On the number of fixed pts of Periodic Flows

Gelis
9/3/2021

Mostly work with

S. Sabatini

or

S. Sabatini + A. Pelayo

Circle actions $\longrightarrow$ Periodic Flows

Fixed points $\longrightarrow$ Equilibrium points

Problem:

Find the minimal number of fixed pts of a circle action on a compact manifold M

Assume Fixed pt set $\to M^{S^1}$ discrete non empty

History:

1. Frankel (1959) – Kähler mflds

A Kähler S^1-action on a compact kähler mfld is Hamiltonian iff it has fixed points

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A Kähler S^1-action on a compact Kähler mfld is Hamiltonian iff it has fixed points

There is an S^1-invariant function $H : M \to \mathbb{R}$ s.t.

$$\iota_{X^\#}\, \omega = dH$$

$H \longrightarrow$ Hamiltonian Function

1. Frankel (1959) – Kähler mflds

A Kähler S^1-action on a compact kähler mfld is Hamiltonian iff it has fixed points

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Fixed pts of the action

$=$

Critical points of the Hamiltonian function

$\downarrow$

Perfect Morse Bott function

1. Frankel (1959) – Kähler mflds

A Kähler S^1-action on a compact Kähler mfld is Hamiltonian iff it has fixed points

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- Isolated fixed pts ⟹ Perfect Morse function
- Morse inequalities become equalities ⟹ $N_k = b_k$

$$\sum_{\substack{k=0 \\ \text{even}}}^{2n} N_k = \sum_{\substack{k=0 \\ \text{even}}}^{2n} b_k(M) \geq n+1$$

- N_k – # of critical pts of Morse index k

$\dim M = 2n$

- Critical pts have even indices
- $[\omega^k] \in H^{2k}(M,\mathbb{R})$ is nontrivial ⟹ $b_{2i} \geq 1$

History:

1. Frankel (1959) – Kähler mflds

A Kähler S^1-action on a compact kähler mfld is Hamiltonian iff it has fixed points

$\#$ Fixed pts $\geq n+1$

(Equality for $\mathbb{C}P^n$)

History:

2. Kosniowski (1979) – Unitary manifolds:

History:

Weakly almost complex

$\parallel\!\!\!\!\!\!\!\parallel$

2. Kosniowski (1979) – Unitary manifolds:

$$\underbrace{TM \oplus \mathbb{R}^{2k}}_{\text{Stable tangent bundle}} \text{ has a fixed complex stz. for some } k$$

Stable tangent bundle + S^1-action preserving the cpx structure

($k=0 \Rightarrow$ almost complex)

(S^1-symp $\Rightarrow S^1$-a.c. $\Rightarrow S^1$-unitary)

History:

2. Kosniowski (1979) - Unitary manifolds:

Conjecture:

- M^{2n} cpt unitary S^1-mfld
- isolated fixed points
- M does not bound equivariantly

M unitary
- cobordant
with the empty
set

$\equiv$ can be realized as the oriented bdy of a unitary oriented $2n+1$ - mfld with bdy s.t. the induced unitary str. on the bdy is isom. to the one on M

History:

2. Kosniowski (1979) – Unitary manifolds:

Conjecture:

- M^{2n} cpt unitary S^1-mfld
- isolated fixed points
- M does not bound equivariantly

$\Rightarrow$ # Fixed points $\geq \underbrace{f(n)}_{\text{linear}}$

$$\boxed{\text{Most likely } f(n) = \frac{n}{2}}$$

History:

2. Kosniowski (1979) – Unitary manifolds:

Conjecture:

- M^{2n} cpt unitary S^1-mfld
- isolated fixed points
- M does not bound equivariantly

$\Rightarrow$ $\#$ Fixed points $\geq \lfloor \frac{n}{2} \rfloor + 1$

History:

3. Hattori (1984) — Almost Complex Manifolds

- $c_j \in H^{2j}(M, \mathbb{Z})$ — Chern classes of TM
- Chern class map

$$c_1^{S^1}(M) : M^{S^1} \longrightarrow \mathbb{Z}$$

$$p \longmapsto c_1^{S^1}(M)(p) \in \mathbb{Z}$$

?

Sum of the weights of the S^1-isotropy representation on T_pM

- $p \in M^{S^1}$ (fixed pt)
- S^1 acts on $T_pM \cong \mathbb{C}^n$ (model for a nbhd of p)

$$\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{a_1} z_1, \ldots, \lambda^{a_n} z_n)$$

$a_1, \ldots, a_n \longrightarrow$ weights of the S^1-action at p

$\mathbin{\rotatebox[origin=c]{90}{$\in$}}$

$\mathbb{Z}$

History:

3. Hattori (1984) –

- M – Almost Complex Manifold +
- S^1-action preserving J
- $c_1^n(M) \neq 0$
- Injective Chern class map

$\Rightarrow$ # Fixed points $\geq n+1$

History:

4. Pelayo-Tolman — Symplectic circle actions

- M - Symplectic manifold +
- S^1-action preserving ω (Symp. circle action)
- Chern class map somewhere injective

there is a value that is attained at only 1 pt

History:

4. Pelayo-Tolman — Symplectic circle actions

- M - symplectic manifold +
- S^1-action preserving ω (symp. circle action)
- Chern class map somewhere injective

$\Rightarrow$ # Fixed points $\geq n+1$

History:

5. Ping Li – Kefeng Liu – Almost Complex mflds

- M^{2m} – Almost Complex Manifold +
- S^1-action preserving J
- $\beta_1, \dots, \beta_k \in \mathbb{Z}^+$ s.t. $\beta_1 + \dots + \beta_k = m$
- $(c_{\beta_1} \cdots c_{\beta_k})^l (M) \neq 0$

$\Rightarrow$ # Fixed points $\geq l + 1$

Rmk:

- $c_1^n \neq 0 \Rightarrow \geq n+1$
- $c_1 c_{n-1} \neq 0 \Rightarrow \geq 2$

History:

6. Cho-Kim-Park – Almost Complex mflds

- M^{2n} – Almost Complex Manifold +
- S^1-action preserving J
- $\beta_1, \dots, \beta_k \in \mathbb{Z}^+$ s.t. $\beta_1 + 2\beta_2 + \dots + n\beta_n = n$
- $(c_1^{\beta_1} \cdots c_n^{\beta_n})(M) \neq 0$

$\Rightarrow$ # Fixed points $\geq \max\{\beta_1, \dots, \beta_n\} + 1$

- the last three results use the Atiyah-Bott-Berline-Vergne localization formula and can be generalized to arbitrary S^1-mflds
- the last two use a nonzero Chern number

We Will use a differeut method but we will also retrieve ifomation from a Cheru number:

$$(c_1 c_{n-1})(M)$$

why $c_1 c_{n-1}$?

— Most importantly...

there is an expression for this Chern number in terms of the numbers of fixed pts with different indices.

