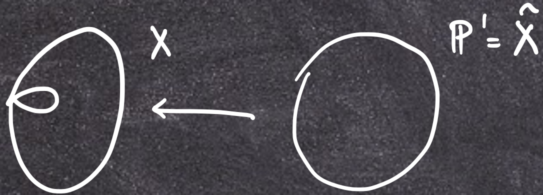
- RHL holds for $IH^\bullet(M_{Dol})$
- $P_\bullet$ is independent of the complex structure on C

Interlude: Intersection cohomology

- X nonsingular, finite quotient sing, $\quad IH^\bullet(X) = H^\bullet(X)$
- Duality, $\quad IH^\bullet(X) = IH_c^{2\dim X - \bullet}(X)$
- $\hat{X} \longrightarrow X$ normalization $\quad IH(X) = IH(\hat{X})$
- $IH(X)$ carry a MHS as the cohomology

Example

$$H^\bullet(X) \longrightarrow IH^\bullet(X) \hookrightarrow H^\bullet(\tilde{X})$$

- 

$\Rightarrow IH(X) = H^\bullet(\mathbb{P}^1)$

Observe that

$H(X) = \mathbb{Q}$

$H^1(X) = \mathbb{Q}$ 0

$H^2(X) = \mathbb{Q}$

- $X =$ 

cone over a smooth manifold M of $\dim_{\mathbb{R}} 2n-1$:

$$IH^i(X) = \begin{cases} H^i(X - o) & \forall\, i < n \\ 0 & \text{otherwise} \end{cases}$$

Rmk:

- IH remembers the cohomology of the smooth locus
- IH restore dualities

Thm [FM] (LIFT OF ψ TO A RESOLUTION)

There exist

- $f_{Dol} : \tilde{M}_{Dol} \longrightarrow M_{Dol}$
- $f_B : \tilde{M}_B \longrightarrow M_B$

  } res. of sing.

- $\tilde{\psi} : \tilde{M}_{Dol} \longrightarrow \tilde{M}_B$ diffeo

such that

$$\begin{tikzcd} \tilde{M}_{Dol} \arrow[r, "\tilde{\psi}"] \arrow[d, "f_{Dol}"'] & \tilde{M}_B \arrow[d, "f_B"] \\ M_{Dol} \arrow[r, "\psi"'] & M_B \end{tikzcd}$$

commutes up to isotopy.

$$\begin{tikzcd} H^\bullet(\tilde{M}_{Dol}) & H^\bullet(\tilde{M}_B) \arrow[l, "\tilde{\psi}^*"'] \\ H^\bullet(M_{Dol}) \arrow[u, "f_{Dol}^*"] \arrow[ur, phantom, "\circlearrowright"] & H^\bullet(M_B) \arrow[l, "\psi^*"] \arrow[u, "f_B^*"'] \end{tikzcd}$$

P=W CONJECTURE FOR RESOLUTION

$$P_k H^\bullet(\tilde{M}_{Dol}) = W_{2k} H^\bullet(\tilde{M}_B)$$

Example

$g=1$

$$\begin{tikzcd}
(C\times\mathbb{A}^1)^{[n]} \arrow[r, dashed, "\tilde{\Psi}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & (\mathbb{C}^*\times\mathbb{C}^*)^{[n]} \arrow[d] \\
(C\times\mathbb{A}^1)^{(n)} \arrow[r, "\Psi"] & (\mathbb{C}^*\times\mathbb{C}^*)^{(n)}
\end{tikzcd}$$

[Trozzo's PhD thesis]

MAIN RESULT

Thm [FM] $P=W$, $PI=WI$, $P=W$ conjecture for resolution hold for character varieties which admit a symplectic resolution, i.e. for

$g=1$ and arbitrary rank

$g=2$ and rank 2

$G = GL, SL$

Def A resolution $f : \widetilde{M} \longrightarrow M$ is <u>symplectic</u> if a symplectic form on M^{sm} extends to a holomorphic form on $\widetilde{M}$.

sympl.

