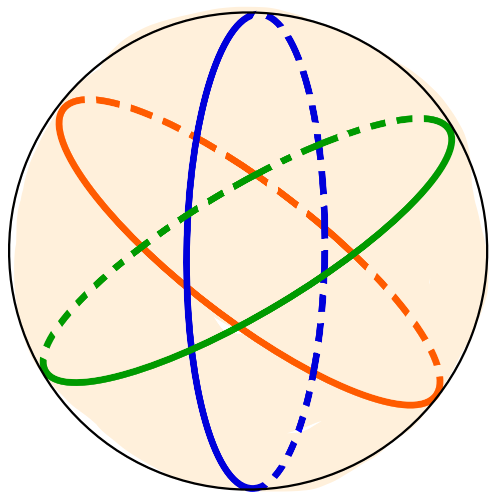
WHAT DOES A BESSE CONTACT SPHERE LOOK LIKE ?

Joint work with Marco Radeschi (Univ Notre Dame)

Arthur Besse "Manifolds all of whose geodesics are closed", 1978



- Closed contact manifold (Y^{2n-1}, λ)

$$\lambda \wedge (d\lambda)^{n-1} \text{ volume form on } Y$$

- Reeb vector field R on Y

$$\lambda(R) \equiv 1, \quad d\lambda(R, \cdot) \equiv 0$$

- Reeb flow ϕ^t $Y \circlearrowleft$

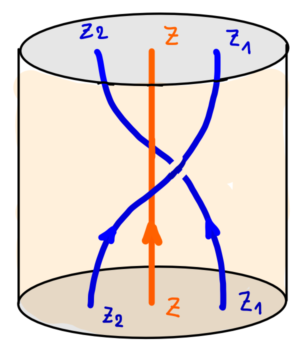
$$\frac{d}{dt}\phi^t = R \circ \phi^t, \quad \phi^0 = id$$

(Y, λ) is **BESSE** when it is connected and every Reeb orbit is closed

Wadsley thm The Reeb flow of a Besse contact manifold is **periodic**

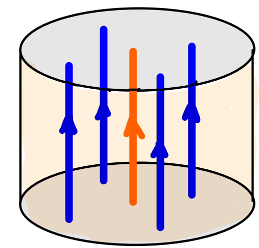
i.e. $\phi^{\tau} = id$ for some $T > 0$

⚠ Not all Reeb orbits must have the same minimal period



(Y, α) is **ZOLL** when every Reeb orbit is closed and has the same minimal period $\tau > 0$

$$\phi^\tau = id, \quad fix(\phi^t) = \emptyset \quad \forall t \in (0, \tau)$$



Rmq A Besse Reeb flow defines a locally free S^1 action on the contact manifold; free in the Zoll case

$$t\, z = \phi^t(z) \qquad \forall t \in \underset{\mathbb{R}/\tau\mathbb{Z}}{\underset{\shortparallel}{S^1}}, \quad z \in Y$$

example Rational ellipsoids

$$E(a_1, .., a_m) = \left\{ z \in \mathbb{C}^m \;\middle|\; \frac{|z_1|^2}{a_1} + . + \frac{|z_m|^2}{a_m} = \frac{1}{\pi} \right\}$$

$\frac{a_j}{a_k} \in \mathbb{Q}$ $\qquad a_j > 0$

$$\lambda_{std} = \frac{1}{4} \sum_{j=1}^{m} \left(z_j \, d\bar{z}_j - \bar{z}_j \, dz_j \right)$$

$$\phi^t(z_1, , z_m) = \left(e^{i2\pi a_1 t} z_1, .., e^{i2\pi a_m t} z_m \right)$$

$\phi^\tau = id$ $\qquad$ with $\quad \tau = lcm(a_1, .., a_m)$

Q Are there Besse spheres (S^{2n-1}, λ) other than the ellipsoids ?

$$\left(\begin{array}{l} \text{i.e. } (S^{2n-1}, \lambda) \text{ Besse such that} \\ \nexists\ \psi : S^{2n-1} \xrightarrow{\cong} E(a_1, \dots, a_n) \text{ with } \psi^* \lambda_{std} = \lambda \end{array} \right)$$

DIMENSION 3

Simple spectrum $\sigma_s(Y^3, \lambda) = \left\{ r > 0 \;\middle|\; \mathrm{fix}(\phi^r) \setminus \bigcup_{k \geq 2} \mathrm{fix}(\phi^{r/k}) \neq \emptyset \right\}$

integer

(set of minimal periods of the closed Reeb orbits)

Thm (Cristofaro Gardiner, Mazzucchelli)

