Correspondence Theorem between Holomorphic Discs and Tropical Discs on (Log)-Calabi-Yau Surfaces

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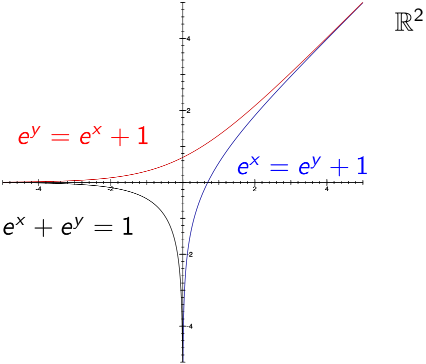
Tropical Geometry

- The image of $\{X + Y + 1 = 0\} \subseteq (\mathbb{C}^*)^2$ under the projection

$(X, Y) \mapsto (\log|X|, \log|Y|)$



Cartoon for Tropical Geometry

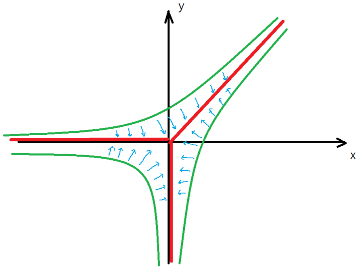
- Mikhalkin considers the following self-diffeomorphism

$$(\mathbb{C}^*)^2 \xrightarrow{H_t} (\mathbb{C}^*)^2$$
$$(X, Y) \mapsto (|X|^{\frac{1}{\log t}} \frac{X}{|X|}, |Y|^{\frac{1}{\log t}} \frac{Y}{|Y|})$$

- This induces a new complex structure J_t.
- Metrically, this is the spirit of SYZ degeneration.

Tropical Geometry

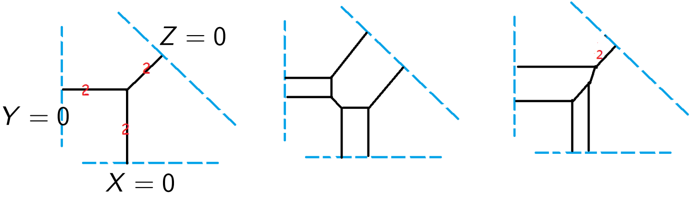
- The image of $X + Y + 1 = 0$ under $Log \circ H_t$



converges (in the sense of Gromov-Hausdorff) to a **tropical curve**.

Tropical Geometry

- Degree 2 tropical rational curves



Tropical Geometry

Observation:

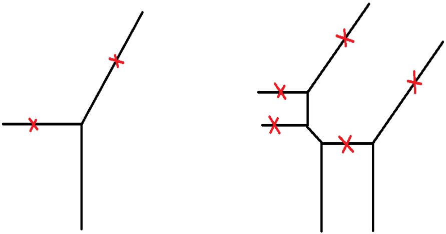
- degree of rational curve $=$ $\#$ of unbounded edges toward a fixed direction
- (**balancing condition**) At each vertex v,
  v_i: primitive integral vector tangent to the edge adjacent to v.

$$\sum_i w_i v_i = 0$$

Tropical Geometry

The tropical curves reflect the geometry.

- Generic two points determine a tropical line.
- Generic 5 points determine a tropical conic.



Toric Correspondence Theorem

Theorem (Mikhalkin '03)

For toric surfaces,
$\#$ of holomorphic curves $=$ $\#$ of trivalent tropical curves counted with weights.

- For a vertex v, define $w_v = w_1 w_2 |v_1 \wedge v_2|$.
- **Mikhalkin's weight** of a tropical curve $= \prod w_v$.

Theorem (Nishinou-Seibert '04)

The correspondence theorem holds for all toric manifolds.

Calabi-Yau Manifolds and Special Lagrangians

- A complex manifold X is **Calabi-Yau *n*-fold** if there exists a Kähler form ω, a holomorphic volume form Ω such that $\omega^n = c\Omega \wedge \bar{\Omega}$.
- Examples:
  1. (Yau) $X =$ compact Kähler w/ trivial canonical bundle.
  2. (Tian-Yau) $X = Y \setminus D$, w/ Y Fano and $D \in |-K_Y|$ smooth.

