On the Yau-Tian-Donaldson conjecture for spherical varieties

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Plan:

1) YTD conjecture
2) Spherical varieties
3) Main results

https://arxiv.org/abs/2009.06463
https://arxiv.org/abs/2011.07135

1) The Yau-Tian-Donaldson conjecture

$$\Big[\ \exists \text{ cscK in } c_1(L) \iff (X,L) \text{ is K-stable}$$ algebro geometric

Central pbm in Kähler geometry:

given a compact complex mfd X

$[\omega]$ Kähler class on X

$\exists$? a constant scalar curvature Kähler metric in $[\omega]$

Recall: ω Kähler form $\iff g(\cdot,\cdot)=\omega(\cdot,J\cdot)$

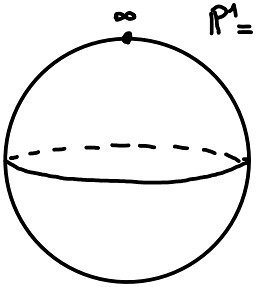
$\Updownarrow$

real 2-form on X which can be written, in local holom coords as

$$\omega = \sum_{j,k} \underbrace{\frac{\partial^2\varphi}{\partial z_j \partial \bar z_k}}_{\text{positive definite}} i\, dz_j \wedge d\bar z_k =: i\partial\bar\partial\varphi$$

$X \supset U$

$\varphi : U \to \mathbb{R}$



$\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$

e.g. Fubini Study metric on $\mathbb{P}^n$

$$\omega = i\partial\bar\partial \ln(1 + |z_1|^2 + \dots + |z_n|^2)$$

on affine chart $[1 : z_1 : \dots : z_n]$

$\omega' \in [\omega]$ if $\omega' - \omega = i\partial\bar\partial\psi$ $\quad\psi : X \to \mathbb{R}$

Ricci curvature: $\mathrm{Ric}(\omega)$ global real 2-form defined locally by $\mathrm{Ric}(\omega) = i\partial\bar\partial\left(-\ln\det\left(\frac{\partial^2\varphi}{\partial z_j \partial \bar z_k}\right)\right)$

$\rightsquigarrow$ Kähler Einstein metric: $\mathrm{Ric}(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$

e.g. Fubini Study metric

Scalar curvature: $S(\omega) = \dfrac{n\,\mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}$

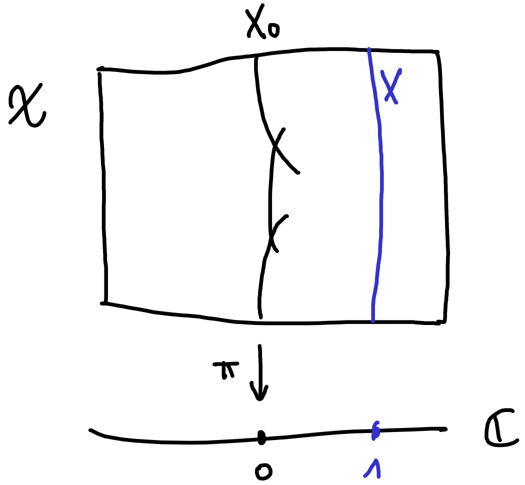
function $X \to \mathbb{R}$

$\hookrightarrow$ cscK metric $\quad S(\omega) \equiv n\dfrac{c_1(X)\cdot[\omega]^{n-1}}{[\omega]^n}$ cohomological constant.

K-stability

$[\omega] = c_1(L)$ L ample line bundle

in part. X is projective.



Test configuration (TC)

$(\mathcal{X}, \mathcal{L})$ flat $\mathbb{C}^*$-equivariant family $\pi : \mathcal{X} \longrightarrow \mathbb{C}$ st

$\mathcal{L}$ π-ample line bundle on $\mathcal{X}$

and $(\mathcal{X}_1, \mathcal{L}_1) = \pi^{-1}(1) \simeq (X, L^r)$ for some $r \in \mathbb{N}^*$

central fiber $(\mathcal{X}_0, \mathcal{L}_0)$ equipped with $\mathbb{C}^*$ action

can be non reduced

have several components

Product TC: when $\mathcal{X}_0 \simeq X$.

Special TC: when $\mathcal{X}_0$ is normal variety.

Donaldson-Futaki invariant

$$H^0(\mathcal{X}_0, \mathcal{L}_0^k) = \bigoplus_{i=1}^{d_k} \mathbb{C}_{\underbrace{\lambda_{i,k}}}$$

as $\mathbb{C}^*$ representation

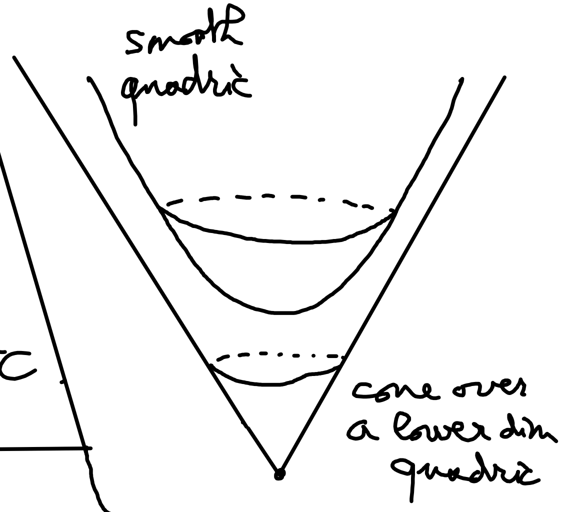
weight of the action of $\mathbb{C}^*$ on $\mathbb{C}_{\lambda_{i,k}}$