If (Y^3, λ_1) is Besse and $\sigma_s(Y, \lambda_1) = \sigma_s(Y, \lambda_2)$

then $(Y, \lambda_1) \cong (Y, \lambda_2)$

strictly contactomorphic

Proof

- Spectral characterization of Besse contact forms (in dimension 3)

$$\sigma(Y^3, \lambda) = \{ t > 0 \mid \mathrm{fix}(\phi^t) \neq \emptyset \}$$ action spectrum

$$\mathrm{rank}(\sigma(Y^3, \lambda)) = 1 \quad \text{iff} \quad \lambda \text{ Besse}$$

$\Rightarrow$ if λ_1 Besse and $\sigma(Y, \lambda_1) = \sigma(Y, \lambda_2)$ then λ_2 Besse as well

- If (Y^3, λ) is Besse, then the quotient projection

$$Y \longrightarrow B = \underbrace{Y/S^1}_{\substack{\text{quotient by the} \\ S^1\text{-action of the} \\ \text{Reeb flow}}}$$

is a Seifert fibration

- If λ_1 Besse and $\sigma_s(Y, \lambda_1) = \sigma_s(Y, \lambda_2)$ then λ_1 and λ_2 define isomorphic Seifert fibrations

$$\exists\, \psi : Y \circlearrowleft \cong \quad \text{such that} \quad \psi_* R_{\lambda_1} = R_{\lambda_2}$$

(This requires the classification of Seifert fibrations, recently completed by Geiges-Lange)

- In dimension 3, $\psi_* R_{\lambda_1} = R_{\lambda_2}$ implies

$$\psi \underset{\text{isotopy}}{\simeq} \tilde{\psi} \quad \text{st} \quad \tilde{\psi}^* \lambda_2 = \lambda_1$$

$\square$

Corollary Every Besse (S^3, λ) is strictly contactomorphic to a rational ellipsoid

- In dimension $2n-1 \geq 5$ it is not even known whether there exist exotic Zoll (S^{2n-1}, λ)

i.e. $\underset{\text{Zoll}}{(S^{2n-1}, \lambda)} \not\cong (E(T, \dots, T), \lambda_{std})$

- $(E(T, \quad ,T)/S^1, d\lambda_{std}) = (\mathbb{C}P^{n-1}, \omega_{std})$

Does $\mathbb{C}P^{n-1}$ admit exotic symplectic structures?

No for $\mathbb{C}P^1$ (Moser) and $\mathbb{C}P^2$ (Taubes)

STRATIFICATION OF A BESSE CONTACT MFD

(Y, λ) Besse, $\quad \phi^T = id$, $\quad T$ = minimal common Reeb period

$\circlearrowleft \phi^t$

$k \in \mathbb{N}$

$$\boxed{Y_k = fix\left(\phi^{T/k}\right)}$$

every connected component is a contact submanifold of (Y, λ)

<u>**example**</u>

$E = E(a_1, .., a_n)$ rational ellipsoid

$T = lcm(a_1, .., a_n)$

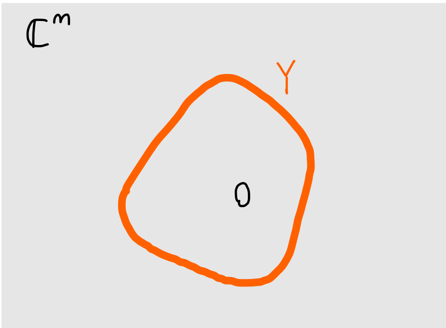
$$\boxed{E_k} = \left\{ z \in E \mid z_j = 0 \text{ if } \frac{T}{k a_j} \notin \mathbb{N} \right\}$$ sub-ellipsoid

Q If $Y = S^{2n-1}$, are the Y_k's spheres as well ?

- Yes for S^3
- In general dimension, Smith theory implies $H_*(Y_k, \mathbb{Z}_p) \cong H_*(S^d; \mathbb{Z}_p)$

$\forall k = p^m$, p prime

CONVEX CONTACT SPHERES



Y convex hypersurface in $\mathbb{C}^n$ enclosing the origin

$$\lambda_{std} = \frac{1}{4} \sum_{j=1}^{n} \left(z_j \, d\bar{z}_j - \bar{z}_j \, dz_j \right)$$

Ekeland-Hofer capacities

$$c_1(Y) \leq c_2(Y) \leq c_3(Y) \leq \ldots$$

$$c_1(Y) \in \sigma(Y, \lambda_{std})$$

example $E = E(a_1, \ldots, a_n)$

$$\sigma(E, \lambda_{std}) = \{ m\, a_j \mid j = 1, \ldots, n, \quad m \in \mathbb{N} \}$$
$$= \{ \sigma_1, \sigma_2, \sigma_3, \ldots \} \qquad \sigma_i < \sigma_{i+1}$$

