The canonical map and the canonical ring of algebraic curves.

LisMath Seminar.

Vicente Lorenzo García

November 8th, 2017

Outline

Outline

Definition

We will use the word curve to mean a complete, nonsingular, one dimensional scheme C over the complex numbers $\mathbb{C}$.

Definition

The canonical sheaf Ω_C of a curve C is the sheaf of sections of its cotangent bundle.

Theorem (Serre's Duality)

Let $\mathcal{L}$ be an invertible sheaf on a curve C. Then there is a perfect pairing

$$H^0(C, \Omega_C \otimes \mathcal{L}^{-1}) \times H^1(C, \mathcal{L}) \to \mathbb{K}.$$

In particular, $h^1(C, \mathcal{L}) = h^0(C, \Omega_C \otimes \mathcal{L}^{-1})$.

Theorem (Riemann-Roch)

Let $\mathcal{L}$ be an invertible sheaf on a curve C of genus g. Then,

$$h^0(C, \mathcal{L}) - h^0(C, \Omega_C \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} + 1 - g.$$

Definition

An invertible sheaf $\mathcal{L}$ on C is said to be generated by global sections if for any point $P \in C$ there is a global section of $\mathcal{L}$ not vanishing at P.

If $\mathcal{L}$ is an invertible sheaf generated by global sections, a basis $s_0, \ldots, s_n$ of $H^0(C, \mathcal{L})$ has no common zeros and induces a map to projective space:

$$C \to \mathbb{P}^n, P \mapsto (s_0(P) : \ldots : s_n(P)).$$

Definition

If the previous map is a closed embedding we say that $\mathcal{L}$ is very ample.

Theorem

Let $\mathcal{L}$ be an invertible sheaf on a curve C. Then:

i) *$\mathcal{L}$ is generated by global sections if and only if for every $P \in C$,*

$$h^0(C, \mathcal{L}(-P)) = h^0(C, \mathcal{L}) - 1.$$

ii) *$\mathcal{L}$ is very ample if and only if for every $P, Q \in C$,*

$$h^0(C, \mathcal{L}(-P-Q)) = h^0(C, \mathcal{L}) - 2.$$

Definition

A curve C of genus $g \geq 2$ is hyperelliptic if there exists a degree 2 morphism $C \to \mathbb{P}^1$.

Theorem

The canonical sheaf Ω_C is always generated by global sections. Moreover, it is very ample if and only if C is not hyperelliptic.

Let C be a non-hyperelliptic curve of genus $g \geq 3$. Our aim is to study its canonical ring:

$$\bigoplus_{n\geq 0} H^0(C, \Omega_C^{\otimes n}).$$

To do so, we are going to consider the map:

$$\varphi : S^* H^0(C, \Omega_C) \to \bigoplus_{n\geq 0} H^0(C, \Omega_C^{\otimes n}).$$

We are going to see that it is surjective and we are going to study its kernel I.

Remark

If C is non-hyperelliptic, Ω_C is very ample and it induces an embedding

$$C \hookrightarrow \mathbb{P}^{g-1}.$$

The homogeneous coordinate ring of C for this embedding is precisely $\bigoplus_{n\geq 0} H^0(C, \Omega_C^{\otimes n})$.

Outline

Lemma

Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on a curve C such that $H^0(C,\mathcal{L}), H^0(C,\mathcal{M}) \neq 0$. Let V be the image of the map

$$H^0(C,\mathcal{L}) \otimes H^0(C,\mathcal{M}) \to H^0(C,\mathcal{L}\otimes\mathcal{M}).$$

Then

$$\dim V \geq h^0(C,\mathcal{L}) + h^0(C,\mathcal{M}) - 1.$$

Theorem (Clifford)

Let $\mathcal{L}$ be an invertible sheaf on a curve C such that $0 \leq \deg\mathcal{L} \leq 2g-2$. Then

$$2(h^0(C,\mathcal{L}) - 1) \leq \deg\mathcal{L}.$$

Furthermore, equality holds if and only if

(i) $\mathcal{L} = \mathcal{O}_C$ *or* $\mathcal{L} = \Omega_C$.