- A n-dim'l submanifold L is **Lagrangian** if $\omega|_L = 0$, **special Lagrangian** if $\Omega|_L = e^{i\theta} vol_L$.
- Examples:
  1. "Straight lines" in elliptic curves.
  2. hyperKähler rotation of holo. curves on HK surfaces.

SYZ Conjecture and Examples

Conjecture (Strominger-Yau-Zaslow '96)

A Calabi-Yau manifold admits a special Lagrangian torus fibration.

Examples:

- Elliptic curve to S^1.
- HyperKähler rotation of elliptic K3 surface.
- (Collins-Jacob-L '19) $X = Y \setminus D$, where Y compact surface w/ $-K_Y$ ample and $D \in |-K_Y|$ smooth.

Complex Affine Structure

B_0 the complement of discriminant locus of SYZ base.

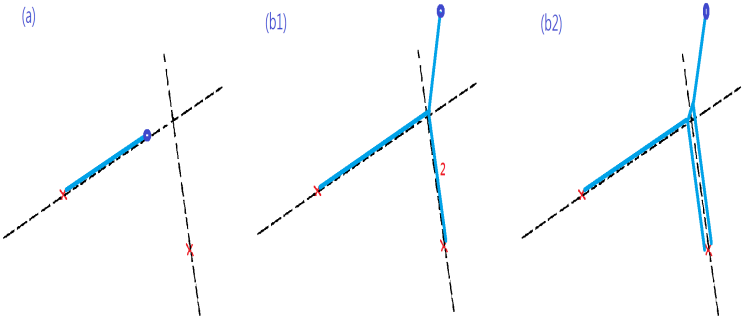
- (Hitchin) $\exists$ integral affine structure on B_0 with coordinates $f_i(u) = \int_{\gamma_i} \mathrm{Im}\Omega$, $\partial\gamma_i \in H_1(L;\mathbb{Z})$ basis.

Lemma (Leung)

L_{u_t} family of SLAG fibres bounding holo. discs of class γ, then u_t sit over an affine line l_γ.

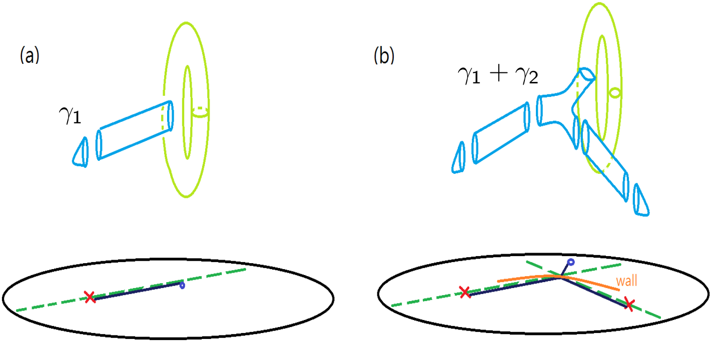
$\int_{\gamma_{u_t}} \mathrm{Im}\Omega \Leftrightarrow f_\gamma(u_t) = 0.$

Examples of tropical discs on HK surfaces



Lifting of Tropical Discs

- Every tropical disc has an associate relative class.



Tropical Counting Invariant

Definition

- *Given a valency ≤ 3 tropical disc (ϕ, T, w), the weight*

$$Mult(\phi) := \prod_{v \in G^{[0]} \setminus \{u\} : \phi(v) \in B_0} Mult_v(\phi) \prod_{v \in G^{[0]} : val(v) = 1} \frac{(-1)^{w_v - 1}}{w_v^2} \prod_{\bar{v} \in \bar{G}^{[0]} \setminus \{u\} : \bar{v} \in B_0} |Aut(\mathbf{w}_{\bar{v}})|.$$

- $\tilde{\Omega}^{trop}(\gamma; u) = \sum_{\phi} Mult(\phi)$.

The red part changes for singular fibres other than type I_1.

