Asymptotic Bridgeland stability on 3-folds

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Stability conditions on triangulated categories: ingredients

A Bridgeland stability condition on a triangulated category $\mathscr{T}$ is a pair $\sigma = (\mathcal{A}, Z)$ consisting of:

- the heart $\mathcal{A}$ of a t-structure on $\mathscr{T}$;
- a homomorphism of abelian groups $Z : K_0(\mathcal{A}) \to \mathbb{C}$, called a central charge.

In addition, we assume that Z factors through a surjective group homomorphism

$$\tau : K_0(\mathcal{A}) \twoheadrightarrow \Lambda,$$

where Λ is a finite dimensional lattice equipped with a norm $||\cdot||$ on $\Lambda \otimes \mathbb{R}$.

Stability conditions on triangulated categories: axioms

(1) $\mathbf{Im}(Z(A)) \geq 0$ for every $A \in \mathcal{A}$, and
$\mathbf{Im}(Z(A)) = 0 \implies \mathbf{Re}(Z(A)) < 0$; set

$$\mu_Z(A) := -\frac{\mathbf{Re}(Z(A))}{\mathbf{Im}(Z(A))}$$

(2) Every $E \in \mathcal{A}$ admits a Harder–Narasimhan filtration;

(3) the support property

$$\inf\left\{ \frac{|Z(E)|}{||\tau(E)||} \;\middle|\; E \in \mathcal{A} \text{ semistable} \right\} > 0$$

An object $E \in \mathscr{T}$ is σ-(semi)stable if $E \in \mathcal{A}$ and every $F \hookrightarrow E$ in $\mathcal{A}$ satisfies $\mu_Z(F) < (\leq)\ \mu_Z(E)$.

Basic example: sheaves on curves

Let X be a smooth projective curve, and let $\mathscr{T} = D^{\mathrm{b}}(X)$.

Take $\mathcal{A} = \mathcal{C}oh(X)$ as the heart of the standard t-structure on $D^{\mathrm{b}}(X)$, and let the central charge be given by:

$$Z(E) := -\deg(E) + \sqrt{-1}\,\mathrm{rk}(E) \quad , \quad E \in \mathcal{C}oh(X).$$

Set $\Lambda = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z})$ with basis $(1,\omega)$, equipped with the usual euclidean norm; set

$$\tau(E) = (\mathrm{rk}(E), \deg(E)\omega).$$

The support property is trivially satisfied because $\dfrac{|Z(E)|}{||\tau(E)||} = 1$ for every sheaf E.

In this context, σ-stability coincides with the usual slope stability.

Stability manifold, walls and chambers

Let $\mathrm{Stab}(\mathscr{T})$ be the set of stability conditions $\mathscr{T}$.

Bridgeland's deformation theorem: The map

$$\mathcal{Z} : \mathrm{Stab}(\mathscr{T}) \to \mathrm{Hom}(\Lambda, \mathbb{C}),$$

sending a stability condition to its central charge, is a local homeomorphism. In particular, $\mathrm{Stab}(\mathscr{T})$ is a complex manifold of complex dimension $\mathrm{rk}(\Lambda)$.

Wall and chamber structure: There are only finitely many walls $\{W_{u_i,v}\}_{i=1}^{n}$ for $v \in \Lambda$ intersecting a compact set $K \subset \mathrm{Stab}(\mathscr{T})$, each of real codimension 1, and any connected component

$$C \subset K \setminus \bigcup_{i=1}^{n} W_{u_i,v}$$

has the following property: E is σ-semistable for some $\sigma \in C$ if and only if E is σ'-semistable for every $\sigma' \in C$.

Existence and moduli spaces

Some of the main problems being studied in the literature are:

Existence: given a projective variety X, are there stability conditions on $D^{\mathrm{b}}(X)$?

Positive answer for surfaces, Fano 3-folds with Picard rank 1, abelian 3-folds, the quintic 3-fold, $\mathbb{P}^n$; following ideas by Bayer, Macrì, Toda, Bertram, among others.