(ii) C *is hyperelliptic and* $\mathcal{L}$ *is* $\frac{1}{2}(\deg\mathcal{L})$*-times the invertible sheaf induced by the unique linear system of dimension* 1 *and degree* 2 *on* C.

Lemma (Base point free pencil trick)

Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on a curve C and s_1 and s_2 two global sections of $\mathcal{L}$ having no common zeros. If V is the subspace of $H^0(C,\mathcal{L})$ generated by s_1 and s_2, then the kernel of the map

$$V \otimes H^0(C,\mathcal{M}) \to H^0(C,\mathcal{L}\otimes\mathcal{M})$$

is isomorphic to $H^0(C,\mathcal{M}\otimes\mathcal{L}^{-1})$.

Lemma (Castelnuovo)

Let $\mathcal{F}$ be a coherent sheaf on a curve C and $\mathcal{L}$ an invertible sheaf on C generated by global sections, such that $H^1(C,\mathcal{F}\otimes\mathcal{L}^{-1}) = 0$. Then the map

$$H^0(C,\mathcal{F}) \otimes H^0(C,\mathcal{L}) \to H^0(C,\mathcal{F}\otimes\mathcal{L})$$

is surjective.

Outline

Let C be a non-hyperelliptic curve of genus 3.

By Riemann-Roch's Theorem,

$$h^0(C,\Omega_C) = 3,$$
$$h^0(C,\Omega_C^{\otimes n}) = 4n-2, \qquad \forall n>1.$$

Now, if $P \in C$ then,

$$H^0(C,\Omega_C(-2P)) \subset H^0(C,\Omega_C(-P)) \subset H^0(C,\Omega_C)$$

and

$$h^0(C,\Omega_C(-2P)) = 1,$$
$$h^0(C,\Omega_C(-P)) = 2,$$
$$h^0(C,\Omega_C) = 3,$$

so we can find a basis $\{r,s,t\}$ of $H^0(C,\Omega_C)$ such that,

$$\mathrm{ord}_P(r) = 2,$$
$$\mathrm{ord}_P(s) = 1,$$
$$\mathrm{ord}_P(t) = 0.$$

Having said that, we consider the map

$$H^0(C,\Omega_C(-P))\otimes H^0(C,\Omega_C)\to H^0(C,\Omega_C^{\otimes 2}(-P))$$

By the base point free pencil trick, it is surjective. Hence,

$$H^0(C,\Omega_C^{\otimes 2}(-P))=\langle r^2,rs,rt,s^2,st\rangle.$$

Now,

$$\left.\begin{array}{l} h^0(C,\Omega_C^{\otimes 2})=6 \\ t^2\in H^0(C,\Omega_C^{\otimes 2})\setminus H^0(C,\Omega_C^{\otimes 2}(-P)) \end{array}\right\}\Rightarrow H^0(C,\Omega_C^{\otimes 2})=\langle r^2,rs,rt,s^2,st,t^2\rangle.$$

On the other hand,

$$H^0(C,\Omega_C(-P))\otimes H^0(C,\Omega_C^{\otimes 2})\to H^0(C,\Omega_C^{\otimes 3}(-P))$$

is also surjective because of the base point free pencil trick. Hence,

$$H^0(C,\Omega_C^{\otimes 3}(-P))=\langle r^3,r^2s,r^2t,rs^2,rst,rt^2,s^3,s^2t,st^2\rangle$$

Now,

$$\left.\begin{array}{l} h^0(C,\Omega_C^{\otimes 3})=10 \\ t^3\in H^0(C,\Omega_C^{\otimes 3})\setminus H^0(C,\Omega_C^{\otimes 3}(-P)) \end{array}\right\}\Rightarrow H^0(C,\Omega_C^{\otimes 3})=\langle r^3,r^2s,r^2t,rs^2,rst, rt^2,s^3,s^2t,st^2,t^3\rangle.$$

Finally, the map

$$H^0(C, \Omega_C) \otimes H^0(C, \Omega_C^{\otimes (n-1)}) \to H^0(C, \Omega_C^{\otimes n})$$

is surjective for every $n \geq 4$ because of Castelnuovo's lemma.

