Periodic Hamiltonian flows on 4-manifolds

Grace Mwakyoma
Instituto Superior Técnico - LisMath Seminar

November 29, 2017

Based on:

- Yael Karshon, “Periodic Hamiltonian flows on four dimensional manifolds”, `arXiv:dg-ga/9510004`.

Symplectic Geometry - Basic Objects

- A **symplectic manifold** (M, ω)

Symplectic Geometry - Basic Objects

- A **symplectic manifold** (M, ω) is a manifold M endowed with a closed, non-degenerate 2-form ω - **symplectic form**.

Symplectic Geometry - Basic Objects

- A **symplectic manifold** (M, ω) is a manifold M endowed with a closed, non-degenerate 2-form ω - **symplectic form**.

Recall:

- **Nondegenerate**:

$$\forall p \in M, \quad \text{if } \omega_p(X_p, Y_p) = 0 \text{ for all } Y_p \in T_pM, \text{ then } X_p = 0.$$

Symplectic Geometry - Basic Objects

- A **symplectic manifold** (M, ω) is a manifold M endowed with

  a closed, non-degenerate 2-form ω - **symplectic form**.

Recall:

- **Nondegenerate**:

$$\forall p \in M, \quad \text{if } \omega_p(X_p, Y_p) = 0 \text{ for all } Y_p \in T_pM, \text{ then } X_p = 0.$$

$$\Rightarrow \begin{cases} \bullet \ \dim M = \dim T_pM \text{ is even.} \\ \bullet \ \omega_p^n \neq 0 \ \forall p \in M - \text{volume form} \end{cases}$$

Symplectic Geometry - Basic Objects

- A **symplectic manifold** (M, ω) is a manifold M endowed with

  a closed, non-degenerate 2-form ω - **symplectic form**.

- A **symplectomorphism** $\rho\colon (M_0, \omega_0) \to (M_1, \omega_1)$ is

  a diffeomorphism s.t. $\rho^*(\omega_1) = \omega_0$.

Symplectic Geometry - Basic Objects

- A **symplectic manifold** (M, ω) is a manifold M endowed with

  a closed, non-degenerate 2-form ω - **symplectic form**.

- A **symplectomorphism** $\rho\colon (M_0, \omega_0) \to (M_1, \omega_1)$ is

  a diffeomorphism s.t. $\rho^*(\omega_1) = \omega_0$.

1. *Locally* symplectic manifolds are indistinguishable:

Symplectic Geometry - Basic Objects

- A **symplectic manifold** (M, ω) is a manifold M endowed with a closed, non-degenerate 2-form ω - **symplectic form**.
- A **symplectomorphism** $\rho\colon (M_0, \omega_0) \to (M_1, \omega_1)$ is a diffeomorphism s.t. $\rho^*(\omega_1) = \omega_0$.

1. *Locally* symplectic manifolds are indistinguishable:

   Darboux Theorem: Local model - $(\mathbb{R}^{2n}, \omega_0)$.

   $p = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$

   $$\omega_0 = \sum_{k=1}^{n} dx_k \wedge dy_k$$

Symplectic manifolds

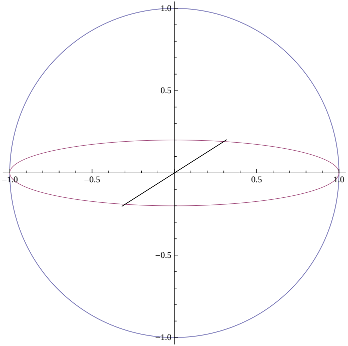
Local model - $(\mathbb{R}^{2n}, \omega_0)$ $\qquad p = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$

$$\omega_0 = \sum_{k=1}^{n} dx_k \wedge dy_k$$

- $d\omega_0 = 0$
- $\omega_0^n = n!\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \neq 0$ - volume form

