On relations between K-moduli and symplectic geometry

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joint with

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§ Introduction

Motivation: Donaldson-Fujiki picture

(M,ω) cpt symp $\quad b_1(M)=0$.

$\mathrm{Symp}(M,\omega) \qquad \mathrm{Lie}\,\mathrm{Symp}(M,\omega) = C_0^\infty(M)$

$$\mathrm{Symp}(M,\omega) \circlearrowright \mathcal{J}_\omega := \{ J \mid J \text{ is compatible with } \omega \}$$

$$g_J = \omega(-, J-)$$

$$\phi . J = d\phi\, J\, d\phi^{-1}$$

$$\exists\, \mu : \mathcal{J}_\omega^{(\mathrm{int})} \longrightarrow C_0^\infty(M)$$

$$J \longmapsto Sc(g_J) - \overbrace{\fint_M Sc(g_J)\,\omega^n}^{\text{(coh. quantity)}} .$$

So $\mu^{-1}(0) \Longleftrightarrow Sc(g_J) = cst$

g_J is cscK metric.

So $\mu^{-1}(0) / \mathrm{Symp}(M,\omega)$ "moduli space of cscK metrics".

$\rightsquigarrow$ "∞-dim GIT picture"

fix J

$J \overset{\text{Moser trick}}{\longleftrightarrow} \{\omega_\phi := \omega + i\partial\bar\partial\phi\}$, $\omega_\phi > 0$ Kähler class.

"complexified orbit"

Thus: search for a cscK metric in a Kähler class $[\omega]$

$\Updownarrow$

find zeros of moment map in the "cplexified orbit".

$\Updownarrow$ Yau-Tian-Donaldson conj

$(M,J) = X$ K-stable $\leftarrow$ (stability for varieties)

$$\text{“}\,\mathcal{J}^{int,\,stable} \big/ Symp^{\mathbb{C}}\,\text{”} \simeq \mathcal{J}^{int} /\!/ Symp(M,\omega)$$

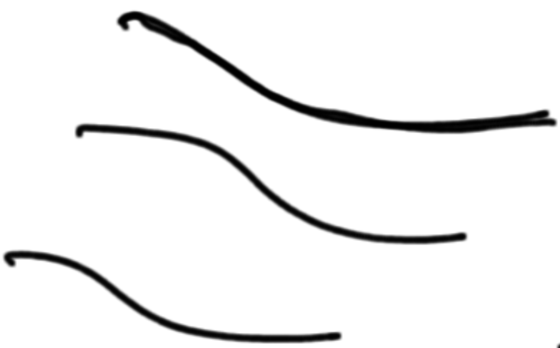
↑ moduli of stable varieties. ↑ moduli of cscK

This picture suggests by studying moduli space of stable varieties then we can find information about more diff. geometric quantity such as the Symp group.

(Such philosophy indeed can work: M. Abreu, G. Granja, N. Kitchloo)

Atiyah-Bott framework

$|\mu|^2$ ← Equivariant Morse-Bott function $\quad G \circlearrowright X$



C_i ← critical set for $|\mu|^2$

$H^*_G(X)$ ← can be relate to top of critical set

(If X is contractible $H^*_G(X) = H^*(BG)$)

$\mathcal{J}^{int}_\omega$

$$|\mu|^2 \rightsquigarrow \underbrace{\int_M Sc^2(g_J)\, dV}_{\text{Calabi functional}}$$

So by looking to critical points of $|\mu|^2$ we are looking to extremal Kähler metric.

AGK (this heuristic works for certain rational surfaces)

$(S^2 \times S^2, \omega_{FS} \oplus \lambda\, \omega_{FS}.)$

$\lambda = 1 \rightsquigarrow$ in this case only $\mathbb{P}^1 \times \mathbb{P}^1$

$\lambda \rightsquigarrow$ certain even Hirzebruch surfaces

$\mathcal{J}^{int}_{\omega_\lambda}$ weakly contractible

$$H^*(B\,Symp) \simeq H^*(\underbrace{BSO(3) \times BSO(3)}_{\text{isometry group of } \omega_{FS} \oplus \omega_{FS}.}) \oplus \sum^{n} H^*(\underbrace{BU(1) \times BSO(3)}_{})$$

$U(1) \times SO(3)$ isometries group of extr. metric on Hir. Sf.

X is Fano complex manifold

$$c_1(X) > 0$$
$$\parallel$$
$$[\omega] \qquad \text{monotone symplectic}$$

J_ω^{int} still corresponds to Fanos

in low dimensions we have complete classification

$n = 2$ del Pezzo surfaces

cplex dim

$n = 3$ Mori-Mukai

$\mu^{-1}(0) \leftrightsquigarrow$ Kähler-Einstein metrics

$$\mathrm{Ric}(g_J) = g_J.$$

$$\mu^{-1}(0)/\mathrm{Symp} \simeq \mathcal{M}^K$$

K-moduli space of K-stable Fanos.