Hence,

$$H^0(C, \Omega_C^{\otimes 4}) = \langle r^4, r^3 s, r^3 t, r^2 s^2, r^2 st, r^2 t^2, rs^3, rs^2 t, rst^2, rt^3, s^4, s^3 t, s^2 t^2, st^3, t^4 \rangle.$$

Since $h^0(C, \Omega_C^{\otimes 4}) = 14$,

$$F = A_1 \cdot r^4 + \ldots + A_{15} \cdot t^4$$

is zero in $H^0(C, \Omega_C^{\otimes 4})$ for some $A_1, \ldots, A_{15} \in \mathbb{K}$, not all zero.

Since

$$\dim \left(\frac{\mathbb{C}[r,s,t]}{(F)} \right)_n = 4n - 2 = h^0(C, \Omega_C^{\otimes n}), \qquad \forall n \geq 4,$$

it follows that,

$$\frac{\mathbb{C}[r,s,t]}{(F)} \simeq \bigoplus_{n \geq 0} H^0(C, \Omega_C^{\otimes n}).$$

Outline

Theorem (Max Noether)

If C is a non-hyperelliptic curve of genus $g \geq 4$, then the canonical map

$$\varphi : S^* H^0(C, \Omega_C) \to \bigoplus_{n \geq 0} H^0(C, \Omega_C^n)$$

is surjective.

There exist $P_3, \ldots, P_g \in C$ such that if we set $D = P_3 + \cdots + P_g$ then,

- $\Omega_C(-D)$ is generated by global sections.
- $h^0(C, \Omega_C(-D)) = 2$.

We can choose $P_1, P_2 \in C$ and a basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(C, \Omega_C)$ such that,

$$\begin{cases} \omega_j(P_j) \neq 0, \\ \omega_j(P_i) = 0 \quad \text{if} \quad i \neq j. \end{cases}$$

Moreover, $H^0(C, \Omega_C(-D)) = \langle \omega_1, \omega_2 \rangle$.

Now, for every $i \geq 2$ the base point free pencil trick implies that the following map is surjective,

$$\Psi_i : H^0(C, \Omega_C(-D)) \otimes H^0(C, \Omega_C^{\otimes(i-1)}) \to H^0(C, \Omega_C^{\otimes i}(-D)).$$

We have that:

- $\omega_3^i, \dots, \omega_g^i \in H^0(C, \Omega_C^{\otimes i}) \setminus H^0(C, \Omega_C^{\otimes i}(-D))$.
- $\omega_3^i, \dots, \omega_g^i$ are linearly independent.
- $h^0(C, \Omega_C^{\otimes i}) - h^0(C, \Omega_C^{\otimes i}(-D)) = g - 2$.

Hence,

$$H^0(C, \Omega_C^{\otimes i}) = \langle H^0(C, \Omega_C^{\otimes i}(-D)), \omega_3^i, \dots, \omega_g^i \rangle.$$

Therefore, the following map is surjective,

$$H^0(C, \Omega_C) \otimes H^0(C, \Omega_C^{\otimes(i-1)}) \to H^0(C, \Omega_C^{\otimes i}).$$

It follows that,

$$S^* H^0(C, \Omega_C) \to \bigoplus_{n \geq 0} H^0(C, \Omega_C^{\otimes n}).$$

is surjective. □

Outline

We know that,

$$H^0(C,\Omega_C^{\otimes 2}) = \langle \omega_1^2, \omega_1\omega_2, \omega_2^2, \omega_1\omega_3, \ldots, \omega_1\omega_g, \omega_2\omega_3, \ldots, \omega_2\omega_g, \omega_3^2, \ldots, \omega_g^2 \rangle.$$

