Introduction to foam evaluation

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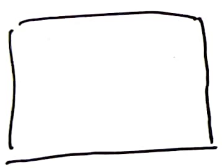
based on joint work with Louis-Hadrien Robert (arxiv 2018)

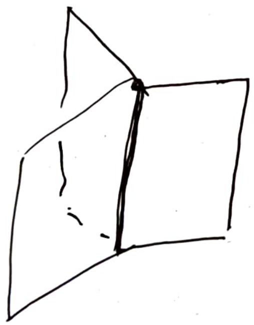
(Unoriented) $sl(3)$ foam F is 2D combinatorial (PL) compact CW-complex with generic singularities + embedding into $\mathbb{R}^3$ + dots on facets

3 types of points

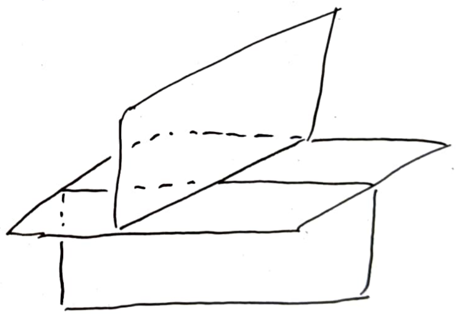
smooth



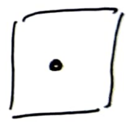
singular edge



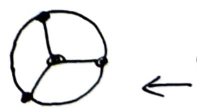
singular vertex



+ dots



link of vertex



cone over

Motivation: $\exists$ various bigraded link homology theories $H(L)$, L link, $L \subset \mathbb{R}^3$

$H(L)$ ← bigraded

$H(L) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(L)$

Best-behaved theories to date:

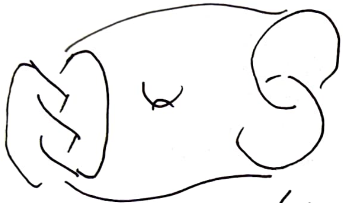
$\mathfrak{g} = \mathfrak{sl}(N)$, components labelled by $\Lambda^k V$ ← fund $\mathfrak{sl}(N)$ rep.

$\chi(H(L)) = \sum (-1)^i q^j \operatorname{rk} H^{i,j}(L) =$

$= P_{\mathfrak{g}}(L)$ ← Witten-Reshetikhin-Turaev invariant

$\mathfrak{g}$ - simple L.A

components of L colored by irreps of $\mathfrak{g}$



L_0 S L_1

$H(L_0) \xrightarrow{H(S)} H(L_1)$

functoriality

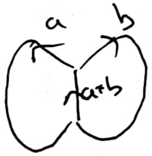
S - cobordism between L_0, L_1.

H: functor from cat. of link cobordisms to (graded) R-modules.

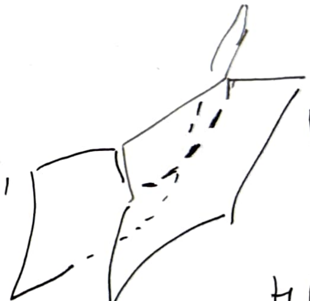
To construct H, reduce L to planar trivalent graphs (MOY, Murakami-Ohtsuki-Yamada graphs)

construct homology of planar graphs $H(\Gamma)$.

Build them into complexes, take homology again.



Γ_1



Γ_0

$H(\Gamma_0) \to H(\Gamma_1)$

Work of many people: Lev Rozansky, MK, Mackaay-Stosic-Vaz, Hao Wu, Yasuyoshi Yonezawa, ...

Matrix factorizations + foams

Recent (early 2017), Louis-Hadrien Robert, Emmanuel Wagner.

Foams only, evaluations. Avoids using MF or similar categories.

Reminiscent of 3D statistical mechanics;

foams (cobordisms between graphs) induce maps of homology groups.

$S(F)$ singular graph of F. 4-valent, circles

$f(F)$ set of facets of F. connected components of $F \setminus S(F)$

<u>Examples</u> 1) Surface S

$S(F) = \emptyset$

2) Θ-foam

$S(F) = S^1$

3 facets

all disks

3)

2 facets

$S(F) = S^1$

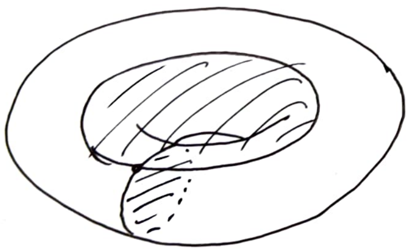
dots

n_1

n_2

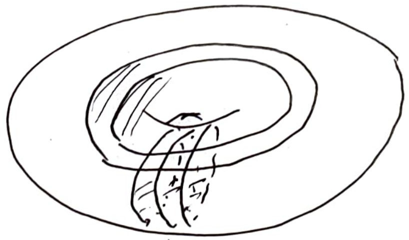
n_3

4)



singular vertex

m disks

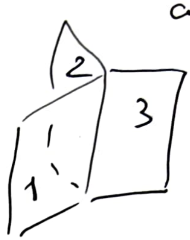


$\underbrace{\quad}_{n \text{ disks}}$

$\text{Torus}_{n,m}$

(admissible) coloring $c \in adm(F)$

map $f(F) \longrightarrow \{1, 2, 3\}$



along each seam the colors are distinct

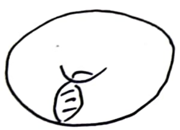
Examples 1) S - connected surface 3 colorings