Remark

- First nontrivial evidence for $PI = WI$

- Any symplectic resolution $\widetilde{M}_{Dol}$ is a degeneration of one of the 4 known examples of compact hyperkähler manifolds.

$g = 1 \quad , rk = n$	$g = 2 \quad , rk = 2$	
$K3^{[n]}$	OG_{10}	$G = GL_n$
$K^{[n]}$	OG_6	$G = SL_n$

- By the Decomposition theorem, a symplectic resolution gives a splitting

$$H^*(\widetilde{M}_{Dol}) = IH^*(M_{Dol}) \oplus \bigoplus H^*\left(\begin{array}{c}\text{strata inside}\\ \text{sing}(M_{Dol})\end{array}\right)$$

Strategy of the proof

$$\boxed{\begin{array}{c}P=W \text{ for}\\ \text{resolutions}\end{array}} = \boxed{PI = WI} \quad + \quad P=W \ \begin{array}{l}\text{for lower dim.}\\ \text{strata}\end{array}$$

$$\Big\Downarrow \quad H \subseteq IH^*$$

$$\boxed{P=W}$$

Sketch of the proof in an extended example

From now on $M := M_{Dol.}(C, SL_2)$ with $g(C) = 2$

- M is a singular 6-fold

- The singular locus has dim 4

$$\Sigma = \left\{ (L,\theta) \oplus (L^{-1}, -\theta) \;\middle|\; \begin{array}{l} L \in Pic^0(C) \\ \theta \in H^0(K) \end{array} \right\} = Pic^0(C) \times H^0(K) \Big/ \underset{(L,\theta) \mapsto (L^{-1}, -\theta)}{\mathbb{Z}_2}$$

$$\rotatebox[origin=c]{90}{$\subseteq$}$$

$$\Omega = \left\{ (L,0) \oplus (L,0) \;\middle|\; L^2 = \mathcal{O} \right\} = 16 \text{ points}$$

- M is endowed with two group actions

  - $\Gamma = \mathrm{Pic}^0(C)[2] \simeq (\mathbb{Z}/2\mathbb{Z})^4 \curvearrowright M$

$$(\mathcal{L}, (E, \phi)) \longmapsto (E \otimes \mathcal{L}, \varphi)$$

$$\Rightarrow \quad \mathbb{H}^{*}(M) = \underbrace{\mathbb{H}^{*}(M)^{\Gamma}}_{\text{invariant part}} \oplus \underbrace{\mathbb{H}^{*}(M)_{var}}_{\text{variant part}}$$

Upshot: $P = W$s for $M \iff \begin{cases} P = W & \text{for variant part} \\ P = W & \text{for invariant part} \end{cases}$

- $\mathbb{C}^* \curvearrowright M \qquad (\lambda, (E, \phi)) \longmapsto (E, \lambda\phi)$

$$M^{\mathbb{C}^*} = N \sqcup \bigsqcup_{j=1}^{16} \theta_j \qquad \left(K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

$\{(E, 0)\}$ = moduli space of rk 2 vb with trivial determinant $\simeq \mathbb{P}^3$

$$H(M) = H^\bullet(N) \oplus \bigoplus_{j=1}^{16} H^{\bullet+6}(\theta_j)$$

trivial Γ-mod

Γ acts reg. rep

d	0	1	2	3	4	5	6
$\dim H^d(M)^\Gamma$	1	0	1	0	1 + 1	0	2

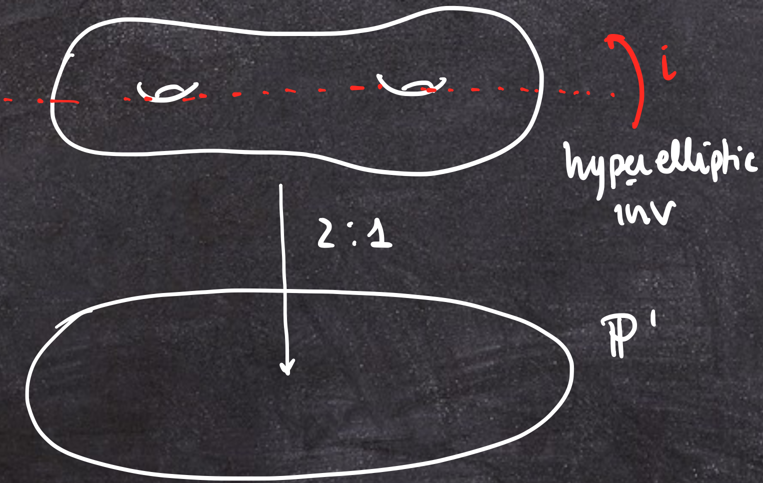
Q: Where does $+1$ come from?