The sequence of Ekeland-Hofer capacities is

$$\underbrace{\overset{\overset{c_1(E)}{\shortparallel}}{\sigma_1}, \overset{\overset{c_2(E)}{\shortparallel}}{\sigma_1}, \ldots, \sigma_1}_{d_1}, \underbrace{\sigma_2, \ldots, \sigma_2}_{d_2}, \underbrace{\sigma_3, \ldots, \sigma_3}_{d_3}, \ldots$$

where $2d_i - 1 = \dim(\mathrm{fix}(\phi^{\sigma_i}))$

Thm (Mazzucchelli - Radeschi)

Every Besse convex contact spere (Y, λ) satisfies

1) Every stratum Y_K is an integral homology sphere

$$H_*(Y_K; \mathbb{Z}) \cong H_*(S^d; \mathbb{Z})$$

2) If we denote $\sigma(Y) = \{\sigma_1, \sigma_2, \sigma_3, .\}$, $\sigma_i < \sigma_{i+1}$, then the sequence of Ekeland-Hofer capacities is

$$\underbrace{\sigma_1, . . , \sigma_1}_{d_1}, \underbrace{\sigma_2, . . , \sigma_2}_{d_2}, ..$$

$$2d_i - 1 = \dim fix(\phi^{\sigma_i})$$

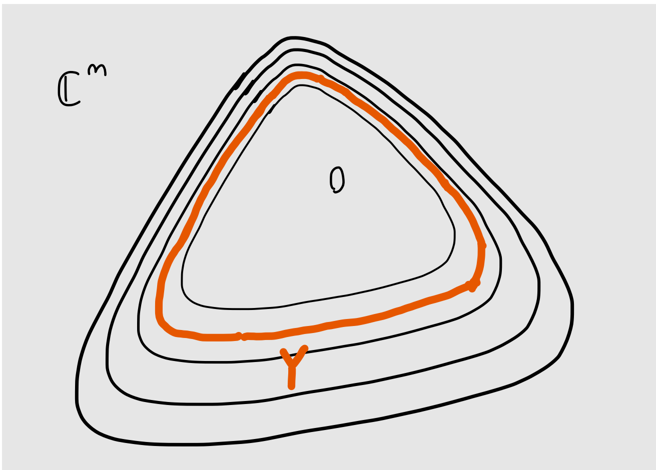
INGREDIENTS OF THE PROOF

Techniques are inspired by Radeschi-Wilking's proof on the Berger conjecture (any Besse Riemannian S^n, $n \geq 4$ is Zoll)

- S^1-equivariant Morse theory
- Morse index formulas

* Torsion of integral S^1-equivariant cohomology of spaces equipped with a non-free S^1 action

- Clarke action functional



$h \; \mathbb{C}^n \to [0,\infty)$

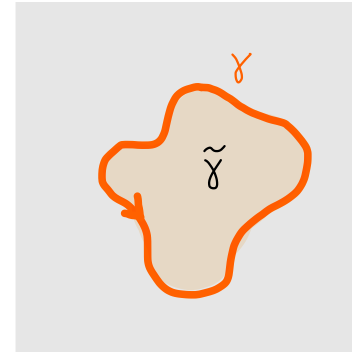
$h|_Y \equiv 1$

$h(\lambda z) = \lambda^2 h(z) \quad \forall \lambda > 0, \; z \in \mathbb{C}^n$

$h^* \; \mathbb{C}^n \to [0,\infty)$ dual to h

$$h^*(w) = \max_z \left(\langle w, z \rangle - h(z) \right)$$

$$\gamma \quad \mathbb{R}/\mathbb{Z} \xrightarrow{W^{1,2}} \mathbb{C}^n$$



$$A(\gamma) = \frac{1}{2}\int_0^1 \langle i\dot\gamma, \gamma \rangle \, dt = \int_{\tilde{\gamma}} d\lambda_{std}$$

$$\mathcal{H}(\gamma) = \int_0^1 h^*(-i\dot\gamma)\, dt$$

Clarke action functional

$$\Psi \quad \Lambda \to (0,\infty), \qquad \Psi(\gamma) = \frac{1}{A(\gamma)}$$

$$\Lambda = A^{-1}(0,\infty) \cap \mathcal{H}^{-1}(1)$$

Variational principle (Clarke)

$\gamma \in \mathrm{Crit}(\Psi)$, $\Psi(\gamma) = c$ iff $t \mapsto c\,\gamma(t/c)$ is a c-periodic Reeb orbit of (Y, λ_{std})

- $\Psi : \Lambda \to (0,\infty)$, $\Lambda \circlearrowleft S^1$, is S^1-invariant (for the S^1 action $t \cdot \dot\gamma = \dot\gamma(t + \cdot)$)