Moduli spaces: Is $M_\sigma(v)$ a projective scheme?

Yes, for certain surfaces like $K3$, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$.
More generally, Piyaratne–Toda showed that, for the so-called geometric stability conditions, $M_\sigma(v)$ is a proper algebraic stack of finite type.

What stable objects look like?

Besides surfaces, some examples on $\mathbb{P}^3$.

Stability conditions on surfaces

Let X be a smooth projective surface with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, with H being the ample generator.

Fix $\beta \in \mathbb{R}$ and consider the full subcategories (a torsion pair)

$$\mathcal{T}_\beta := \{E \in \mathcal{C}oh(X) \mid \forall\ E \twoheadrightarrow G \text{ satisfies } \mu(G) > \beta\}, \quad \text{and}$$

$$\mathcal{F}_\beta := \{E \in \mathcal{C}oh(X) \mid \forall\ F \hookrightarrow E \text{ satisfies } \mu(F) \leq \beta\}.$$

Next, take the subcategory of $D^{\mathrm{b}}(X)$ generated by $\mathcal{T}_\beta$ and $\mathcal{F}_\beta[1]$

$$\mathcal{B}^\beta(X) := \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$$

This is the heart of a t-structure on $D^{\mathrm{b}}(X)$, and the procedure above is known as tilting.

The central charge is given as follows, for $B \in \mathcal{B}^\beta$ and $\alpha \in \mathbb{R}^+$:

$$Z^{\mathrm{tilt}}_{\alpha,\beta}(B) := -\left(\mathrm{ch}_2^\beta(B) - \frac{1}{2}\alpha^2 \,\mathrm{ch}_0(B)\right) + \sqrt{-1}\,\mathrm{ch}_1^\beta(B)$$

Stability conditions for 3-folds, I

The idea of Bayer–Macrì–Toda is to tilt $\mathcal{B}^\beta$ on the torsion pair

$$\mathcal{T}_{\alpha,\beta} := \{E \in \mathcal{B}^\beta(X) \mid \forall\ E \twoheadrightarrow G \text{ satisfies } \nu_{\alpha,\beta}(G) > 0\}, \text{ and}$$

$$\mathcal{F}_{\alpha,\beta} := \{E \in \mathcal{B}^\beta(X) \mid \forall\ F \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta}(F) \leq 0\}.$$

where $\nu_{\alpha,\beta}$ is the slope function for the central charge $Z^{\mathrm{tilt}}_{\alpha,\beta}$:

$$\nu_{\alpha,\beta}(B) := \begin{cases} \dfrac{\mathrm{ch}_2^\beta(B) - \alpha^2\,\mathrm{ch}_0(B)/2}{\mathrm{ch}_1^\beta(B)}, & \text{if } \mathrm{ch}_1^\beta(B) \neq 0; \\ +\infty, & \text{if } \mathrm{ch}_1^\beta(B) = 0. \end{cases}$$

The category $\mathcal{A}^{\alpha,\beta} := \langle \mathcal{F}_{\alpha,\beta}[1], \mathcal{T}_{\alpha,\beta} \rangle$ is the heart of another t-structure on $D^{\mathrm{b}}(X)$.

Stability conditions 3-folds, II

BMT define the central charge, for objects $A \in \mathcal{A}^{\alpha,\beta}$, as follows

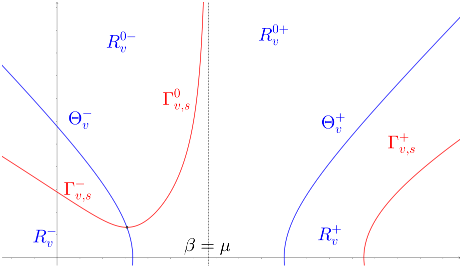
$$Z_{\alpha,\beta,s}(A) := -\operatorname{ch}_3^\beta(A) + (s+1/6)\alpha^2 \operatorname{ch}_1^\beta(A) + \\ + \sqrt{-1}\left(\operatorname{ch}_2^\beta(A) - \alpha^2 \operatorname{ch}_0(A)/2\right)$$

They prove that the pair $(\mathcal{A}^{\alpha,\beta}, Z_{\alpha,\beta,s})$ with $s > 0$ is a Bridgeland stability condition on X provided a certain generalized Bogomolov inequality is satisfied; this is related to the support property.