Comparison of New and Old Tropical Geometry

Toric surfaces	HyperKähler surfaces
Log map $\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}$ $t \to \infty$	SYZ fibration integral affine structure Large complex structure limit
adiabatic limit of projection of the holo. curves	Locus of SLAG fibres bounding holo. discs
edges going to $\infty \Rightarrow$ curve class	trivalent vertex $\rightsquigarrow$ pair-of-pant edge $\rightsquigarrow$ cylinder 1-vertex $\rightsquigarrow$ cap
Mikhalkin weight	trivalent vertex $\rightsquigarrow$ Mikhalkin weight 1-valent vertex $\rightsquigarrow \frac{(-1)^{d-1}}{d^2}$/" Aut"

Lagrangian Floer Theory

- (Fukaya-Oh-Ohta-Ono) A_∞ **structure** on $H^*(L_u, \Lambda)$:

$$m_{k,\gamma} : H^*(L_u, \Lambda)^{\otimes k} \to H^*(L_u, \Lambda)$$

  satisfying the A_∞ relations.

- (Maurer-Cartan equation) $b \in H^1(L_u, \Lambda_+)$

$$m(e^b) := m_0 + m_1(b) + m_2(b, b) + \cdots = 0.$$

- (**Maurer-Cartan space**)

$$\mathcal{MC}(L_u) := \{b \in H^1(L_u) | m(e^b) = 0\}/ \sim = H^1(L_u, \Lambda_+).$$

Pseudo-Isotopy and Wall-Crossing

- (**pseudo-isotopy**, Fukaya '09) ϕ path between u and u'

$$H^1(L_u, \Lambda_+) = \mathcal{MC}(L_u) \stackrel{F_\phi}{\to} \mathcal{MC}(L_{u'}) = H^1(L_{u'}, \Lambda_+).$$

- F_ϕ records $MI = 0$ discs w/ boundaries on $L_{\phi(t)}$.
- F_ϕ **is a homotopy invariant.**
- Choose a basis $e_1, e_2 \in H_1(L_u, \mathbb{Z})$ and write $b = x_1 e_1 + x_2 e_2$, $F_\phi(b) = F_\phi(b)_1 e_1 + F_\phi(b)_2 e_2$.

$$\begin{aligned} \rightsquigarrow \Lambda[[H_1(L_u, \mathbb{Z})]] &\to \Lambda[[H_1(L_u, \mathbb{Z})]] \\ z_i = \exp x_i &\mapsto \exp F_\phi(b)_i \end{aligned}$$

KS Transformations and Open GW

Write $z^{\partial\gamma} = z_1^{\langle \partial\gamma, e_1 \rangle} z_2^{\langle \partial\gamma, e_2 \rangle}$

Theorem (L '17)

The path ϕ passing through l_γ, then

$$\mathcal{K}_\gamma := F_\phi : z^{\partial\gamma'} \mapsto z^{\partial\gamma'} f_\gamma^{\langle \gamma', \gamma \rangle},$$

where $f_\gamma \in 1 + \mathbb{R}[[z^{\partial\gamma} T^{\omega(\gamma)}]]$.

Define the **open Gromov-Witten invariants** $\tilde{\Omega}(\gamma; u)$

$$\log f_\gamma(u) = \sum_{d \geq 1} d\tilde{\Omega}(d\gamma; u) T^{d\omega(\gamma)},$$

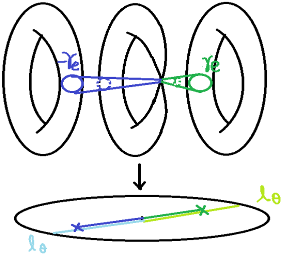
if ϕ passes through l_γ at u.