- Λ is S^1-equivariantly homotopy equivalent to the unit sphere of L^2

$$\Rightarrow \quad H^*_{S^1}(\Lambda, R) \cong R[e], \quad e \text{ generator of } H^2_{S^1}(\Lambda, R)$$

PROPERTIES OF Ψ IN THE BESSE CASE

Assume (Y, λ_{std}) Besse, T = min common Reeb period

Then

- $\mathrm{Crit}(\Psi) \cap \Psi^{-1}\left(\frac{T}{K} + mT\right) \underset{S^1\text{-equiv.}}{\cong} Y_K$ $= \mathrm{fix}(\Phi^{T/K})$

- Ψ Morse-Bott (every critical mfd is non-degenerate)

- Morse indices are all even

  (linearized Poincaré map of any periodic orbit is a root of the identity)

* Every critical mfd is homologically visible

$\left(K \subset \mathrm{Crit}(\Psi), \quad \text{negative bundle } \begin{array}{c} E^- \\ \downarrow \\ K \end{array} \text{ is orientable} \right)$

$\Rightarrow$ $\forall$ K connected component of $\mathrm{Crit}(\Psi) \cap \Psi^{-1}(c)$, we have

$$H^*_{S^1}\left(\{\Psi < c+\varepsilon\}, \{\Psi < c-\varepsilon\}\right) \supset H^{*-i}_{S^1}(K)$$

$i = \mathrm{ind}(K)$
Morse index

* $\forall$ critical mfd $K \subset \mathrm{Crit}(\Psi)$

$$H^{odd}_{S^1}(K, \mathbb{Q}) = 0$$

(if $H^{2d+1}_{S^1}(K; \mathbb{Q}) \neq 0$

$\Rightarrow$ $H^{r}_{S^1}(K^p, \mathbb{Z})$ has p-torsion $\forall$ large prime p, large odd r

$\Rightarrow$ $H^{r}_{S^1}(\Lambda; \mathbb{Z}) \neq 0$ for some large odd r

but $H^{*}_{S^1}(\Lambda, \mathbb{Z}) \cong \mathbb{Z}[e]$ $\lightning$)

This shows

Thm If (Y, λ_{std}) Besse, then Ψ is perfect for S^1-equivariant Morse theory with $\mathbb{Q}$ coefficients

i.e. with $\mathbb{Q}$ coefficients, we have short exact sequences

$$0 \to H^*_{S^1}(\{\Psi < b\}, \{\Psi < a\}) \to H^*_{S^1}(\{\Psi < b\}) \to H^*_{S^1}(\{\Psi < a\}) \to 0$$

$\forall\, a < b$

- $H^*_{S^1}(\Lambda, \mathbb{Q}) = \langle \underset{\overset{\shortparallel}{1}}{e^0}, e, e^2, e^3, e^4, \dots \rangle$

Ekeland-Hofer spectral invariants

$$s_i = \inf \{ c \in \mathbb{R} \mid e^{i-1} \in \operatorname{Ker} \left(H^*_{S^1}(\Lambda) \to H^*_{S^1}(\{\Psi \leq c\}) \right) \}$$

$$(i = 1, 2, 3, \dots)$$

$s_1 \leq s_2 \leq s_3 \leq$ are critical values of Ψ

Putting everything together

cohomology classes	$e^0 = 1$	e	e^2	e^3	e^4	
spectral values	s_1	s_2	s_3	s_4	s_5	$\cdot$
carrier critical mfd	K_1	K_1	K_1	K_2	K_2	
	$\underbrace{\qquad\qquad}_{d_1}$			$\underbrace{\qquad\qquad}_{d_2}$		

$\Rightarrow$ S^1-equiv cohomology $\quad H^*_{S^1}(K_i, \mathbb{Q}) = \langle 1, e, \dots, e^{d_i - 1} \rangle$

$\Rightarrow$ By the classical Gysin sequence.

$$H^*(K_i, \mathbb{Q}) \cong H^*(S^{2d_i - 1}, \mathbb{Q})$$

With more work (using that Λ is contractible), we obtain this isomorphism with $\mathbb{Z}$ coefficients as well

$\Rightarrow$ point 1) and 2) for Ekeland-Hofer spectral values

- Ekeland-Hofer spectral invariants $s_i(Y)$ vs Ekeland-Hofer capacities $c_i(Y)$

Prop $\forall$ convex contact sphere Y

- (Sikorav) $s_1(Y) = c_1(Y)$
- (Baracco-Bernardi-Mazzucchelli) $s_i(Y) \geq c_i(Y) \quad \forall i \geq 2$

$\Rightarrow \quad s_i(Y) = c_i(Y) \quad \forall i \geq 1 \quad$ if Y Besse $\square$

Thank you for attending the talk!

(and see you hopefully soon in Lyon, Lisbon, or anywhere else)