Let $i,k \in \{3,\ldots,g\}$ be distinct. Then $\omega_i\omega_k \in H^0(C,\Omega_C^{\otimes 2})$ and therefore there exist $\lambda_{isk}, \mu_{isk}, b_{ik} \in \mathbb{C}$ such that:

$$\omega_i\omega_k = b_{ik}\omega_1\omega_2 + \sum_{s=3}^{g}(\lambda_{isk}\omega_1 + \mu_{isk}\omega_2)\omega_s \in H^0(C,\Omega_C^{\otimes 2}).$$

It follows that

$$f_{ik} := \omega_i \cdot \omega_k - b_{ik}\omega_1 \cdot \omega_2 - \sum_{s=3}^{g}(\lambda_{isk}\omega_1 + \mu_{isk}\omega_2) \cdot \omega_s \in S^* H^0(C,\Omega_C),$$

is in the kernel I of φ. Hence, the f_{ik}'s are $\frac{(g-2)(g-3)}{2}$ linearly independent elements in the $\frac{(g-2)(g-3)}{2}$-dimensional vector space I_2. So, I_2 is generated by the f_{ik}'s.

On the other hand,

$$W := \langle \omega_1^2\omega_3, \dots, \omega_1^2\omega_g, \omega_1\omega_2\omega_3, \dots, \omega_1\omega_2\omega_g, \omega_2^2\omega_3, \dots, \omega_2^2\omega_g, \omega_1^3, \omega_1^2\omega_2, \omega_1\omega_2^2, \omega_2^3 \rangle$$

is a $(3g-2)$-dimensional subspace of the $(3g-1)$-dimensional space $H^0(C, \Omega_C^{\otimes 3}(-2D))$.

Let us take $\eta \in H^0(C, \Omega_C^{\otimes 3}(-2D)) \setminus W$, so that:

$$H^0(C, \Omega_C^{\otimes 3}(-2D)) = \langle W, \eta \rangle.$$

For each $i \in \{3, \dots, g\}$ we can find $\alpha_i \in H^0(C, \Omega_C(-D))$ such that

$$\alpha_i \omega_i^2 \in H^0(C, \Omega_C^{\otimes 3}(-2D)) \setminus W.$$

Therefore, there exists $\theta_i \in W$ such that

$$\alpha_i \omega_i^2 = \eta + \theta_i.$$

It follows that given distinct $k, l \in \{3, \dots, g\}$,

$$G_{kl} := \alpha_k \omega_k \cdot \omega_k - \alpha_l \omega_l \cdot \omega_l + \theta_l - \theta_k \in S^* H^0(C, \Omega_C),$$

is in the kernel I of φ.It turns out that I_3 is generated by the $\omega_l \cdot f_{ik}$'s and the G_{kl}'s.

Now we are going to see that I_n can be reduced to the f_{ij}'s and the G_{kl}'s for every $n \geq 4$.

Firstly, by the base point free pencil trick we have that

$$\Theta_n : H^0(C, \Omega_C(-D)) \otimes H^0(C, \Omega_C^{\otimes(n-1)}((2-n)D)) \to H^0(C, \Omega_C^{\otimes n}((1-n)D))$$

is surjective for every $n \geq 4$. Hence, we can prove by induction on n that,

$$H^0(C, \Omega_C^{\otimes n}((1-n)D)) = \langle \omega_1^l \omega_2^m, \omega_1^s \omega_2^t \omega_i, \omega_1^h \omega_2^k \eta : \\ i \in \{3, \ldots, g\}, l+m=n, s+t=n-1, h+k=n-3 \rangle.$$

Now, for each $i \in \{1, \ldots, g\}$ let us choose $\beta_i \in H^0(C, \Omega_C(-D))$ such that $\{\alpha_i, \beta_i\}$ generates $H^0(C, \Omega_C(-D))$. In particular, $\mathrm{ord}_{P_i}(\beta_i) = 1$. For each $j \in \{2, \ldots, g\}$ we denote,

$$\mathcal{B}_j = \{\beta_3^{n-j} \omega_3^j, \ldots, \beta_g^{n-j} \omega_g^j\}.$$