Near an I_1 Singular Fibre

Theorem (L. '13)

Let $u \in B_0$ be closed to a singularity. Let γ_e be the Lefschetz thimble and $d \in \mathbb{Z}$, then

$$\tilde{\Omega}(d\gamma_e, u) = \frac{(-1)^{d-1}}{d^2}.$$



Weak Correspondence Theorem

Theorem (L.'17)

If $\tilde{\Omega}(\gamma; u) \neq 0$*, then there exists a tropical disc represent* γ*.*

- If $\tilde{\Omega}(\gamma; u)$ constant along l_γ, then γ is a parallel transport of (multiple cover of) a Lefschetz thimble.
- Otherwise, $\gamma = \sum_i \gamma_i$ with $\tilde{\Omega}(\gamma_i; u_1) \neq 0$ for some $u_1 \in l_\gamma$.
- Finite procedure due to Gromov compactness theorem and reduce to the first case.

A Different Point of View

Given the input as elliptic fibration $X \to$ via HK rotation.

- $\exists S^1$-family $X_\vartheta \to B$ SLag fibrations.
- **central charge** $Z_\gamma(u) = \int_\gamma \Omega$, $\gamma \in H_2(X, L_u)$.
- **wall of marginal stability** divided B_0 into chambers.

$$\mathrm{Arg} Z_{\gamma_1} = \mathrm{Arg} Z_{\gamma_2}$$

Theorem (L '13)

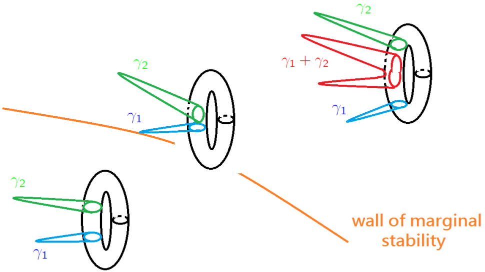
1. $\tilde{\Omega}(\gamma; u)$ *is locally constant.*
2. *(reality condition)* $\tilde{\Omega}(-\gamma; u) = \tilde{\Omega}(\gamma; u)$.

Wall-Crossing Phenomenon

Theorem (L. '13)

- *An explicit example that* $\tilde{\Omega}(\gamma; u)$ *jumps when* u *varies.*
- *If* γ_1, γ_2 *primitive, then the jump is*

$$\Delta\tilde{\Omega}(\gamma_1 + \gamma_2) = |\langle \partial\gamma_1, \partial\gamma_2 \rangle| \tilde{\Omega}(\gamma_1)\tilde{\Omega}(\gamma_2).$$



Tropical/Holomorphic Correspondence

Theorem (L.'17)

The open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$ satisfies the Kontsevich-Soibelman wall-crossing formula as u varies.

This is the mathematical realization of the proof of KS WCF of Cecotti-Vafa '09.

Theorem (L. '17)

Open Gromov-Witten invariants= Tropical discs counting

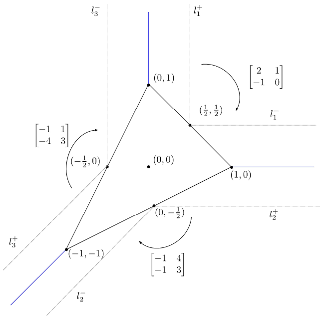
$$\tilde{\Omega}(\gamma; u) = \tilde{\Omega}^{trop}(\gamma; u).$$

1. Same initial data
2. Satisfy the same wall-crossing formula

Towards the Equivalence of AG/SG Mirrors

Theorem (Bouseau '19, Lau-Lee-L., '20)

The complex affine structure of the SYZ fibration of $\mathbb{P}^2 \backslash E$ coincides with the one of Carl-Pomperla-Siebert.

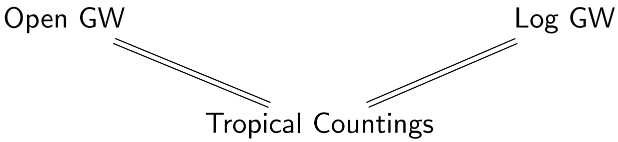


Equivalence of Open GW/Log GW

Together with the result of Gräfnitz '20

Theorem (L. '20)

The open Gromov-Witten invariants of $\mathbb{P}^2$ coincide with log Gromov-Witten invariants w/ maximal tangency.



Advantage: bypassing the comparison of VFCs.

THANK YOU!