Theorem (G. – Sabatini)

- M^{2n} – compact a.c. S^1-mfld
- Discrete fixed pt set
- N_k – # of fixed pts with k negative weights

$$(c_1 c_{n-1})(M) := \int_M c_1 c_{n-1} =$$

$$= \sum_{k=0}^{n} \left(6k(k-1) + \frac{5n-3n^2}{2} \right) N_k$$

Hirzebruch genus $\chi_y(M)$

$\rightarrow$ genus corresponding to the power series

$$Q_y(x) = \frac{x(1+y\,e^{-x(1+y)})}{1-e^{-x(1+y)}}$$

Ping Li (Rigidity) $\rightarrow$ $\chi_y(M) = \chi_y^{S^1}(M) = \sum_{j=0}^{n} N_j(-y)^j$

Salamon 1993 $\rightarrow$ $(c_1 c_{n-1})(M) = 6\left.\frac{d^2\chi_y(M)}{dy^2}\right|_{y=1} + \frac{5n-3n^2}{2}\chi_{-1}(M)$

- $(C_1 C_{n-1})(n) = 0$
- $(C_1 C_{n-1})(n) \neq 0$

Let's start with the case $(c_1 c_{n-1})(H) = 0$

- Satisfied, for example when $c_1 = 0$

  Ex: Symplectic Calabi-Yau mflds

  In this case the action cannot be Hamiltonian

why?

why? $c_1 = 0 \Rightarrow c_1^{S^1}\big|_{M^{S^1}}$ is cte (equiv. ext.)

$$0 = \int_M c_1^{S^1} = \sum_{p \in M^{S^1}} c_1^{S^1}(M)\Big|_p \Rightarrow \text{cte} = 0$$

In particular, M cannot have a minimum $\cancel{z}$

$$c_1^{S^1}\Big|_{min} = (a_1 + \dots + a_n)\,x$$

$$a_1, \dots, a_n > 0$$

Long Time Open Question (McDuff)

Does there exist a symplectic circle action with isolated fixed pts that is not Hamiltonian?

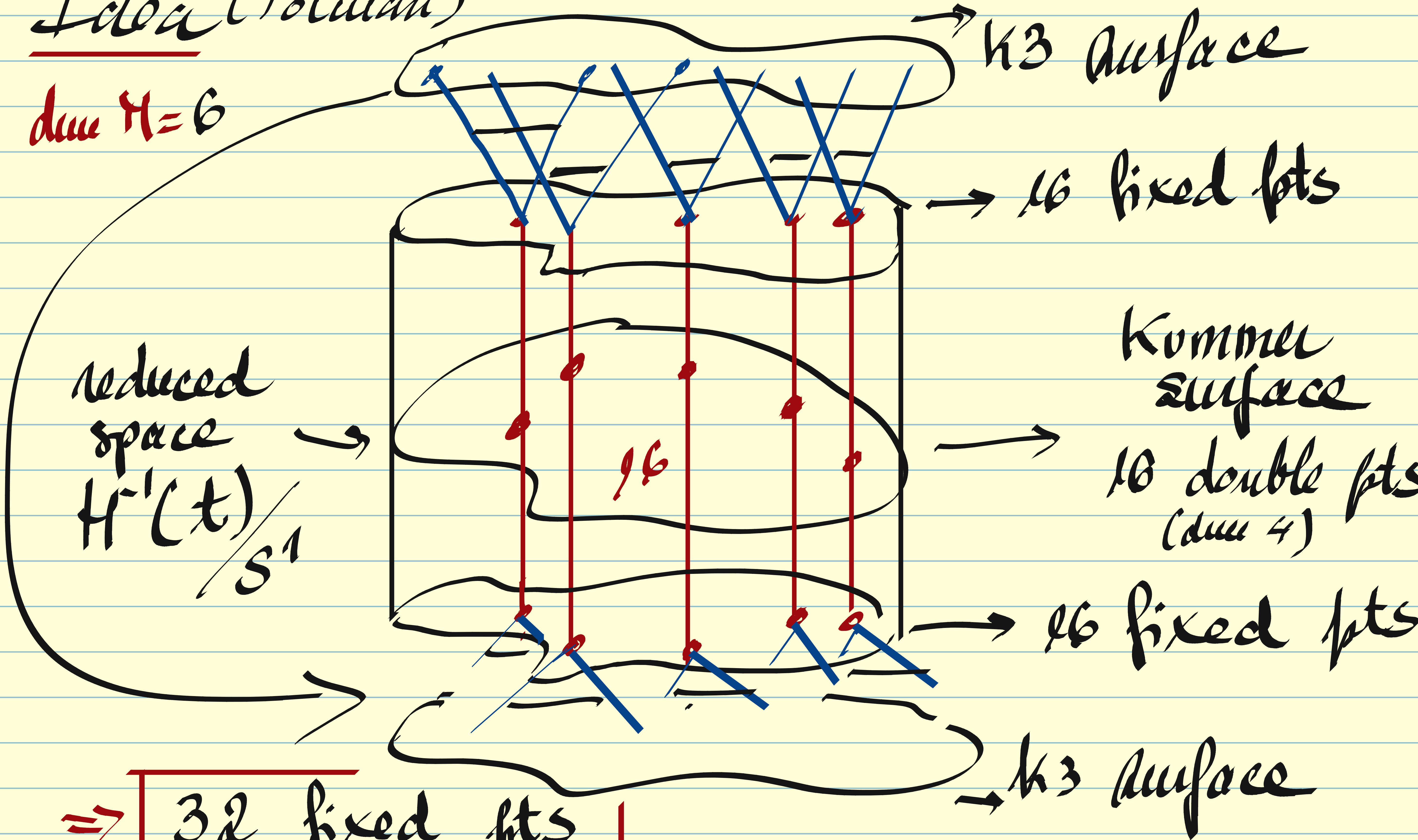
Answer: Yes (Tolman - 2017)

Construction of an example with

$c_1 = 0$

Idea (Tolman)