$z \cdot s = z^{\lambda_{i,k}} s$

$$\frac{\sum_i \lambda_{i,k}}{k\, d_k} = F_0 + F_1 k^{-1} + o(k^{-1})$$

$$DF(\mathcal{X}, \mathcal{L}) := -F_1 .$$

example of special TC



Defn: (X, L) is K-stable if

$$DF(\mathcal{X}, \mathcal{L}) \geqslant 0 \quad \text{for all TC}$$

$$= 0 \quad \text{iff product TC}$$

Rem: when a group G acts on X, one can consider only G-equivariant test configuration

Best partial results on the YTD conjecture:

i) Fano case

Chen Donaldson Sun ~2015

$\exists$KE on a Fano mfd $\Longleftrightarrow$ $(X, c_1(X))$ is K-stable wrt special TC

ii) Donaldson 2009: YTD conj holds for toric surfaces

+ simplification & convex geometric translation of the K-stability condition.

2) Spherical varieties

a) Toric manifolds

X is toric if it admits an effective holomorphic action of a compact torus $(S^1)^n$ where $n = \dim_{\mathbb{C}} X$.

Key property: (X^n, L) polarized toric mfd $\Longleftrightarrow$ (Delzant) integral convex polytope Δ in $\mathbb{R}^n$ (up to translation)

+ integral points in $k\Delta$ encode $H^0(X, L^k)$ as a $(S^1)^n$-representation

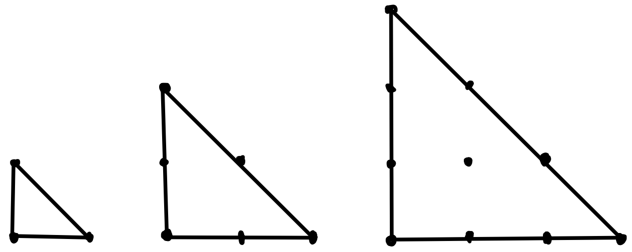
$(\mathbb{C}^*)^n$-representation

e.g.

- $\mathbb{C}P^2$

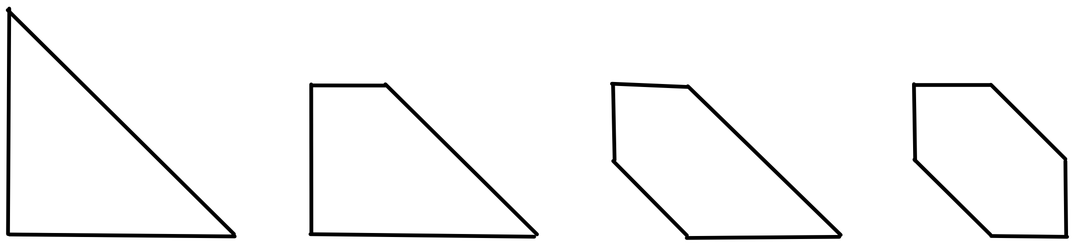
  $\circlearrowright$

  $(S^1)^2$ by choice of a affine chart



$H^0(\mathbb{P}^2, \mathcal{O}(k))$

- can always blow up codim ≥ 2 orbits $(\mathbb{C}^*)^n$ from a toric mfd



Rem: a proj mfd of $\dim_{\mathbb{C}} X = n$ cannot admit an effective action of $(S^1)^{n+1}$.

b) Cohomogeneity one manifolds

K compact real Lie grp

if $K \circlearrowright X^n$ with orbits of real dimension $2n$, X is homogeneous

$\longrightarrow$ cscK in all Kähler classes

if $K \circlearrowright X^n$ admits at least one orbit of real dim $2n-1$

$\rightsquigarrow$ "cohomogeneity one manifold"

→ Classification by Akhiezer | Huckleberry & Snow

$\exists \tilde{X} \longrightarrow X$ K-equiv blowup st

$\tilde{X}$ is a K-homogeneous bundle over a homogeneous K-mfd with fiber a "primitive" cohomogeneity 1 mfd.

Primitive cases:

- $\mathbb{P}^1 \circlearrowleft S^1$ → Hirzebruch surfaces e.g. $Bl_{pt}\mathbb{P}^2 \circlearrowleft SU(2)$
- $\mathbb{P}^n \times (\mathbb{P}^n)^* \circlearrowleft SU(n+1)$
- Q^n projective quadric $\circlearrowleft SO(n+1)$
- ...

all almost-homogeneous under the action $K^{\mathbb{C}}$

can have as open orbit all complexified rk 1 symmetric spaces among the primitive cases.

