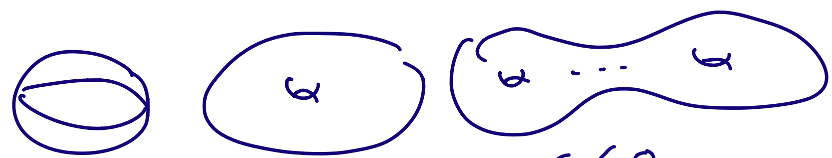
Geometria em Lisboa, March 16 2021

Fano & Hyper-Kähler manifolds

jt work w/ L. Flapan, E. Macrì, K. O'Grady

(classification of $\mathbb{C}$ proj varieties "positivity" properties of c_1 ($\longleftrightarrow$ negativity properties of canonical class)

dim 1



$c_1 > 0$ $\quad$ $c_1 = 0$ $\quad$ $c_1 < 0$

In higher dim, not all case are "definite".

		Fano	Calabi-Yau	General type
$\{F_d = 0\} \subseteq \mathbb{P}^n$		$d \le n$	$d = n+1$	$d \geq n+2$
hol forms		No non trivial hol forms	up to a finite étale, $\exists$ nowhere van hol form of top degree	$\dim H^0(X, K_X^{\otimes m}) \sim m^{\dim X}$
		only finitely many in each dim	Beauville-Bogomolov	$\exists$ natural deformation. ....

Thm [BB] Y cmpt Kähler mfld $c_1(Y) = 0$ up to a finite étale cover

strict CY mflds

$$T \times \prod M_i \times \prod N_i \longrightarrow Y$$

$\mathbb{C}$ torus $\quad$ irr hol sympl. mfld.

M cmpt Kahler is called <u>IHS</u>

$\begin{cases} \pi_1(M) = 1 \\ H^0(\Omega^2_M) = \mathbb{C}\,\sigma \end{cases}$ hol sympl. form

$\to$ hyper-Kähler mflds (IHS = HK)

N cmpt Kahler is called strict CY

$\begin{cases} \pi_1(N) = 1 \\ H^0(\Omega^p_N) = \begin{cases} 0 & \text{—} \\ \mathbb{C} & p = 0, \dim \end{cases} \end{cases}$

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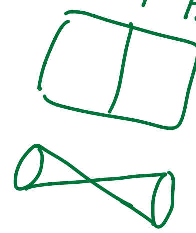
dim 2   K3 surfaces S

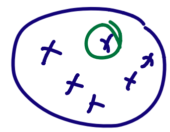
<u>Eg</u> $\{F_4 = 0\} \subseteq \mathbb{P}^3$,

$T^*\mathbb{P}^1$



A = ab. surface

$$\begin{array}{c} S \quad K3 \\ \downarrow \text{ sympl. res.} \\ A \longrightarrow A/\pm 1 \end{array}$$



<u>In higher dim?</u>

Most geom constructions starting from K3 give rise to IHS.

<u>Eg</u> Hilbert scheme of points on a K3, <u>$S^{[n]}$</u>

Beauville, Fujiki: $S^{[n]}$ are IHS of dim $2n$

& their deformations

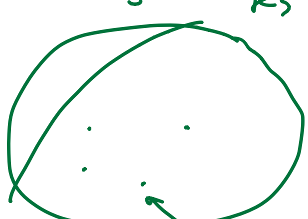
def. class $K3^{[n]}$-type

<u>More generally</u>: Fix chern classes $\underline{v} \in H^*(S,\mathbb{Z})$

$H \in \mathrm{Amp}(S)$ $M_{v,H}(S)$ when semistable locus = stable locus

$S^{[n]}$ proj K3 $\Rightarrow$ Mukai is IHS $\sim S^{[n]}$

Yoshiotas
O'Grady
Huybrechts



21-dimensl

$2n \geq 4$

$M_{v,H}(S)$ have codim ≥ 2

Rmk Another series of examples in dim $2n \geq 4$ come from A = ab. surfaces

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Two exceptional examples

dim 6	dim 10
OG6	OG10
A	S

$\longrightarrow I_{z_1+z_2}$ $\quad v(I_{z_1+z_2}) = v_0$

$\widetilde{M}_{2v_0}(S) \xrightarrow[\text{sympl. res.}]{\exists} M_{2v_0,H}(S)$