Given $j \in \{2, \dots, n\}$, we have that:

- $\mathcal{B}_j \subseteq H^0(C, \Omega_C^{\otimes n}((j-n)D)) \setminus H^0(C, \Omega_C^{\otimes n}((j-n-1)D))$.
- $\mathcal{B}_j$ is a set with $g-2$ linearly independent elements.
- $h^0(C, \Omega_C^{\otimes n}((j-n)D)) - h^0(C, \Omega_C^{\otimes n}((j-n-1)D)) = g-2$.

Hence, for every $j \in \{2, \dots, n\}$,

$$H^0(C, \Omega_C^{\otimes n}((j-n)D)) = \langle H^0(C, \Omega_C^{\otimes n}((j-n-1)D)), B_j \rangle.$$

In particular,

$$H^0(C, \Omega_C^{\otimes n}) = \langle H^0(C, \Omega_C^{\otimes n}((1-n)D)), \mathcal{B}_2, \dots, \mathcal{B}_g \rangle.$$

This explicit basis allows us to eliminate generators of an arbitrary element of I_n to write it in terms of the f_{ij}'s and the G_{kl}'s.

We conclude that I is generated by the f_{ij}'s and the G_{kl}'s.

Theorem (Max Noether-Enriques-Petri)

Let C be a non-hyperelliptic curve of genus $g \geq 4$. Then:

(1) *The following map is surjective,*

$$\varphi : S^* H^0(C, \Omega_C) \to \bigoplus_{n \geq 0} H^0(C, \Omega_C^n).$$

(2) *The kernel I of φ is generated by its elements of degree 2 and of degree 3.*

(3) *I is generated by its elements of degree 2 except in the following cases:*

- (i) *C is a nonsingular plane quintic.*
- (ii) *C is a trigonal curve.*

Outline

Let C be a hyperelliptic curve of genus $g \geq 3$.

The degree two morphism $C \to \mathbb{P}^1$ is induced by a degree two invertible sheaf $\mathcal{L}$ such that $\Omega_C \simeq \mathcal{L}^{\otimes(g-1)}$. We have that,

$$h^0(C, \mathcal{L}^{\otimes n}) = \begin{cases} n+1 & \text{if } 1 \leq n \leq g-1, \\ 2n+1-g & \text{if } n \geq g. \end{cases}$$

By the base point free pencil trick, the map

$$\Psi_n : H^0(C, \mathcal{L}) \otimes H^0(C, \mathcal{L}^{\otimes(n-1)}) \to H^0(C, \mathcal{L}^{\otimes n})$$

is surjective for $n \neq g+1$ and its image has codimension 1 in the case $n = g+1$.

It follows that we may choose $x_1, x_2 \in H^0(C, \mathcal{L})$, $y \in H^0(C, \mathcal{L}^{\otimes(g+1)})$ such that $H^0(C, \mathcal{L}^{\otimes n})$ equals,

$$\begin{cases} \langle x_1^{n-j} x_2^j : 0 \leq j \leq n \rangle & \text{if } 1 \leq n \leq g, \\ \langle x_1^{n-j} x_2^j, x_1^{n-(g+1)-k} x_2^k y : 0 \leq j \leq n, 0 \leq k \leq n-(g+1) \rangle & \text{if } n \geq g+1. \end{cases}$$

Since $y^2 \in H^0(C, \mathcal{L}^{\otimes(2g+2)})$, we have that

$$F := y^2 - (A_1 \cdot x_1^{2g+2} + \cdots + A_{3g+5} \cdot y x_2^{g+1})$$

is zero for some $A_1, \ldots, A_{3g+2} \in \mathbb{C}$ not all zero.