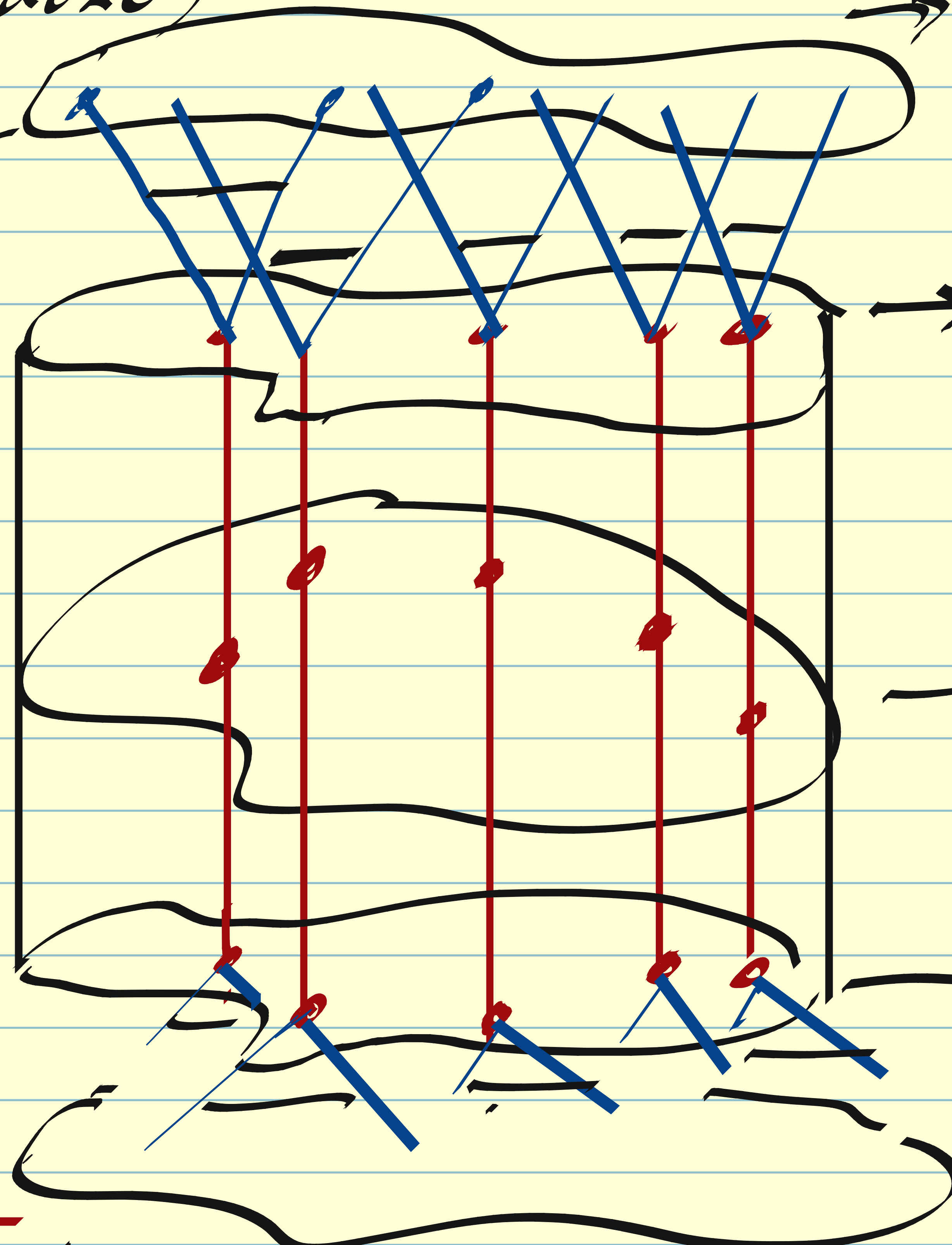
$\dim H = 6$



$\Rightarrow$ $\boxed{\text{32 fixed pts}}$

Jang - Tolman (2020)

dim $M = 6$



K3 surface

5 fixed pts

reduced space $H^{-1}(t)/S^1$

generalized K3 surface

5 double pts (dim 4)

5 fixed pts

K3 surface

$\Rightarrow$ $\boxed{10 \text{ fixed pts}}$

trying to reduce the # fixed pts

Case $\quad c_1 c_{n-1}(M) = 0$

- verified when $c_1 = 0$ ✓
- Considered by Hirzebruch:

$$c_n(M) = ?$$

Remark: $\quad c_n(M) = \#$ fixed pts

Want: Minimize

$$\sum_{k=0}^{n} N_k$$

Knowing that

$$(c, c_{n-1})(H) = \sum_{k=0}^{n} \left(6k(k-1) + \frac{5n - 3n^2}{2} \right) N_k = 0$$

Note: Hattori (1984)

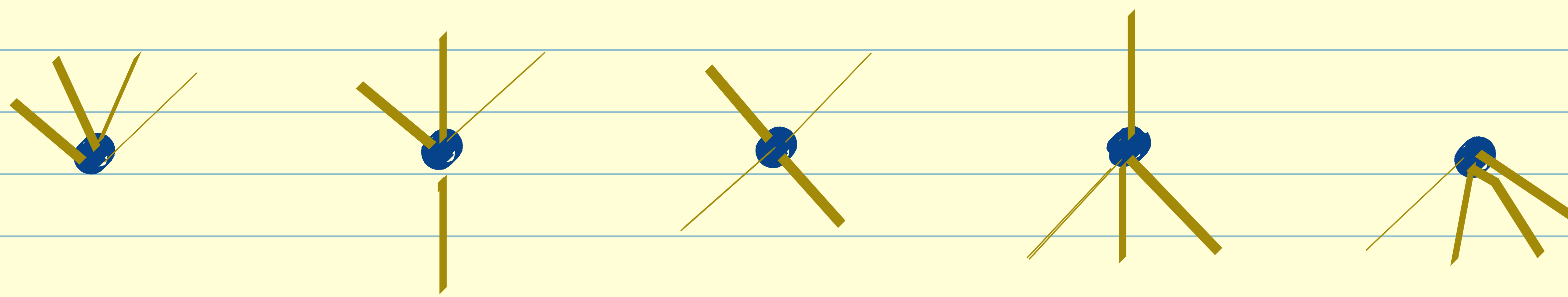
(Pelago - Toluman) dim $\mathcal{H} = 2n$

symp.

$$\boxed{N_k = N_{n-k}}$$

$n = 2m$ even (ex: $\dim M = 8$, $n = 4$, $m = 2$)

$$\sum_{k=0}^{n} N_k = N_m + 2 \sum_{k=1}^{m} N_{m-k}$$



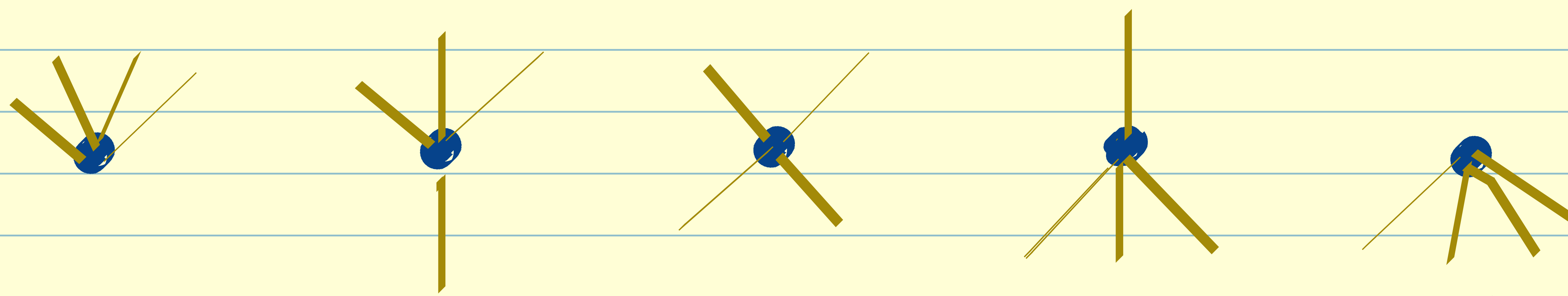
$N_0 \qquad N_1 \qquad N_2 \qquad N_3 \qquad N_4$