This inequality was proved to hold for Fano 3-folds with Picard rank 1, abelian 3-folds and the quintic 3-fold; counter-examples have also been found.

These are the so-called geometric stability conditions.

The plane parametrizing geometric stability conditions



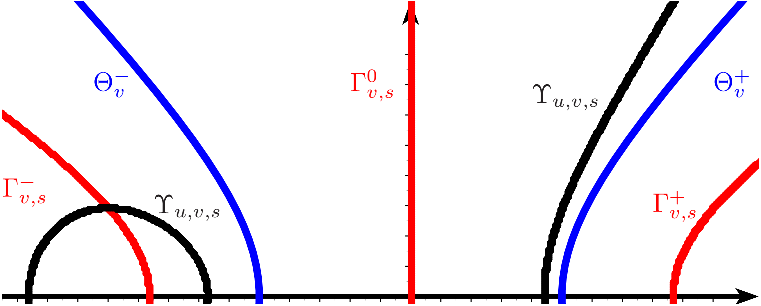
$$s = 1/3$$
$$v = (3, 4, 2, 2/3)$$
$$= \mathrm{ch}(T\mathbb{P}^3)$$

The blue hyperbola is the curve $\mathbf{Re}(Z^{\mathrm{tilt}}_{\alpha,\beta}(v)) = 0$; we call it Θ_v.

The red curve is given by $\mathbf{Re}(Z_{\alpha,\beta,s}(v)) = 0$; we call it $\Gamma_{v,s}$.

Θ_v and $\beta = \mu$ divide the plane into 4 regions, labeled (from left to right) R_v^-, R_v^{0-}, R_v^{0+} and R_v^+.

The wall $\Upsilon_{u,v,s}$ has two connected components: one bounded and the other unbounded.



Here, $v = (2, 0, -1, 0)$ (null correlation bundle on $\mathbb{P}^3$)
$u = (1, 0, -1, 1)$ (ideal sheaf of a line), and $s = 1/3$.

Asymptotic Bridgeland stability

Let $\gamma : (0, \infty) \to \mathbb{R} \times \mathbb{R}^+$ be an unbounded path.

Let $\lambda_{\alpha,\beta,s}$ be the slope associated with the central charge $Z_{\alpha,\beta,s}$.

An object $A \in D^{\mathrm{b}}(X)$ is asymptotically λ-(semi)stable along γ if the following two conditions hold for a given $s > 0$:

(i) there is $t_0 > 0$ such that $A \in \mathcal{A}^{\gamma(t)}$ for every $t > t_0$;

(ii) for every sub-object $F \hookrightarrow A$ within $\mathcal{A}^{\gamma(t)}$ with $t > t_0$, there is $t_1 > t_0$ such that $\lambda_{\gamma(t),s}(F) < \ (\leq)\ \lambda_{\gamma(t),s}(A)$ for $t > t_1$.

Item (ii) implies that A has a last wall along γ, which in principle depends on the object A.

Unbounded Θ-curves

A path $\gamma(t) = (\alpha(t), \beta(t))$ is called an unbounded Θ^--curve if

$$\lim_{t\to\infty} \beta(t) = -\infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\dot{\alpha}(t)}{\dot{\beta}(t)} > -1.$$

That is, $\gamma(t)$ is asymptotically bounded by Θ_v^-

Similarly, we say that $\gamma(t) = (\alpha(t), \beta(t))$ is an unbounded Θ^+-curve if $\gamma^*(t) := (\alpha(t), -\beta(t))$ is an unbounded Θ^--curve

Asymptotic λ-stability on $R_v^{\pm}$

Let v be a numerical Chern character with $v_0 \neq 0$.