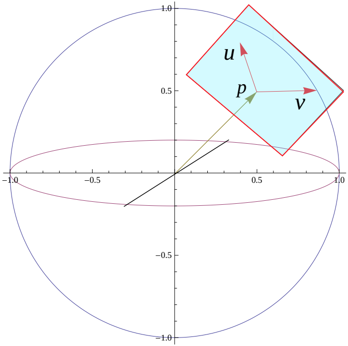
Symplectic manifolds: example

Example: The sphere $S^2 = \{p \in \mathbb{R}^3 : ||p|| = 1\}$



Symplectic manifolds: example

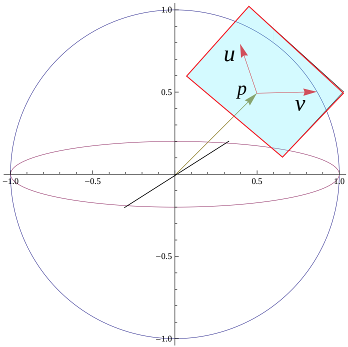
Example: The sphere $S^2 = \{p \in \mathbb{R}^3 : ||p|| = 1\}$



$u, v \in T_pS^2 = \{p\}^\perp$

Symplectic manifolds: example

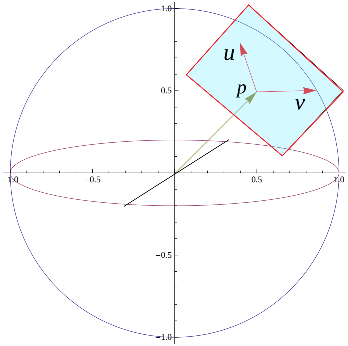
Example: The sphere $S^2 = \{p \in \mathbb{R}^3 : ||p|| = 1\}$



$u, v \in T_pS^2 = \{p\}^\perp \qquad \omega_p(u, v) := \langle p, u \times v\rangle = \det(p, u, v).$

Symplectic manifolds: example

$$\omega_p(u, v) := \langle p, u \times v \rangle = \det(p, u, v)$$



- $d\omega = 0$
- non-degenerate:

$$\omega_p(u, u \times p) = \langle p, \underbrace{u \times (u \times p)}_{/\!/ p.} \rangle \neq 0 \text{ when } u \neq 0$$

Symplectic manifolds with symmetries

G: Lie group

(M, ω): symplectic manifold

Symplectic manifolds with symmetries

G: Lie group

(M, ω): symplectic manifold

$G \curvearrowright (M, \omega)$: $\exists$ group homomorphism

$$G \longrightarrow Sympl(M, \omega)$$

Symplectic manifolds with symmetries

G: Lie group

(M, ω): symplectic manifold

$G \curvearrowright (M, \omega)$: $\exists$ group homomorphism

$$G \longrightarrow Sympl(M, \omega)$$

Assumptions:

For now, $G = S^1$, M compact.

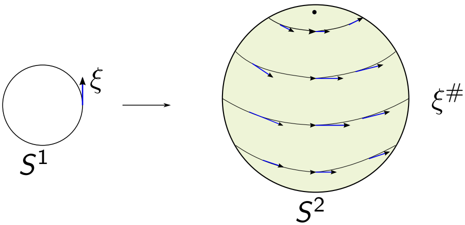
Symplectic S^1-actions

Example:
$S^1 \curvearrowright (S^2, \omega)$

$\xi^{\#}$: vector field associated to the 1-parameter flow of symplectomorphisms.

Example:
$S^1 \curvearrowright (S^2, \omega)$

$\xi^{\#}$: vector field associated to the 1-parameter flow of symplectomorphisms.



Symplectic S^1-actions

Flow of symplectomorphisms $\Longrightarrow$

Symplectic S^1-actions

Flow of symplectomorphisms $\implies \iota_{\xi^\#}\omega$ is **closed**.

Symplectic S^1-actions

Flow of symplectomorphisms $\implies \iota_{\xi^\#}\omega$ is **closed**.

$$0 = \mathcal{L}_{\xi^\#}\omega = \iota_{\xi^\#}d\omega + d(\omega(\xi^\#, \cdot))$$

Hamiltonian S^1-actions

Flow of symplectomorphisms $\implies$ $\iota_{\xi^\#}\omega$ is **closed**.

Hamiltonian S^1-actions

Flow of symplectomorphisms $\implies$ $\iota_{\xi^\#}\omega$ is **closed**.