$=$

$=$

• $n = 2m$ even (ex: $\dim M = 8$, $n = 4$, $m = 2$)

$$\sum_{k=0}^{n} N_k = N_m + 2 \sum_{k=1}^{m} N_{m-k} =: F_1$$



$N_0 \qquad N_1 \qquad N_2 \qquad N_3 \qquad N_4$

=

=

On the other hand,

$$(c_1 c_{n-1})(H) = \sum_{k=0}^{n}\left(6k(k-1) + \frac{5n-3n^2}{2}\right) N_k$$

$$\dots$$

$$= -n\, N_n + 2\sum_{k=1}^{n}\left(6k^2 - n\right) N_{n-k}$$

Also using $N_k = N_{n-k}$

On the other hand,

$$(c_1 c_{n-1})(H) = \sum_{k=0}^{n} \left(6k(k-1) + \frac{5n-3n^2}{2} \right) N_k$$

$$\dots$$

$$= -m N_m + 2 \sum_{k=1}^{m} \left(6k^2 - m \right) N_{m-k}$$

σ_1

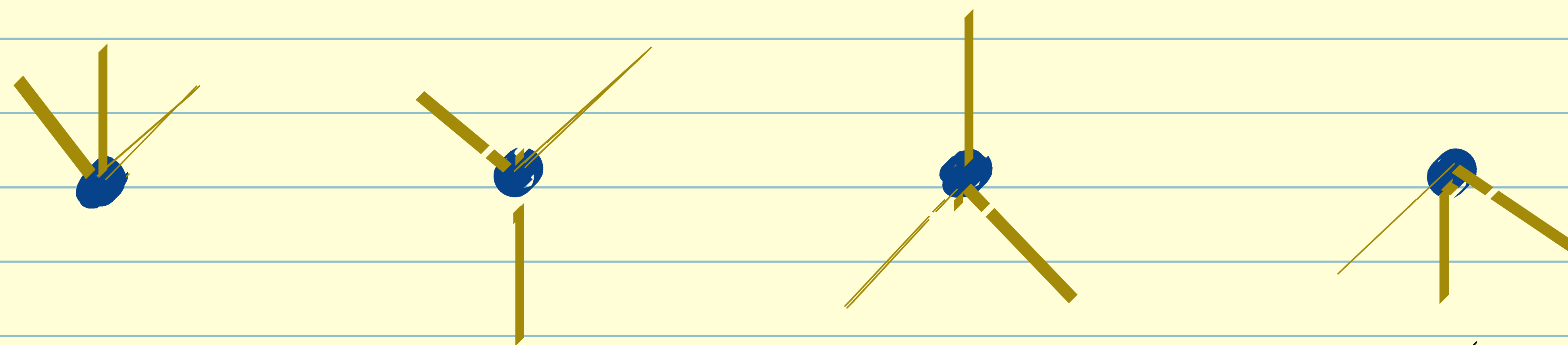
When $n=2m$ even

Want to minimize F_1 on

$$Z_1 := \left\{ (N_0, \dots, N_m) \in \mathbb{Z}_{\geq 0}^{m+1} : \quad G_1 = 0 \text{ and } F_1 > 0 \right\}$$

• $n = 2m+1$ odd (ex: $\dim M = 6$, $n = 3$, $m = 1$)

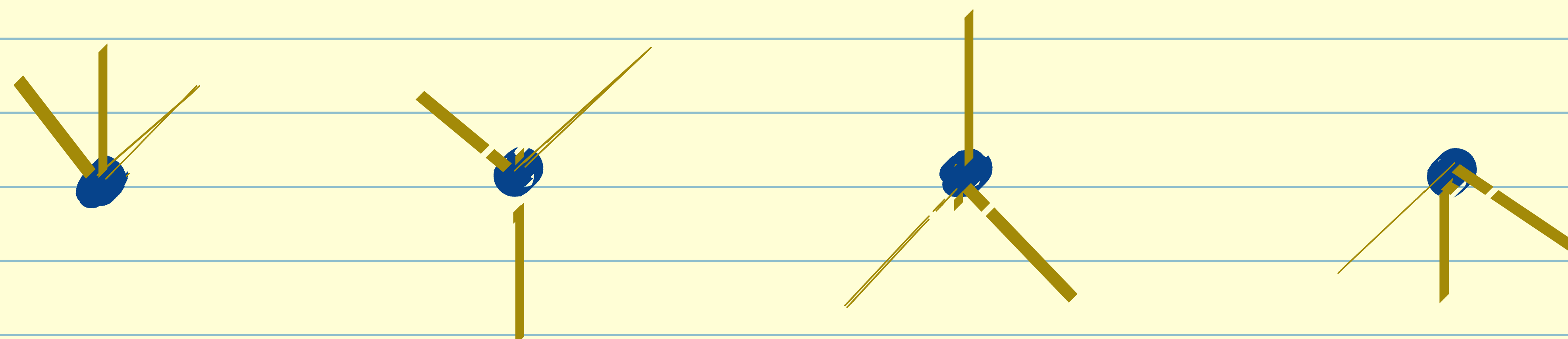
$$\sum_{k=0}^{n} N_k = 2 \sum_{k=1}^{m} N_k$$



$N_0 \qquad N_1 \qquad N_2 \qquad N_3$

$n = 2m+1$ odd ($\text{ex: } \dim M = 6,\ n = 3,\ m = 1$)

$$\sum_{k=0}^{n} N_k = 2 \sum_{k=1}^{m} N_k =: F_2$$



$$N_0 \qquad N_1 \qquad N_2 \qquad N_3$$

$=$

$=$

On the other hand,

$$(c_1 c_{n-1})(H) = \sum_{k=0}^{n} \left(6k(k-1) + \frac{5n-3n^2}{2} \right) N_k$$

$$\dots$$

$$= 2 \sum_{k=1}^{m} \left(6k(k+1) - (m-1) \right) N_{m-k}$$

σ_2

$n = 2m+1$ odd

Want to minimize F_2 on

$$Z_2 := \left\{ (N_0, \dots, N_m) \in \mathbb{Z}_{\geq 0}^{m+1} : \quad G_2 = 0 \text{ and } F_2 > 0 \right\}$$