- Tautological classes: in the twisted case the generators are Künneth factors of $c_2(\mathbb{P}(\mathcal{E}))$ where $\mathcal{E}$ is a universal Higgs bdl on $M^{tw} \times C$, i.e.

$$\mathcal{E}|_{\{(E,\varphi)\} \times C} \cong (E, \phi)$$

- no universal bundle on M

Solution: The additional class in IH^4 is a tautological class coming from a quasi-étale cover of M



Def

$$M_i = \left\{ \begin{array}{l} \text{equivariant} \\ \text{Higgs bdls} \end{array} \right\} = \left\{ (E,\phi) + \boxed{\begin{array}{ccc} E & \xrightarrow{h} & i^*E \\ \downarrow & & \downarrow \\ C & \xrightarrow[i]{} & C \end{array}} \begin{array}{l} \\ \\ \\ \text{lift of } i\text{-action} \\ i^*h \circ h = \mathrm{id} \end{array} \right\}$$

$$q \downarrow \qquad\qquad \downarrow$$

$$M \qquad\qquad (E,\phi)$$

Prop [FM]

- $q : M_i \longrightarrow M$ is a quasi-étale cover branched along Σ
- M_i has isolated singularities $(q^{-1}(\Omega))$
- q is the only nontrivial quasi-étale cover of $M(C, SL_2)$ with $g(C) \geq 2$

Thm $\exists$ a universal equivariant bundle on $M_i^{sm} \times C$

$$\begin{array}{c} M_i^{sm} \times C \\ \uparrow \\ \mathbb{P}(\mathcal{E}) \end{array}$$

let $\Gamma_i = \langle \Gamma, \text{deck transf. of } q \rangle$.

- $$\begin{aligned} c_2(\mathbb{P}(\mathcal{E})) \in H^4(M_i^{sm})^{\Gamma_i} \\ \| \\ IH^4(M_i)^{\Gamma_i} \\ \| \\ IH^4(M)^{\Gamma} \end{aligned}$$

Thm $c_2(\mathbb{P}(\mathcal{E}))$ has perversity < 4

weight $= 4$

The weight filtration

[Logares-Munoz-Newstead]: $\quad E(\mathcal{M}_B^s) = q^6 + 16q^4 - 5q^2 \quad , \quad q = uv$

$\Sigma_B = (\mathbb{C}^*)^4 / \mathbb{Z}_2$

$$\begin{cases} IP_t(\mathcal{M}_B)^\Gamma = 1 + t^2 + 2t^4 + 2t^6 \\ IE(\mathcal{M}_B)^\Gamma = 1 + 2q^2 + 2q^4 + q^6 \end{cases}$$

Poincaré duality:

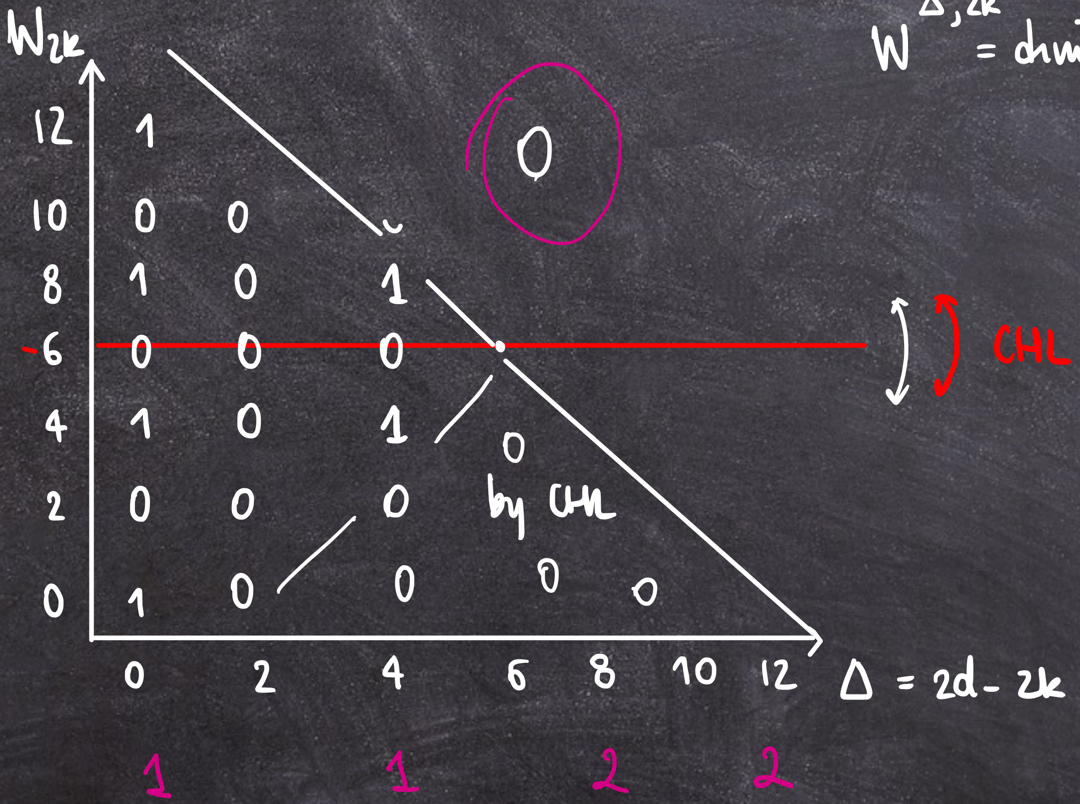
1 class of weight 12 $\longrightarrow H^6$