Since

$$\dim\left(\frac{\mathbb{C}[x_1, x_2, y]}{(F)}\right)_n = 2n + 1 - g = h^0(C, \mathcal{L}^{\otimes n}), \qquad \forall n \geq g,$$

it follows that

$$\frac{\mathbb{C}[x_1, x_2, y]}{(F)} \simeq \bigoplus_{n \geq 0} H^0(C, \mathcal{L}^{\otimes n}).$$

On the other hand, we have that.

$$h^0(C, \Omega_C^{\otimes n}) = h^0(C, \mathcal{L}^{\otimes n(g-1)}) = \begin{cases} g & \text{if } n = 1, \\ (2n-1)g - (2n-1) & \text{if } n \geq 2. \end{cases}$$

Let us write,

- $\omega_i := x_1^{g-i} x_2^{i-1} \in H^0(C, \mathcal{L}^{\otimes(g-1)}) = H^0(C, \Omega_C)$, $1 \leq i \leq g$.
- $\alpha_i := x_1^{g-2-i} x_2^{i-1} y \in H^0(C, \mathcal{L}^{\otimes 2(g-1)}) = H^0(C, \Omega_C^{\otimes 2})$, $1 \leq i \leq g-2$.

Castelnuovo's lemma and the explicit description of the generators of $H^0(C, \mathcal{L}^{\otimes n})$ yield that,

$$\mathbb{C}[\omega_1, \ldots, \omega_g, \alpha_1, \ldots, \alpha_{g-2}] \to \bigoplus_{n \geq 0} H^0(C, \Omega_C^n)$$

is surjective.

Moreover, the kernel of this map contains the ideal I generated by:

- $\{x_1^{i_1} \cdots x_g^{i_g} \cdot F : i_1 + \cdots + i_g = 2(g-3)\} \subseteq \mathbb{C}[\omega_1, \ldots, \omega_g, \alpha_1, \ldots, \alpha_{g-2}]_4$.
- The 2×2 minors of the matrix

$$\begin{pmatrix} \omega_1 & \cdots & \omega_{g-1} & \alpha_1 & \cdots & \alpha_{g-3} \\ \omega_2 & \cdots & \omega_g & \alpha_2 & \cdots & \alpha_{g-2} \end{pmatrix}.$$

Evenmore, counting dimensions degree by degree, we see that I coincides with the kernel of this map and therefore,

$$\frac{\mathbb{C}[\omega_1, \ldots, \omega_g, \alpha_1, \ldots, \alpha_{g-2}]}{I} \simeq \bigoplus_{n \geq 0} H^0(C, \Omega_C^n).$$

Theorem

Let C be a hyperelliptic curve of genus $g \geq 3$. Then:

(1) *There exist a basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(C, \Omega_C)$ and $\alpha_1, \ldots, \alpha_{g-2} \in H^0(C, \Omega_C^{\otimes 2})$ such that the following map is surjective,*

$$\varphi : \mathbb{C}[\omega_1, \ldots, \omega_g, \alpha_1, \ldots, \alpha_{g-2}] \to \bigoplus_{n \geq 0} H^0(C, \Omega_C^n).$$

(2) *For each $i, j, k \in \{1, \ldots, g-2\}$ there exist homogeneous $f_2^{kij}, f_4^{ij} \in \mathbb{C}[\omega_1, \ldots, \omega_g]$ of degrees 2 and 4 respectively, such that the kernel I of φ is generated by:*

- *The polynomials of the form*

$$\alpha_i \alpha_j - \left(f_4^{ij} + \sum_{k=1}^{g-2} f_2^{kij} \alpha_k \right).$$

- *The 2×2 minors of the matrix*

$$\begin{pmatrix} \omega_1 & \cdots & \omega_{g-1} & \alpha_1 & \cdots & \alpha_{g-3} \\ \omega_2 & \cdots & \omega_g & \alpha_2 & \cdots & \alpha_{g-2} \end{pmatrix}.$$

References

- E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of Algebraic Curves*, Springer-Verlag, 1984.
- R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.
- B. Saint-Donat, *On petri's analysis of the linear system of quadrics through a canonical curve*, Mathematische Annalen, 206 (1973), pp. 157–175.

Thank you very much for your attention.