• $n = 2m$ <u>even</u>

$$(C_1 C_{n-1})(H) = 0 \iff \sigma_1 = 0$$

$$\iff -m N_m + 2 \sum_{k=1}^{m} (6k^2 - m) N_{m-k} = 0$$

$$N_m = 2 \sum_{k=1}^{m} \left(\frac{6k^2}{m} - 1 \right) N_{m-k}$$

$$N_m = 2\sum_{k=1}^{m}\left(\frac{6k^2}{m} - \cancel{1}\right) N_{m-k}$$

Substituting in

$$F_1 = N_m + 2\cancel{\sum_{k=1}^{m}} N_{m-k}$$

$$F_1 = \frac{12}{m}\sum_{k=1}^{m} k^2 N_{m-k}$$

$$F_1 = \frac{12}{m} \sum_{k=1}^{m} k^2 N_{m-k} \in \mathbb{Z}$$

$$\sum_{k=1}^{m} k^2 N_{m-k} \equiv 0 \mod \frac{m}{r}$$

$$r = \gcd(m, 12) \in \{1, 2, 3, 4, 6, 12\}$$

Note: this also implies that

$$\# \text{ Fixed pts} \equiv 0 \mod \frac{12}{r}$$

$$F_1 = \frac{12}{m} \sum_{k=1}^{m} k^2 N_{m-k} \qquad \Big| \qquad \sum_{k=1}^{m} k^2 N_{m-k} \equiv 0 \bmod \frac{m}{r}$$

Want: Find the smallest positive (integer) value of

$$\sum_{k=1}^{m} k^2 N_{m-k}$$

which is a multiple of m/r and s.t.

$$N_m = 2 \sum_{k=1}^{m} \left(\frac{6k^2}{m} - 1 \right) N_{m-k} \geq 0$$

Then,

Fixed pts $\geq \frac{12}{m} \times$ (this value)

Smallest $\sum_{k=1}^{m} k^2 N_{m-k} \in \mathbb{Z}_{\geq 0}$ and $\equiv 0 \mod \frac{m}{2}$

s.t. $N_m = 2\sum_{k=1}^{m}\left(\frac{6k^2}{m} - 1\right) N_{m-k} \geq 0$

$\Longleftrightarrow$

$$\sum_{k=1}^{m} k^2 N_{m-k} \geq \frac{m}{6} \sum_{k=1}^{m} N_{m-k}$$

Smallest $\sum_{k=1}^{m} k^2 N_{m-k} \in \mathbb{Z}_{\geq 0}$ and $\equiv 0 \mod \frac{m}{r}$

s.t. $\sum_{k=1}^{m} k^2 N_{m-k} \geq \frac{m}{6} \sum_{k=1}^{m} N_{m-k}$ $(N_m \geq 0)$

Smallest positive multiple of m/r that can be written as

$$\sum k^2 N_{m-k}$$

and is

$$\geq \frac{m}{6} \sum_{k=1}^{m} N_{m-k}$$

Smallest $l \in \mathbb{Z}_{\geq 0}$ s.t.

$$l \cdot \frac{m}{r} = \sum_{k=1}^{m} k^2 N_{m-k} \geq \frac{m}{6} \sum_{k=1}^{m} N_{m-k}$$

Smallest $l \in \mathbb{Z}_{>0}$ s.t.

$$l \cdot \frac{m}{2} = \sum_{k=1}^{m} k^2 N_{m-k} \geq \frac{m}{6} \sum_{k=1}^{m} N_{m-k}$$

Sum of squares
(possibly with repetitions)

# of squares
used to write $l \cdot \frac{m}{2}$
as a sum of squares

Smallest $l \in \mathbb{Z}_{>0}$ s.t

$$l \cdot \frac{\cancel{m}}{2} = \sum_{k=1}^{m} k^2 N_{m-k} > \frac{\cancel{m}}{6} \sum_{k=1}^{m} N_{m-k}$$

Smallest $l \in \mathbb{Z}_{>0}$ s.t $\displaystyle\sum_{k=1}^{m} N_{m-k} \leq \frac{6l}{2}$

$\downarrow$

Smallest # of squares that one needs to write $l \cdot \frac{m}{2}$ as a sum of squares then our lower bound is

$$B(n) = \frac{12}{\cancel{m}} \cdot \frac{\ell \cancel{m}}{r} = \frac{12\ell}{r}$$

We need to find the

smallest # of squares
that one needs to write
$l \cdot \frac{m}{r}$ as a sum of squares ...

Fermat (XVII century)

Every positive integer is the sum of at most 4 squares

Fermat (XVII century)

Every positive integer is the sum of at most 4 squares

Proved by Lagrange (1770)

Lagrange's 4 square theorem

Ex:

$$60 = 6^2 + 4^2 + 2^2 + 2^2$$

$$105 = 10^2 + 2^2 + 1^2$$

$$245 = 14^2 + 7^2$$

there are of course numbers that can be written as a sum of 3, 2 or 1 square

Legendre's 3-square Theorem (1789)

The set of positive integers that are not sums of 3 or fewer squares is

$$\{ m \in \mathbb{Z}_+ : m = 4^k(8t+7),\ k,t \in \mathbb{Z}_{\geq 0} \}$$

these need 4 squares

Ex:

$$60 = 6^2 + 4^2 + 2^2 + 2^2 = 4(8+7)$$

$$105 = 10^2 + 2^2 + 1^2 = 8 \cdot 13 + 1$$

$$245 = 14^2 + 7^2 = 30 \cdot 8 + 5$$

Euler

A positive integer $m > 1$ can be written as a sum of 2 squares iff every prime factor of m congruent to 3 mod 4 has an even exponent

Ex:

$$60 = 6^2 + 4^2 + 2^2 + 2^2 = 3 \cdot 4 \cdot 15$$

$$105 = 10^2 + 2^2 + 1^2 = 3 \cdot 5 \cdot 7$$

$$245 = 14^2 + 7^2 = 5 \cdot 7^2$$

$$16 = 4^2 = 2^4$$

A positive integer $m > 1$ can be written as a sum of 2 squares iff every prime factor of m congruent to 3 mod 4 has an even exponent

Back to our problem

Goal: Find the smallest $l \in \mathbb{Z}^+$ s.t. $\sum_{k=1}^{m} N_{m-k} \le \frac{6l}{r}$

smallest # of squares needed to write $\frac{l \cdot m}{r}$ as a sum of squares

$\rightarrow$ $B(n) = 12l/r$ $\quad r = \gcd(m, 12)$

Ex: $r = \gcd(m,12) = 1$
$m = \frac{dn+1}{4}$

$\left.\begin{array}{l} r = \gcd(m,12)=1 \\ m = \frac{dn+1}{4} \end{array}\right\} \Rightarrow \sum_{k=1}^{m} N_{m-k} \le 6l$

- $l = 1 \Rightarrow \sum_{k=1}^{m} N_{m-k} \le 6$ always true $\boxed{B(n) = 12}$

Goal: Find the smallest $l \in \mathbb{Z}^+$ s.t. $\sum_{k=1}^{m} N_{m-k} \le \frac{6l}{r}$

smallest # of squares needed to write $\frac{l \cdot m}{r}$ as a sum of squares

$\rightarrow$ $B(n) = 12l/r$ $\quad r = \gcd(m, 12)$

Ex: $r = \gcd(m,12) = 4$

$m = \frac{\text{dim } M}{4}$ $\Rightarrow \sum_{k=1}^{m} N_{m-k} \le \frac{3l}{2}$

• $\boxed{l=1} \Rightarrow \sum_{k=1}^{m} N_{m-k} \le \frac{3}{2}$

If $\frac{l \cdot m}{r} = \frac{m}{4}$ is a square then ok $\Rightarrow$ $\boxed{B(n) = 12}$

o.w. ...