$H^*_{\mathrm{Symp}}(J_\omega^{int})$ some relations $H^*(\mathcal{M}^K)$

"moduli stack"

E.g. K-mod of $\mathbb{P}^n$

$$[\cdot/PGL(n)] \longrightarrow \{p\}$$

" coarse moduli"

$$H^*([\cdot/PGL(n)])$$
$$\parallel$$
$$H^*(BPGL(n))$$

Rmk: $H^*_{Symp}(\mathcal{J}^{int}_\omega)$ can be considered as a "symplectic invariant".

$\rightsquigarrow$ get course invariants by integrating interesting natural classes on the moduli

e.g. $c_1(\lambda_{CM}) \in H^2(\mathcal{M}^K)$

$\uparrow$ CM-line bundle.

$$\underbrace{\int_{\mathcal{M}^K} c_1^*(\lambda_{CM})}_{} = \int_{\overline{\mathcal{M}}^K} c_1^*(\lambda_{CM})$$

(degree of descended CM-line bundle on the coarse $\overline{M}^K$

Tamborow '20 dP quartic in any dimension (...)

K-moduli cptifications (constructed via analysis using Gromov-Hausdorff limits)

$X_{n}, \ldots$ (algebro geo construction)

§ Some results

Study aspects of these relations on explicit examples (del Pezzo surfaces).

Thm 1 ($BG_\circ$, S-D)

$$B\Gamma(\mathcal{L}_{[\omega]}, \mathrm{Diff}_{[\omega]}) \underset{\substack{\text{weakly}\\ \text{homot.}\\ \text{equivalent}}}{\xrightarrow{\ \sim\ }} B\mathrm{Symp}(\omega)$$

$\Leftarrow$

$\mathcal{J}^{int}_{\omega}$ is weakly homot. contractible.

Here: $\mathcal{L}_{[\omega]} := \{ J \mid \exists\, \phi \in \mathrm{Diff}_{[\omega]}(M) \text{ s.t. } J \in \mathcal{J}^{int}_{\phi^* \omega_0} \}$

$\circlearrowleft \mathrm{Diff}_{[\omega]}(M)$

$\Gamma(\mathcal{L}_{[\omega]}, \mathrm{Diff}_{[\omega]}(M))$

∞-dim groupoid

$B\Gamma(\mathcal{L}_{[\omega]}, \mathrm{Diff}_{[\omega]})$

Borel construction.

Thm 2 $\mathcal{J}^{int}_{\omega}$ is weakly homotopically contractible for del Pezzo 4 and 5

(n° of blow-up points).

(Monotone case!)

Ideas:

$$K = \{(\omega, J) \mid \omega, J \text{ are compatible}, \ \omega \in \mathcal{O} = \{\phi^{*}\omega_0 \mid \phi \in \mathrm{Diff}_{[\omega]}\}\}$$

$$\mathcal{O} \xleftarrow{p_1} K \xrightarrow{p_2} \mathcal{L}_{[\omega]}$$

$\mathrm{Diff}[\omega]$

fiber bundle with fibers $\mathcal{J}_{\omega}^{int}$

sections.

p_2 has convex fibers

equi.

$B\Gamma(K, \mathrm{Diff}([\omega]) \longrightarrow B\Gamma(\mathcal{L}_{[\omega]}, \mathrm{Diff}_{[\omega]})$

w. ho. equ.

$B\Gamma(\Theta, \mathrm{Diff}(\omega))$

$\cong$

$B\Gamma(\mathrm{h.i.}, \mathrm{Symp}(\omega))$

$\cong$

forms w. h. cont of $\int_\omega^{\mathrm{int}}$ con

dP_5

moduli space of marked dP

$\mathcal{U}$

$\downarrow$

$B^{ord}/PSL(3)$ $\circlearrowleft$ $W(E_8)$

5 non collinear points on $\mathbb{P}^2$

Seidel: $\pi_1(B^{ord}/PSL(3)) \xrightarrow[\text{inj}]{} \pi_0(\mathrm{Symp}_0(D_5, \omega))$

induced by Dehn twist around formation of ODP (sing. appearing in k-mod cpt)

Evans

$\alpha: B^{ord}/PSL(3, \mathbb{C}) \longrightarrow B\mathrm{Symp}_0(D_5, \omega)$

weakly hom. equ.

$B(B^{ord}/PSL(3,\mathbb{C}), W) \longrightarrow B\mathrm{Symp}(D_5, \omega)$

w.h. eq.