- **Hamiltonian**: $\iota_{\xi^\#}\omega$ is **exact**,

Hamiltonian S^1-actions

Flow of symplectomorphisms $\implies$ $\iota_{\xi^\#}\omega$ is **closed**.

- **Hamiltonian**: $\iota_{\xi^\#}\omega$ is **exact**, i.e. $H \in C^\infty(M)$ s.t.

$$\iota_{\xi^\#}\omega = dH.$$

Hamiltonian S^1-actions

Flow of symplectomorphisms $\implies$ $\iota_{\xi^\#}\omega$ is **closed**.

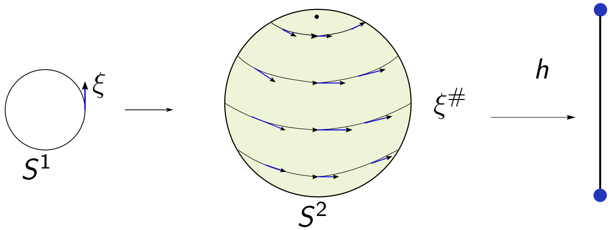
- **Hamiltonian**: $\iota_{\xi^\#}\omega$ is **exact**, i.e. $H \in C^\infty(M)$ s.t.

$$\iota_{\xi^\#}\omega = dH.$$

$H\colon M \to \mathbb{R}$ - **moment map**

$$S^1 \curvearrowright (S^2, \omega), \quad \omega = d\theta \wedge dh$$

$$\xi^\# = \frac{\partial}{\partial \theta}, \quad \iota_{\xi^\#}\omega = \omega(\frac{\partial}{\partial \theta}, \cdot) = dh$$



Hamiltonian S^1-actions

(M, ω) compact symplectic manifold.
$H : M \to \mathbb{R}$ Hamiltonian function = moment map
Hamiltonian vector-field $\iota_\xi \omega = dH$ that generates a circle action, i.e. the corresponing flow is 2π-periodic.

$\Rightarrow (M, \omega, H)$ is called **Hamiltonian S^1-space**

(We will always assume the group action to be effective.)

Symplectic reduction

Theorem - Marsden-Weinstein and Meyer

(M, ω, μ), compact Lie group G.
Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. G acts freely on $\mu^{-1}(0)$. Then

- the orbit space $M_{red} := \mu^{-1}(0)/G$ is a symplectic manifold with symplectic form ω_{red} defined by $i * \omega = \pi * \omega_{red}$, where $\pi : \mu^{-1}(0) \to M_{red}$

(M_{red}, ω_{red}) is called the reduced space

Toric action

Convexity Theorem - Atiyah, Guillemin and Sternberg '82

- (M, ω) compact, connected symplectic manifold.
- $\mathbb{T}^k \curvearrowright (M, \omega)$ Hamiltonian $(\mathbb{T}^k = \overbrace{S^1 \times \cdots \times S^1}^{k\,-\text{times}})$
- $\mu : M \to \mathbb{R}^k$ - Moment map

Then $\mu(M)$ is a convex polytope of dimension k

Symplectic toric manifolds

- "Best case": $\dim(\mathbb{T}^k) = k = \frac{\dim(M)}{2}$.

Symplectic toric manifolds

- "Best case": $\dim(\mathbb{T}^k) = k = \frac{\dim(M)}{2}$.
- (M, ω, μ) is called a **symplectic toric manifold**, and $\mu(M) =: \Delta$ a **Delzant polytope**:

Symplectic toric manifolds

- "Best case": $\dim(\mathbb{T}^k) = k = \frac{\dim(M)}{2}$.
- (M, ω, μ) is called a **symplectic toric manifold**, and $\mu(M) =: \Delta$ a **Delzant polytope**:

1. Δ **simple**: exactly n edges meeting at each vertex.
2. Δ **rational**: edges meeting at each v: $e_i^v = \{v + tu_i^v \mid u_i^v \text{ primitive in } \mathbb{Z}^n, t \in [0, l(e_i^v)]\}$.
3. Δ **smooth**: rational and $\mathbb{Z}\langle u_1^v, \ldots, u_n^v \rangle = \mathbb{Z}^n$ for all v.