Goal: Find the smallest $\ell \in \mathbb{Z}^+$ s.t. $\sum_{k=1}^{m} N_{m-k} \le \frac{6\ell}{r}$

smallest # of squares needed to write $\frac{\ell \cdot m}{r}$ as a sum of squares

$\rightarrow$ $B(n) = 12\ell/r$ $r = \gcd(m, 12)$

Ex: $r = 4$

$$\sum_{k=1}^{m} N_{m-k} \le \frac{3\ell}{2}$$

- If $\frac{\ell \cdot m}{r} = \frac{m}{4}$ is a square then $\Longrightarrow$ OK $\boxed{B(n) = 12}$

O.W.

- $\boxed{\ell = 2} \Rightarrow \sum_{k=1}^{m} N_{m-k} \le 3$

If $\frac{\ell \cdot m}{r} = \frac{m}{2}$ is a sum of at most 3 squares $\Longrightarrow$ OK $\boxed{B(n) = \frac{12 \cdot 2}{4} = 6}$

O.W.

Goal: Find the smallest $\ell \in \mathbb{Z}^+$ s.t. $\sum_{k=1}^{m} N_{m-k} \le \frac{6\ell}{r}$

Smallest # of squares needed to write $\frac{\ell \cdot m}{r}$ as a sum of squares

$\rightarrow$ $B(n) = 12\ell / r$ $\qquad r = \gcd(m, 12)$

Ex: $r = 4$ $\qquad \sum_{k=1}^{m} N_{m-k} \le \frac{3\ell}{2}$

• If $\frac{\ell \cdot m}{r} = \frac{m}{4}$ is a square then $\Rightarrow$ OK $\qquad \boxed{B(n) = 12}$

o.w.

• If $\frac{\ell \cdot m}{r} = \frac{m}{2}$ is a sum of at most 3 squares $\Rightarrow$ OK $\qquad \boxed{B(n) = 6}$

o.w.

$\boxed{\ell = 3} \Rightarrow \sum_{k=1}^{m} N_{m-k} \le 9/2 \Rightarrow$ OK $\qquad \boxed{B(n) = \frac{12 \cdot 3}{4} = 9}$

- n even ✓

What if n is odd?

Recall: When n is odd we want to

minimize
$$F_2 = 2 \sum_{k=0}^{m} N_k$$

on
$$\mathcal{Z}_2 = \{ (N_0, \dots, N_m) \in \mathbb{Z}^{m+1}_{\geq 0} : G_2 = 0 \text{ and } F_2 > 0 \}$$

where
$$G_2 = 2 \sum_{k=0}^{m} \big(6k(k+1) - (m-1) \big) N_{m-k}$$

No squares!

$$(C_1 C_{m-1})(t) = 0 \iff \theta_2 = 0$$

$$\iff \sum_{k=0}^{m} \Big(6k(k+1) - (m-1)\Big) N_{m-k} = 0$$

$$\iff N_m = \sum_{k=1}^{m} \left(\frac{6k(k+1)}{m-1} \cancel{-1}\right) N_{m-k}$$

Substituting in $F_2 = 2 N_m + 2 \cancel{\sum_{k=1}^{m}} N_{m-k}$

$$F_2\Big|_{\theta_2=0} = 2 \cdot \frac{12}{m-1} \sum_{k=1}^{m} \frac{k(k+1)}{2} N_{m-k} \in \mathbb{Z}_+$$

$$(C_1 C_{m-1})(H) = 0 \iff \theta_2 = 0$$

$$\iff \sum_{k=0}^{m} \big(6k(k+1) - (m-1)\big) N_{m-k} = 0$$

$$N_m = \sum_{k=1}^{m} \left(\frac{6k(k+1)}{m-1} \cancel{-1} \right) N_{m-k}$$

Substituting in $F_2 = 2 N_m + \cancel{2 \sum_{k=1}^{m} N_{m-k}}$

$$F_2 \Big|_{\theta_2 = 0} = 2 \cdot \frac{12}{m-1} \sum_{k=1}^{m} \frac{k(k+1)}{2} N_{m-k} \in \mathbb{Z}_+$$

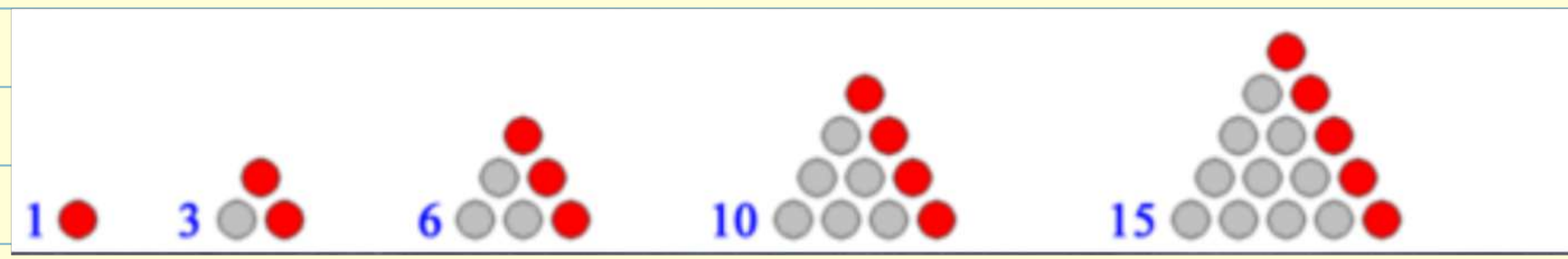
$\equiv 0 \bmod \frac{m-1}{r}$

$r = \gcd(m-1, 12)$

\# Fixed pts $\equiv 0 \bmod 24/r$

$$F_2\Big|_{\theta_2=0} = 2 \cdot \frac{12}{m-1} \sum_{k=1}^{m} \frac{k(k+1)}{2} N_{m-k} \in \mathbb{Z}_+$$

triangular number



$$F_2\Big|_{\theta_2=0} = 2\cdot\frac{12}{m-1}\sum_{k=1}^{m}\frac{k(k+1)}{2}N_{m-k} \in \mathbb{Z}_+$$

$$\equiv 0 \bmod \frac{m-1}{r}$$

$r = \gcd(m-1, 12)$

Want: smallest $\ell \in \mathbb{Z}_{\geq 0}$ s.t.