[O'G]

$M_{2v_0,H}(S) \supseteq \mathrm{Sym}^2 M_{v_0}(S) \ni I_{z_1+z_2} \oplus I_{w_1+w_2}$

$\widetilde{M}_{2v_0}(S)$ is an IHS of dim 10

$b_2(\widetilde{M}) > b_2(S^{[n]})$

HK mflds & cubic 4folds

$X \subseteq \mathbb{P}^5$ sm cubic 4fold, Fano

Center of rationality problems

From $X \rightsquigarrow HK(X)$ IHS

1) Beauville-Donagi '80s

"the Fano variety of lines in X"

$F(X) \subseteq Gr(2,6)$ $\quad F(X)$ sm, 4dim

$F(X) = \{\ell \subseteq X\}$

IHS $\quad F(X) \underset{\text{def.}}{\sim} S^{[2]}$

2) Lehn-Lehn-Sorger-van Straten 2015-6

$Z(X) = \left\{ \begin{array}{c} C \subseteq X \\ \text{twisted cubics} \end{array} \right\} / \sim$

$\mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$[1,t] \longmapsto [1,t,t^2,t^3]$

smooth IHS 8fold, $Z(X) \sim S^{[4]}$

$\mathrm{Hilb}_{3t+1}(X) \supseteq \mathcal{H}$
$\quad\downarrow \mathbb{P}^2$
$\quad Z'(X)$

$C \subseteq X \qquad \langle C \rangle = \mathbb{P}^3$

$\langle C \rangle \cap X = \Sigma$

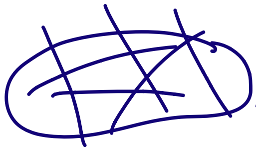
$|\mathcal{O}_\Sigma(C)| = \mathbb{P}^2$

3) Laza-S.-Voisin '16
S. '20 $\quad J(X)$ intermediate Jacobian fibration
$\quad\downarrow$
$\quad\mathbb{P}^5$ $\quad$ IHS 10-dim
$\sim$ OG10

4) Kuznetsov, Bayer-Lahoz-Macrì-Stellari
Perry Neur

moduli spaces of obj in $Ku(X) \subseteq D^b(X)$
recover (1), (2) & many more

(but for v.g. X not recover 3))

(0) Hassett:  $\mathcal{C} \supseteq \mathcal{C}_{K3} = \{X$ w/ an associated K3 surface$\}$
20-dim
$X \longleftrightarrow S$
$H^4(X) \longleftrightarrow H^2(S)$

Why? $\quad H^4(X) = \cancel{H^{4,0}} \oplus H^{3,1} \oplus H^{2,2} \oplus \dots$
$\quad 1 \quad 22 \quad 1 \quad 0$

$H^2(IHS) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$
$\quad 1 \quad b_2 - 2 \quad 1$

Where does the hol sympl. form on $F(X)$, $Z(X)$, $J(X)$ come from?

Ex: $\Gamma \subseteq X \times M$ fam of curves on X par. by M (sm proj)

$M \xleftarrow{p} \Gamma \xrightarrow{q} X$

$p^* = p_* q^* : H^4(X) \longrightarrow H^2(M)$

$$H^{3,1}(X) \longmapsto H^{2,0}(M)$$
$$\sigma_M \longmapsto \text{hol } 2\text{-form.}$$

Kuznetsov Debarre Perry

Rmk $\exists$ other Fano manifolds w/ a cohomology gp of K3-type

eg Gushel-Mukai manifolds of dim 4, 6
Debarre-Voisin varieties $\longrightarrow$ IHS 4 $S^{[2]}$
....