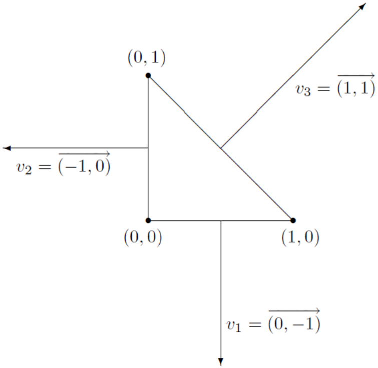
Symplectic toric manifolds

- "Best case": $\dim(\mathbb{T}^k) = k = \frac{\dim(M)}{2}$.
- (M, ω, μ) is called a **symplectic toric manifold**, and $\mu(M) =: \Delta$ a **Delzant polytope**:

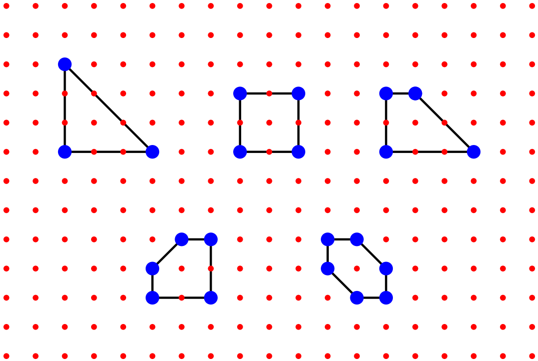
1. Δ **simple**: exactly n edges meeting at each vertex.
2. Δ **rational**: edges meeting at each v: $e_i^v = \{v + tu_i^v \mid u_i^v$ primitive in $\mathbb{Z}^n, t \in [0, l(e_i^v)]\}$.
3. Δ **smooth**: rational and $\mathbb{Z}\langle u_1^v, \ldots, u_n^v \rangle = \mathbb{Z}^n$ for all v.

Delzant '88: Every Delzant polytope Δ has an associated symplectic toric manifold (M, ω, μ), i.e. $\mu(M) = \Delta$.

Delzant Polytopes

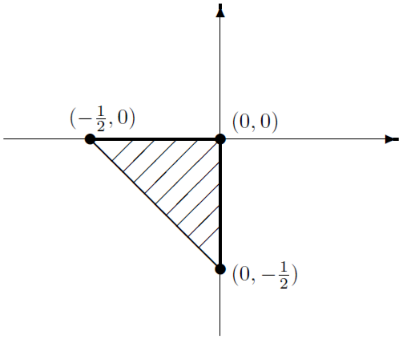


Delzant Polytopes



Delzant Polytope

$\mathbb{T}^2 \curvearrowright (\mathbb{CP}^2, \omega_{FS})$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2]$,
$\mu[z_0, z_1, z_2] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}\right)$



Hamiltonian S^1-action, dim=4

(M, ω, H) Hamiltonian S^1-space, $dim(M) = 4$

Neighbourhood of a fixed point

Let $p \in M$ be a fixed point. There exist complex coordinates z, w on a nbhd of p in M and unique integers m and n, such that

- the circle action is $\lambda \cdot (z, w) = (\lambda^m z, \lambda^n w)$
- the symplectic form is $\omega = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$
- the moment map is $H(z, w) = H(p) + \frac{m}{2}|z|^2 + \frac{n}{2}|w|^2$

Hamiltonian S^1-action, dim=4

(M, ω, H) Hamiltonian S^1-space, $dim(M) = 4$

Neighbourhood of a fixed point

Let $p \in M$ be a fixed point. There exist complex coordinates z, w on a nbhd of p in M and unique integers m and n, such that

- the circle action is $\lambda \cdot (z, w) = (\lambda^m z, \lambda^n w)$
- the symplectic form is $\omega = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$
- the moment map is $H(z, w) = H(p) + \frac{m}{2}|z|^2 + \frac{n}{2}|w|^2$

$\Rightarrow$for $m, n \neq 0$ moment map is Morse function.
m, n are called the **isotropy weights**.

Hamiltonian S^1-action, dim=4

Each component of the fixed point set is either a single point or a symplectic surface. The max and min of the moment map is each attained on exactly one component of the fixed point set.