$$\ell\cdot\frac{m-1}{r} = \sum_{k=1}^{m}\frac{k(k+1)}{2}N_{m-k} \geq \underbrace{\frac{m-1}{12}\sum_{k=1}^{m}N_{m-k}}$$

to have $N_m \geq 0$

Sum of triangular numbers

Fermat

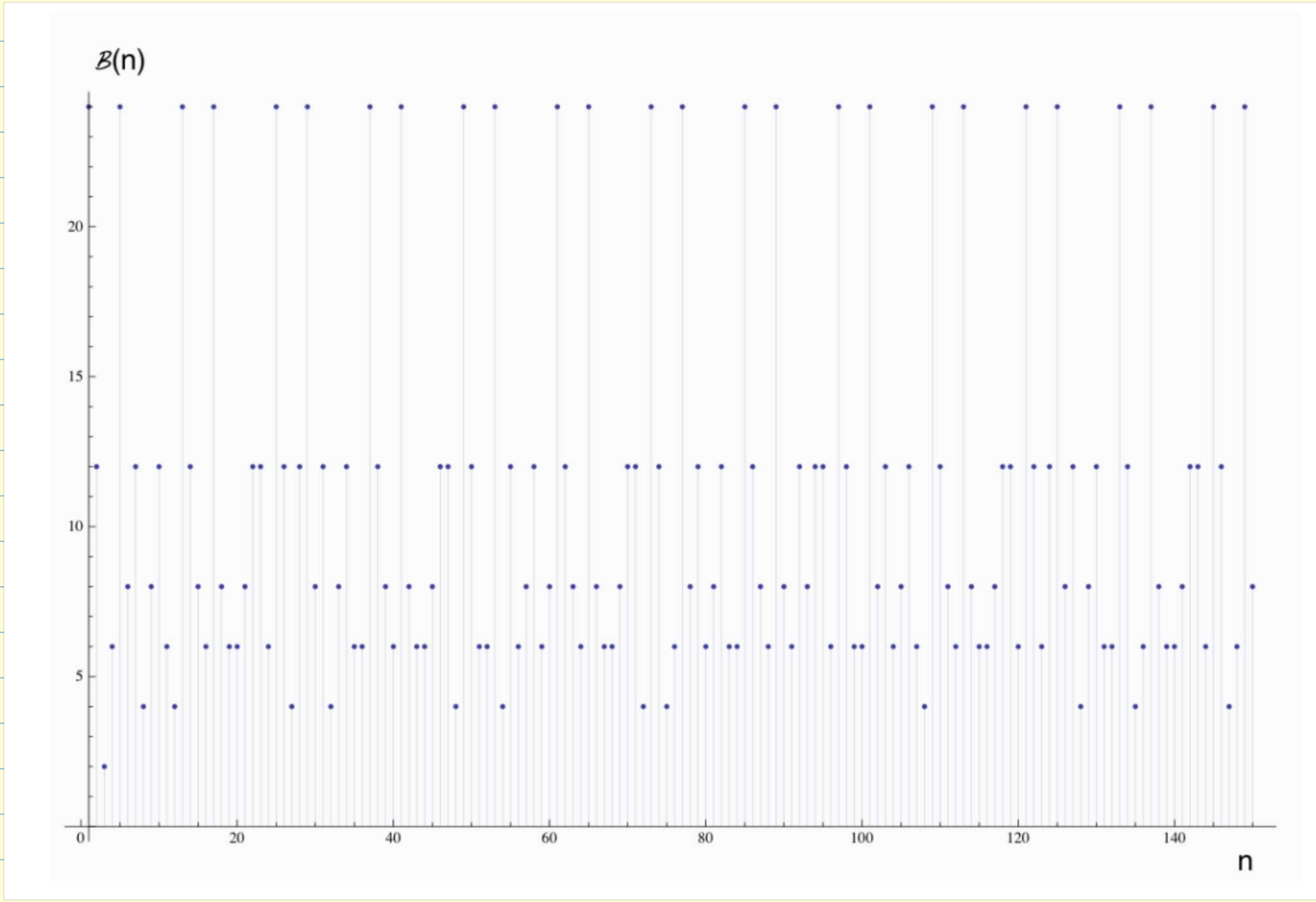
Every positive integer is a sum of at most 3 triangular numbers

Proved by Gauss (1796)

Famous Eureka Theorem

Ewell (1992)

A positive integer n can be represented as a sum of 2 triangular numbers iff every prime factor of $4n+1$ which is congruent to 3 mod 4 occurs with even exponent.



$\mathcal{B}$(n)
20
15
10
5
0
20
40
60
80
100
120
140
n

In particular

$$B(n) \geq \lfloor \tfrac{n}{2} \rfloor + 1 \quad \text{iff}$$

$$\dim M \in \{4, 6, 8, 10, 12, 14, 18, 20, 22, 26, 28, 34, 44, 46, 50, 58, 74, 82\},$$

and so Kosniowski's conjecture is true for these dimensions whenever $c_1 c_{n-1} \neq 0$

Moreover, $B(n) \geq n+1$ iff

$$\dim M \in \{4, 8, 10, 14, 20, 26, 34\},$$

Divisibility Conditions for the # Fixed pts

(or, equivalently for $C_n(M)$) when $c_1 c_{n-1} = 0$

Hirzebruch:

- *if* $n \equiv 1$ *or* $5 \pmod 8$, *the Chern number* $c_n[M]$ *is divisible by* 8;
- *if* $n \equiv 2, 6$ *or* $7 \pmod 8$, *the Chern number* $c_n[M]$ *is divisible by* 4;
- *if* $n \equiv 3$ *or* $4 \pmod 8$, *the Chern number* $c_n[M]$ *is divisible by* 2.

($\dim M = 2n$)

We improve Hirzebruch's divisibility factors for $\chi(M) = |M^{S^1}|$ whenever $n \not\equiv 0 \mod 3$ and obtain the same factors otherwise.

In particular if $n \not\equiv 0 \mod 3$ (and $c_1 c_{n-1} = 0$) then

- *if* $n \equiv 0 \pmod 8$, *then* $|M^{S^1}|$ *is divisible by* 3;
- *if* $n \equiv 1$ *or* $5 \pmod 8$, *then* $|M^{S^1}|$ *is divisible by* 24;
- *if* $n \equiv 2, 6$ *or* $7 \pmod 8$, *then* $|M^{S^1}|$ *is divisible by* 12;
- *if* $n \equiv 3$ *or* $4 \pmod 8$, *then* $|M^{S^1}|$ *is divisible by* 6.