2 " " 8 $\longrightarrow 1H^4 + 1H^6$

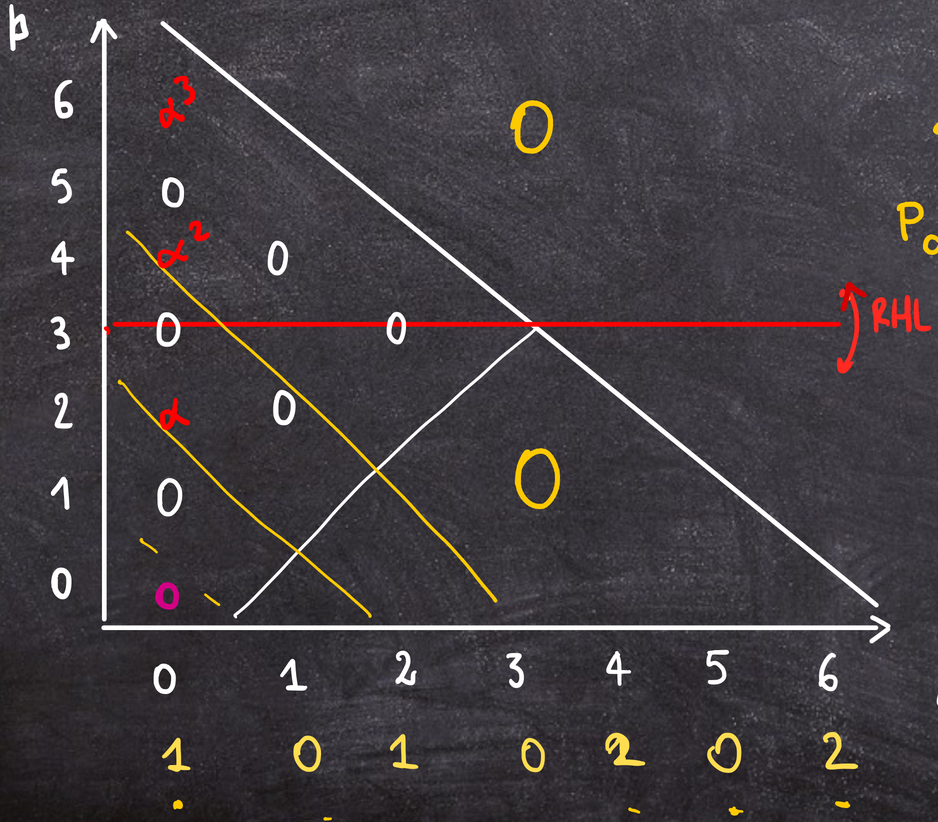
2 " " 4 $\longrightarrow 1H^2 + 1H^4$

1 " " 0 $\longrightarrow H^0$

$$W^{\Delta,2k} = \dim \mathrm{Gr}^{2k}_W \mathrm{IH}^{d+k}(M_B)$$



Perverse filtration



$$p^{\Delta,p} = \dim \mathrm{Gr}^P_p \, IH^{p+\Delta}(M)$$

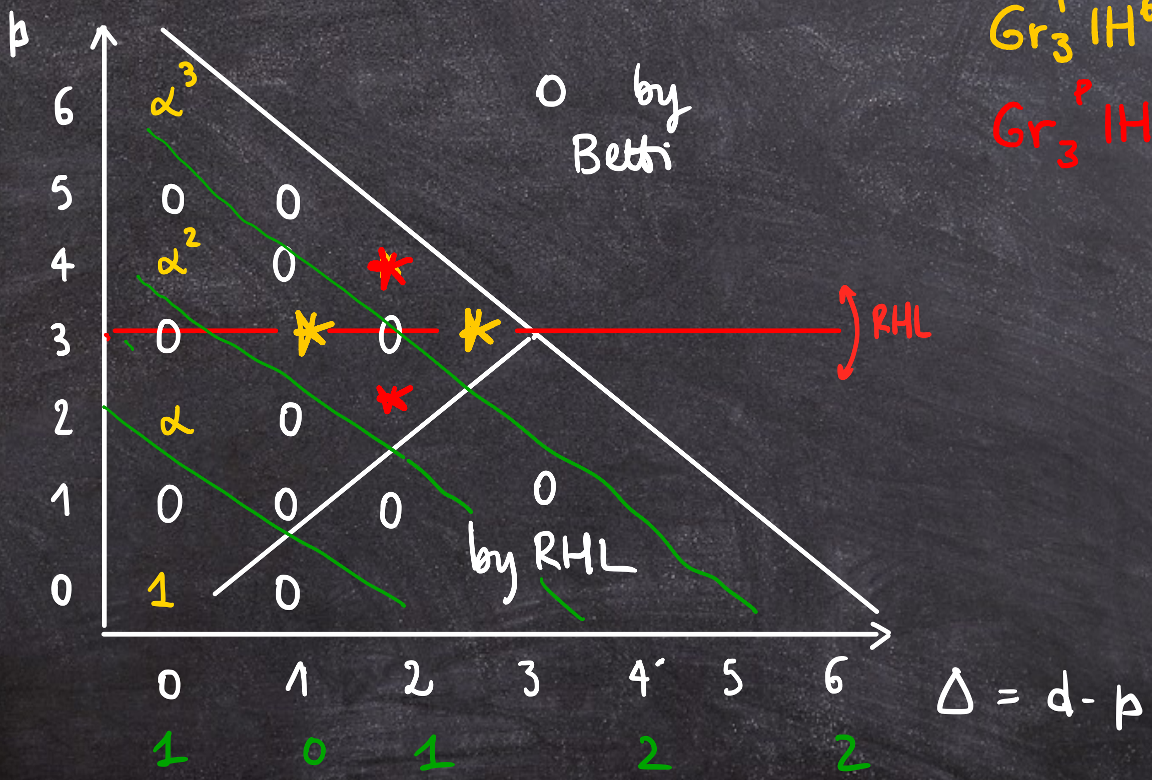
- $\alpha \in H^2$ χ-ample

$$P_{d-1}H^d = \mathrm{Ker}\{H^d(M) \to H^d(\chi^{-1}(s))$$

$$\alpha^d \longmapsto \alpha|_{\chi^{-1}(s)} \neq 0$$

$$\Delta = d - p$$

Perverse filtration



$$Gr_3^P IH^6 \neq 0$$

$$Gr_3^P IH^6 = 0$$

Intersection form

[de Cataldo - Migliorini]