Theorem B

(X, L) cohomogeneity one is cscK iff it is K-stable wrt its (essentially unique) equivariant special test configuration.

c) Strategy for the proof

→ Differential geometric approach using Chen-Cheng

→ via uniform K-stability

Chi Li + Yuji Odaka

⇓

"uniform" YTD holds for spherical manifolds

$\cup$

toric mfds, cohomogeneity-1 mfds

Defn:

$X \circlearrowleft G$ complex connected reductive group

is <u>spherical</u> if a Borel subgroup $B \subset G$ acts with an open dense orbit on X.

3) Main results

toric mfd	spherical mfd
(X^n, L) $\circlearrowleft$ $T = (\mathbb{C}^*)^n$	(X^n, L) $\circlearrowleft$ G complex reductive group $\cup$ T maximal torus
• $\Delta \subset \mathbb{R}^n$ convex polytope (moment polytope) • $\mathfrak{X}(T) =$ lattice of characters of T $\simeq \mathbb{Z}^n$	• $\Delta_+ \subset \mathfrak{X}(T) \otimes \mathbb{R}$ of dimension $r \leq n$ • M lattice $\subset$ direction(Δ_+) • $\mathcal{V}$ valuation cone $\subset$ direction$(\Delta_+)^*$

Theorem A

toric	spherical
	choose $\chi \in \Delta_+ \cap M$, set $\Delta = -\chi + \Delta_+$
TC ‖ piecewise linear rational convex function f on Δ	piecewise linear rational convex function f on Δ, with slopes in $-\mathcal{V}$.
$DF(f) = \int_{\partial\Delta} f \, d\sigma - 2a \int_\Delta f \, d\mu$	$\int_{\partial\Delta} f P \, d\sigma - \int_\Delta 2(aP - Q) f \, d\mu$
	$P(x) = \prod_{\alpha \in R^+ \setminus \Delta_+^\perp} \frac{\langle \alpha, x+\chi \rangle}{\langle \alpha, \rho \rangle}$ Duistermaat–Heckman polynomial
	$Q(x) = d_x P(\rho)$
	$a > 0$ st $DF(1) = 0$.

Theorem A

toric	spherical
K-stable: $DF(f) \geqslant 0$, $= 0$ iff f linear	$DF(f) \geqslant 0$, $= 0$ iff $f \in Lin(V)$
uniformly K-stable: $\exists\, \varepsilon > 0$ st $DF(f) \geqslant \varepsilon \inf_{\ell \text{ linear}} \int_\Delta (f + \ell - \inf(f+\ell))\, d\mu$	$\exists\, \varepsilon > 0$ st $DF(f) \geqslant \varepsilon \inf_{\ell \in Lin(V)} \int_\Delta (f + \ell - \inf(f+\ell))\, P\, d\mu$

cohomogeneity one manifold $\Longleftrightarrow$ spherical with $\dim(\Delta_+)=1$

$$\Delta = [s_-, s_+] \subset \mathbb{R} \qquad f : [s_-, s_+] \longrightarrow \mathbb{R}$$

$$DF(f) = f(s_-)P(s_-) + f(s_+)P(s_+) - \int_{s_-}^{s_+} 2f(t)\,(aP(t) - Q(t))\,dt$$

Theorem B

$KG(X,L)$ cohomogeneity one admits cscK iff

$$\begin{cases} DF(t)=0 & \text{if } 3 \text{ orbits under } K^{\mathbb{C}} \\ DF(t)>0 & \text{if } 2 \text{ orbits under } K^{\mathbb{C}} \end{cases}$$

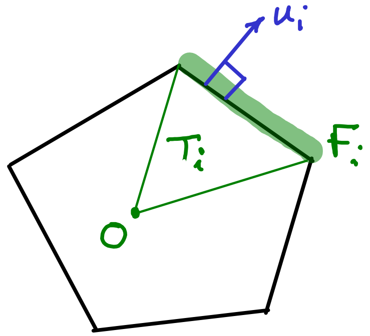
primitive

$\mathbb{P}^1 \supset \mathbb{C}^*$

all other cases

A general combinatorial condition

Decompose Δ into pyramids T_i with base the facet F_i
vertex the origin 0.



$$\Delta = \{x \mid u_i(x) \leqslant n_i\}$$

Theorem C

Assume $\forall i$,

$$\frac{1}{n_i}\left(d_x P(x) + (n+1)P(x)\right) + 2Q(x) - 2a\,P(x) \geqslant 0 \quad \forall x \in T_i$$

Then (X, L) cscK iff K-stable wrt special TC

Why is it useful?

in toric case, (inspired by Zhou Zhu 2008)

all $n_i = 1 \iff \Delta$ reflexive $\iff X$ Fano and $L = K_X^{-1}$

then $2a =$ scalar curvature $= n \dfrac{c_1(X)\cdot c_1(X)^{n-1}}{c_1(X)^n} = n$

thus $\dfrac{n+1}{n_i} - 2a > 0$

$+$ this condition is <u>open</u> as the Kähler class varies.

Expectation: condition holds on neighborhood of $c_1(X)$ always

Theorem D: Proof for some families of spherical mflds