Hamiltonian S^1-action, dim=4

Each component of the fixed point set is either a single point or a symplectic surface. The max and min of the moment map is each attained on exactly one component of the fixed point set.

For each $k \geq 2$ consider the set of points whose stabilizer is $Z_k = \{\lambda \in S^1 | \lambda^k = 1\}$.
Each connected component of the closure of this set is a closed symplectic 2-sphere, on which the quotient circle, S^1/Z_k, acts with two fixed points.
Such a sphere is called Z_k-**sphere**.

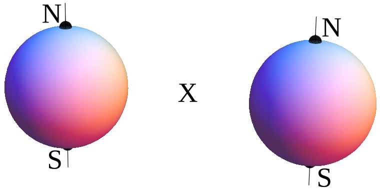
(We restrict to only isolated fixed points.)

Construction of the Graph

- isolated fixed point $\Rightarrow$ vertex, label: value of moment map
- Z_k-sphere $\Rightarrow$ edge connecting corresponding vertices, label: isotropy weight k
- label: cohomology class of symplectic form

Symplectic manifolds with symmetries

Example: $G = S^1 \curvearrowright (S^2 \times S^2, \omega \oplus \omega)$ by $\lambda(u, v) = (\lambda u, \lambda^2 v)$



4 **Fixed points**: $S \times S$, $S \times N$, $N \times S$ and $N \times N$.

Classification

Uniqueness Theorem

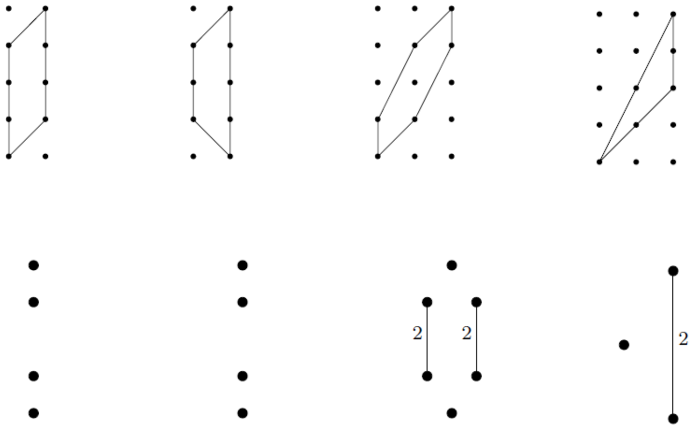
Let (M, ω, ψ) and (M', ω', ψ') be two compact four dimensional Hamiltonian S^1 spaces. Then any isomorphism between their corresponding graphs is induced by an equivariant symplectomorphism.

Classification

Existence Theorem

Every four dimensional, compact Hamiltonian S1-space with isolated fixed points comes from a toric action by restricting the action to a sub-circle

Classification



Symplectic blow up

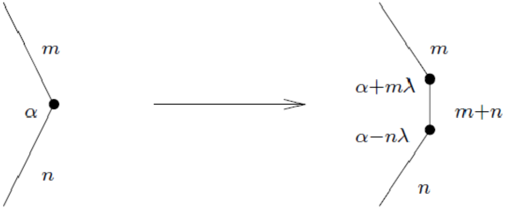
Which spaces occur?

Symplectic blow up

Which spaces occur? $\Rightarrow$ Symplectic blow up:

Symplectic blow up

Which spaces occur? $\Rightarrow$ Symplectic blow up:

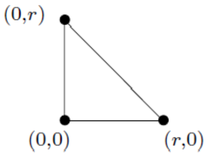


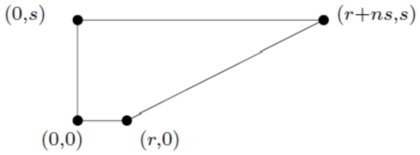
Symplectic blow up

Theorem

Every compact 4-dimensional Hamiltonian S^1 space can be obtained by a sequence of S^1-equivariant symplectic blow ups from $\mathbb{CP}^2$ or a Hirzebruch surface, with a symplectic form and a circle action that comes from a toric action.

Delzant polygon of $\mathbb{CP}^2$ and Hirzebruch surface





Obrigada!