For each $s \geq 1/3$, we have:

- An object $A \in D^{\mathrm{b}}(X)$ with $\mathrm{ch}(A) = v$ is asymptotically λ-(semi)stable along an unbounded Θ^--curve if and only if A is a Gieseker (semi)stable sheaf.
- An object $A \in D^{\mathrm{b}}(X)$ is asymptotically $\lambda_{\alpha,\beta,s}$-(semi)stable objects along an unbounded Θ^+-curve if and only if $A^\vee$ is a Gieseker (semi)stable sheaf.

If we follow the curves $\Gamma^{\pm}_{v,s}$, then we can take $s > 0$.

Victor Pretti recently proved a similar result for the case $v_0 = 0$.

Vague idea of the proof

Steps to prove asymptotic stability $\implies$ Gieseker stability, $s > 0$.

Let $\gamma = (a, t)$ be a horizontal line for a fixed $a \in \mathbb{R}^+$.

✓ If $A \in \mathcal{A}^{a,t}$ for $t \gg 0$, then A is a sheaf;

✓ Asymptotic semistability implies that A is torsion free;

✓ If A is not Gieseker semistable, let $F \hookrightarrow A$ be its maximal destabilizing subsheaf; one can check that $F \in \mathcal{A}^{a,t}$ for $t \gg 0$, so we get a morphism $F \to A$ in $\mathcal{A}^{a,t}$ for $t \gg 0$ as well.

✓ Look at limits to show that A is not asymptotically stable.

$$\lim_{t \to -\infty} \left(\lambda_{a,t,s}(F) - \lambda_{a,t,s}(A) \right) = \frac{\mu(F) - \mu(E)}{3} \geq 0$$

Gieseker semistable sheaves N on $\mathbb{P}^3$ with $\mathrm{ch}(N) = (2, 0, -1, 0)$ are null correlation sheaves, defined as cokernels of sections $\sigma \in H^0(\Omega_{\mathbb{P}^3}(2))$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\sigma} \Omega_{\mathbb{P}^3}(2) \to N \to 0.$$

The following two exact sequences will induce walls

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to N \to I_S(1) \to 0 \quad , \quad S = \text{ pair of skew lines}$$

$$0 \to K \to N \to \mathcal{O}_L(-1) \to 0 \quad ,$$

where L is a jumping line for N, and K is a strictly semistable torsion free sheaf with $\mathrm{ch}(N) = (2, 0, -2, 2)$.

Let $v = (2, 0, -1, 0)$ be the numerical Chern character corresponding to null correlation sheaves on $\mathbb{P}^3$, and fix $s = 1/3$.

The region R_v^- is divided into three stability chambers C_i within which the $\lambda_{\alpha,\beta,s}$-stable objects are described as follows:

(C_1) null correlation sheaves;

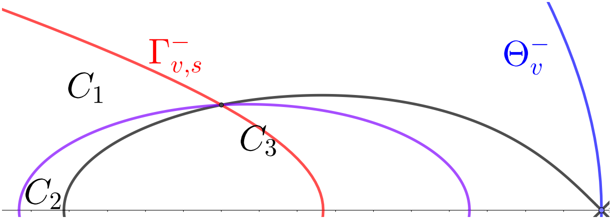
(C_2) nontrivial extensions of a semistable torsion free sheaf K with $\mathrm{ch}(K) = (2, 0, -2, 2)$ by $\mathcal{O}_L(-1)$, where L is a line;

(C_3) no stable objects.

$s = 1/3$

Crossing point:

$(\alpha, \beta) = (1/\sqrt{3}, -2)$



Issues

- show that the last Bridgeland wall (along $\Gamma^-_{v,s}$ or horizontal lines) for a Gieseker semistable sheaf E only depends on $\mathrm{ch}(E)$ and not on E itself.
- study the asymptotics along curves of the form $\Theta^-_v \pm \epsilon$;
- understand the vertical asymptotics and other non Θ-curves;

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Thanks!