$$\dim \mathrm{Gr}^P_3 H^6(\tilde{M})^\Gamma \leq \text{rank intersection form}$$

$$H^6_c(\tilde{M})^\Gamma \times H^6_c(\tilde{M})^\Gamma \longrightarrow \mathbb{Q}$$

$$\|$$

$$\dim \mathrm{Gr}^P_3 IH^6(M)^\Gamma + 1 \quad \text{as} \quad H^6(\tilde{M})^\Gamma = IH^6(M)^\Gamma \oplus \underbrace{H^2(\Sigma)^\Gamma}_{\text{perv}>3} \oplus H^0(\Omega)^\Gamma$$

$$\text{if rank} = 1 \quad \Rightarrow \quad \dim \mathrm{Gr}^P_3 IH^6(M)^\Gamma = 0$$

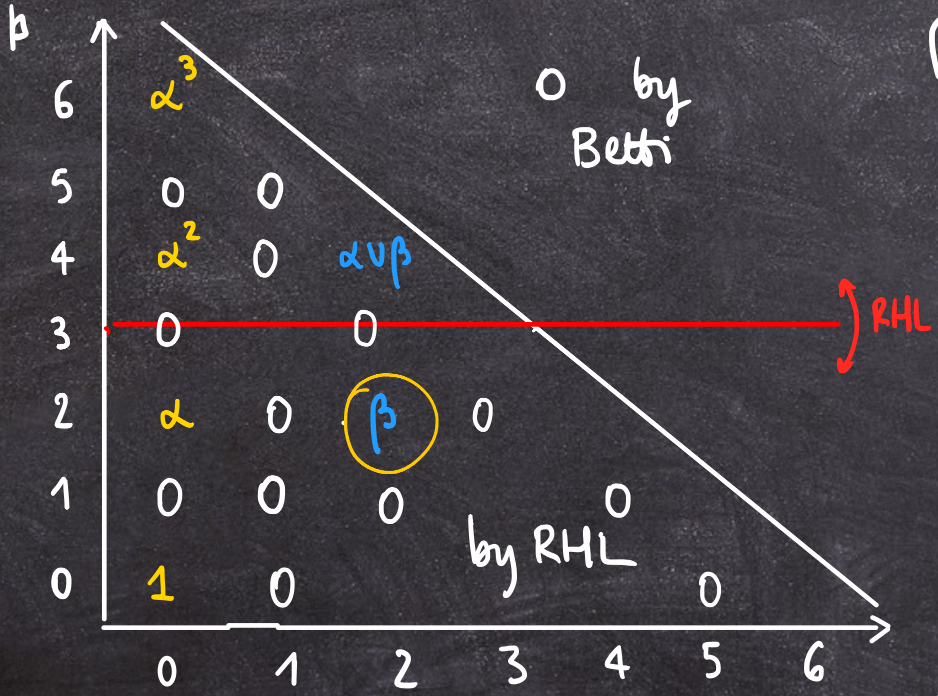
Thm [FM]

The intersection form on $H^6_c(\tilde{M})^{\Gamma}$ can be represented by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 16 & -8 \\ 0 & 16 & -64 & 32 \\ 0 & -8 & 32 & -16 \end{bmatrix}$$

in the basis $[(x \circ f)^{-1}(0)]$, $[\tilde{N}]$, $[f^{-1}(\Omega)]$, $[f^{-1}(\Sigma \setminus \Omega)]$

Perverse filtration



Summarizing

$\beta := c_2(\mathbb{P}(\mathcal{E}))$ has weight 4

α has weight 4

$\Downarrow$

$P I = W I$ for invariant part

$\Delta = d - p$

THANK YOU

FOR THE

ATTENTION !