$\longrightarrow$ IHS

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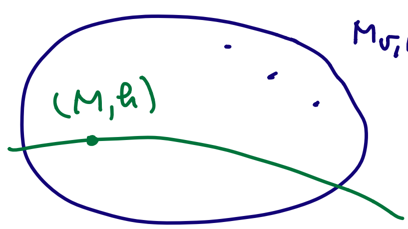
these constructions give codim 1 families



$M_{v,H}(S)$ codim ≥ 2

$\mathcal{C}$ 20-dim

IHS $\sim S^{[n]}$ 21-dim

codim 1 families
Noether-Lefschetz loci

$S^{[2]}$ $(M \simeq F(X))$ $Z(X)$

$F(X) \overset{i}{\subseteq} Gr(2,6)$
i^*Plucker $=: h$

$(H^2(S,\mathbb{Z}), \cup)$

$(H^2(IHS,\mathbb{Z}), q)$

$$\boxed{\begin{array}{ll} Z(X) \overset{\varphi}{\dashrightarrow} Gr(4,6) & \varphi^*\text{Plucker} =: d \\ \quad\; \Psi \\ [C] \longmapsto \langle C\rangle = \mathbb{P}^3 & q(d) = 2 \end{array}}$$

Look at codim 1 families (M,d), $d \in H^2(M,\mathbb{Z})$
when does this family come from a Fano?

L. Flapan - E. Macrì - K. O'Grady - G.S.

$\boxed{\text{First case } q(d) = 2}$

Eg $(Z(X), \delta)$ $q(\delta) = 2$
IHS
$\mathrm{div}(\delta) = 2$

can you recover X from $F(X)$?

$\tau : Z(X) \to Z(X)$ LLSvS

$\exists$ anti-sympl. involution

$\tau^* \sigma = -\sigma$

$F_1 \sqcup F_2 = \mathrm{Fix}(\tau) \subseteq Z(X)$ Lagrangian subvar.

and $F_1 \simeq X$

$C \mapsto \tau(C) = C'$

$\langle C \rangle = \mathbb{P}^3$

$C \subseteq \Sigma = X \cap \langle C \rangle \subseteq \mathbb{P}^3$

$C \subseteq Q$ $\quad Q \cap \Sigma = C \cup C'$

Eg (S, δ) $\mathrm{div}(\delta) = 1$

$\delta^2 = 2$

$S \xrightarrow{2:1} \mathbb{P}^2$, τ on S, $\Gamma \subseteq S$, $\Gamma \subseteq \mathbb{P}^2$ sextic curve $\mathrm{Fix}(\tau) = \Gamma$ general type

the difference is the DIVISIBILITY of δ

$q(\delta) = 2$ $\to$ $\mathrm{div}(\delta) = 1$ or $\mathrm{div}(\delta) = 2$

$x \in \Lambda$ $\quad x^2$

$(x, \Lambda) \subseteq \mathbb{Z}$
$= (\mathrm{div}(x))$

Thm [LMO'GS] Let (M, δ) be a polarized IHS $\sim S^{[n]}$

$q(\delta) = 2$

Then $\exists$ anti-sympl. involution $\tau : M \to M$

s.t. $\operatorname{Fix}(\tau)$ has $\operatorname{div}(d)$ connected components

$\nearrow$ conn $\quad \operatorname{div} = 1$

$\searrow$ 2 conn comp. $\quad \operatorname{div} = 2$

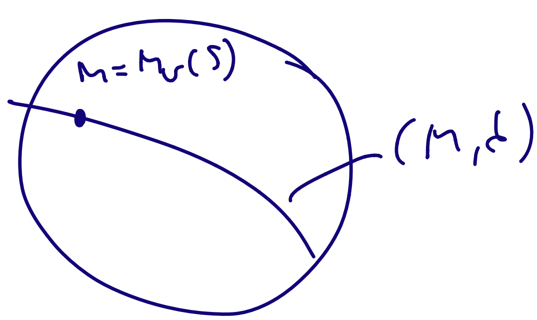
In $\operatorname{div}(d) = 2$ case one component is a Fano manifold of dim n & index 3.

The moduli space of (M, d) $\quad q(d) = 2$

$S^{[n]}$ $\quad \operatorname{div}(d) = \varepsilon$

is non empty $\forall n$ in $\varepsilon = 1$

$\iff 4 \mid n$ in $\varepsilon = 2$



$S \to \mathbb{P}^2$ $\quad \mathbb{P}_t \rightsquigarrow \mathbb{P}_0$

$S_t \circlearrowleft \tau_t \quad S_0 \leftarrow \tilde{S} \circlearrowleft \tilde{\tau}$