3

What if we restrict to Hamiltonian actions?

$$N_i > 0 \qquad N_0 = N_n = 1$$

Theorem (G. - Pelayo - Sabatini)

- M - compact, connected symplectic
- $c_1 c_{n-1}(M) = 0$

$\Rightarrow$ # Fixed pts of a Hamiltonian S^1-action on M is at least

$$\begin{cases} (n+1)(n+2), & n \text{ even} \\ n^2+6n+17+\dfrac{24}{\gcd\left(\frac{n-3}{2}, 12\right)}, & n>3 \text{ odd} \end{cases}$$

$N^4 \equiv \mathbb{CP}^2 \# 9\, \mathbb{CP}^2$

$c_1 c_{n-1} = c_1^2 = 0$

S^6

$\cap$

$\mathbb{R} \times \mathbb{C}^3$

$\lambda \cdot (t, z_1, z_2, z_3)$

$''$

$(t, \lambda^a z_1, \lambda^b z_2, \lambda^{-(a+b)} z_3)$

$\frac{1}{2}\dim M$	A priori possible values of $\lvert M^{S^1}\rvert$ if $c_1c_{n-1}[M]=0$		Kosniowski's conjectural lower bound	Lower bound Ham. actions
n	general	Ham. actions	$\lfloor n/2 \rfloor + 1$	$n+1$
2	$12^*, 24, 36, \dots$	$12, 24, 36, \dots$	2	3
3	$2, 4, 6, \dots$	—	2	4
4 ?	$6, 12, 18, \dots$	$30, 36, 42, \dots$	3	5
k=l=1 5	$24, 48, 72, \dots$	$96, 120, 144, \dots$	3	6
k=0 l=2 6	$4, 8, 12, \dots$	$56, 60, 64, \dots$	4	7
7 $\times S^6$	$12, 24, 36, \dots$	$120, 132, 144, \dots$	4	8
8	$6, 9, 12, \dots$	$90, 93, 96, \dots$	5	9
9	$8, 16, 24, \dots$	$160, 168, 176, \dots$	5	10
10	$12^*, 24, 36, \dots$	$132, 144, 156, \dots$	6	11
11	$6, 12, 18, \dots$	$210, 216, 222, \dots$	6	12
12	$4, 6, 8, \dots$	$182, 184, 186, \dots$	7	13
13	$24, 48, 72, \dots$	$288, 312, 336, \dots$	7	14
14	$12, 24, 36, \dots$	$240, 252, 264, \dots$	8	15
15	$4, 8, 12, \dots$	$336, 340, 344, \dots$	8	16
...				
k=0 ⇐ l=3 18	$8, 12, 16, \dots$	$380, 384, 388, \dots$	10	19
28	$12, 18, 24, \dots$	$870, 876, 882, \dots$	15	29
99	$6, 8, 10, \dots$	$10414, 10416, \dots$	50	100
112	$9, 12, 15, \dots$	$12882, 12885, \dots$	57	113
144	$6, 7, 8, \dots$	$21170, 21171, \dots$	73	145
252	$8, 10, 12, \dots$	$64262, 64264, \dots$	127	253
1008	$7, 8, 9, \dots$	$1019090, 1019091, \dots$	505	1009

* if $c_1 = 0$ then, a priori, the possible values of $\lvert M^{S^1}\rvert$ are $24, 48, 72, \dots$

$(N^4)^k \times (S^6)^\ell$

$2^\ell \times 12^k$

fixed pts

the story continues...

What if $c_1 c_{n-1} \neq 0$?

Let's see for example where $n = 2m$ (is even)

$$(C_1 C_{n-1})(H) = -m N_m + 2\sum_{k=1}^{m}(6k^2 - m) N_{m-k}$$

$$\cdots$$

$$N_m = \frac{1}{m}\left(2\sum(6k^2 - \cancel{m}) N_{m-k} - \underline{\underline{C_1 C_{n-1}}}\right) \geq 0$$

$$F_1 = N_m + 2\cancel{\sum_{k=1}^{m}} N_{m-k}$$

$$\Downarrow$$

$$F_1 \geq 2\sum_{k=1}^{m} N_{m-k}$$

$$\boxed{F_1 = \frac{1}{m}\left(12\sum_{k=1}^{m} k^2 N_{m-k} - C_1 C_{n-1}\right)}$$

$$\boxed{F_1 = \frac{1}{m}\left(12\sum_{k=1}^{m} k^2 N_{m-k} - C_1 C_{n-1}\right) \notin \mathbb{Z}}$$

$$\boxed{\frac{F_1}{2} \not\geq \sum N_{m-k}}$$

$$m \underbrace{F_1}_{l} + C_1 C_{n-1} = \underbrace{12\sum_{k=1}^{m} k^2 N_{m-k}}_{n}$$

$$\boxed{12n - ml = C_1 C_{n-1}}$$

linear

Diophantine equation

$\Rightarrow (C_1 C_{n-1})(\ast) = 0 \mod r$

$r = \gcd(12, m)$

$$\boxed{l_2 x - m\, l = c_1 c_{n-1}} \qquad (A)$$

$: r$

$$\frac{l_2}{r} x - \frac{m}{r} l = \frac{c_1 c_{n-1}}{r} \qquad (B)$$

- $\frac{l_2}{r} x - \frac{m}{r} l = 1$ (note that $\gcd\left(\frac{l_2}{r}, \frac{m}{r}\right) = 1$)

Get a solution (x_0, l_0) (Euclide's algorithm)

$\Rightarrow \left(\frac{c_1 c_{n-1}}{r} x_0, \frac{c_1 c_{n-1}}{r} l_0\right)$ is a solution of (A) and (B)

$$\boxed{12x - ml = c_1 c_{n-1}} \qquad (A)$$

All solutions of (A) are of the form

$$\begin{cases} x = \dfrac{c_1 c_{n-1}}{r}\, x_0 + \dfrac{m}{r}\, k \\[2ex] l = \dfrac{c_1 c_{n-1}}{r}\, l_0 + \dfrac{12}{r}\, k \end{cases} \qquad k \in \mathbb{Z}$$

we then look for the smallest positive value of l

s.t. $$\frac{l}{2} \geq \sum_{k=1}^{m} N_{m-k}$$

\# squares needed to write x as a sum of squares

...

Example

1) $(C_1 C_{n-1})(M) = \frac{1}{2} n(n+1)^2$ (Same as $\mathbb{C}P^n$)

	$C_1 C_{n-1}$	Hamilt.	a.c.	$[n/2]+1$
$n=2$	9	3	3	2
$n=4$	50	5	5	3
$n=6$	147	7	7	4
$n=8$	324	9	6	5

Kosniowski

2) $(c_1 c_{n-1})(M) = n(n^2-n+2)$ (Same as in $\mathbb{C}P^n \# \overline{\mathbb{C}P}^n$)

	$c_1 c_{n-1}$	Hamilt.	a.c.	$\lfloor n/2 \rfloor + 1$
$n=2$	8	4	4	2
$n=4$	56	8	8	3
$n=6$	192	12	4	4
$n=8$	464	13	4	5

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Kosniowski

Thanks!