2) Θ- foam

6 colorings

S_3 acts on $adm(F)$

3)



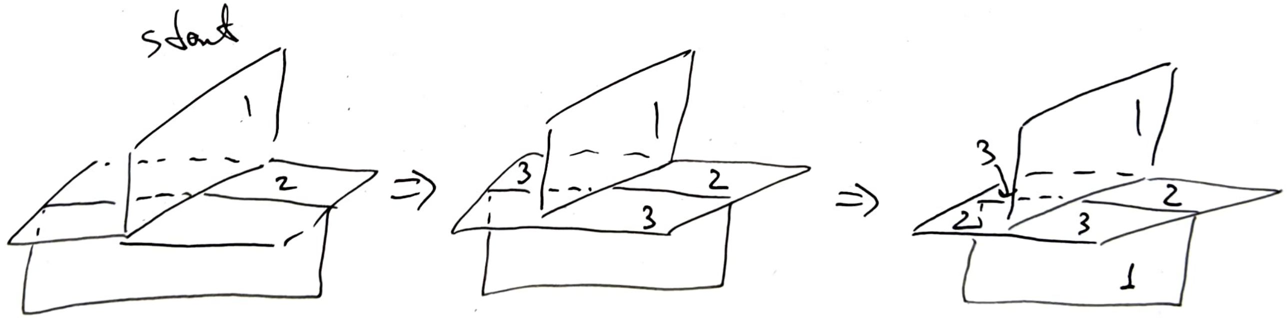
no admissible colorings

4)

no admissible colorings

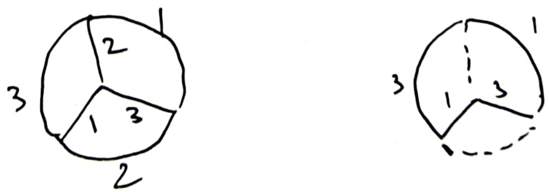
Exercise: determine admissible colorings for $\text{Torus}_{n,m}$

admissible coloring near a singular vertex



pairs of opposite corners carry the same coloring.

Unique up to permutation

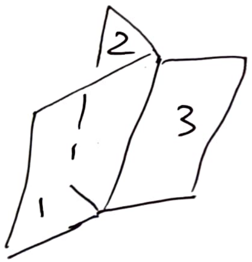


circle for $\forall$ 2 colors out of 3

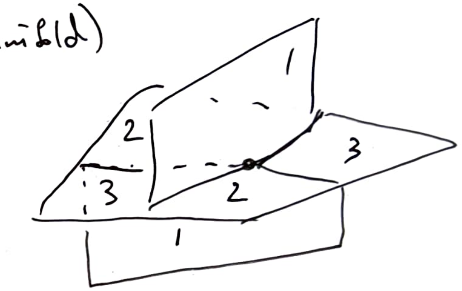
For $\{i,j\} \subset \{1,2,3\}$ & coloring $c \in adm(F)$ define $F_{ij}(c)$ = union of facets colored i or j

<u>Prop</u> $F_{ij}(c)$ is an orientable closed surface, $\chi(F_{ij}(c))$ -even

<u>Proof</u> $i,j = 1,2$ $F_{12}(c)$ 'smooth' (PL manifold) along seams



at vertices $\rightarrow$



$F_{12}(c) \subset \mathbb{R}^3 \Rightarrow$ orientable $\Rightarrow$ even $\chi(F_{12}(c))$

for each component, $\chi \in 2, 0, -2, -4 \dots$

S^2, T^2, genus 2 . .

χ 'mostly' negative

Rings: $\mathbb{k}$, char $\mathbb{k}=2$ a field ($\mathbb{F}_2$ is ok)

easier

$+ = -$

$$\underset{\substack{R\\ \|}}{\mathbb{k}[E_1,E_2,E_3]} \subset \underset{\substack{R'\\ \|}}{\mathbb{k}[x_1,x_2,x_3]} \subset \underset{\substack{R''\\ \|}}{R'\left[\frac{1}{x_i+x_j}\right]_{i<j}}$$

formal variables

$x_i+x_j = x_i - x_j$

$E_1 = x_1+x_2+x_3$

$E_2 = x_1x_2+x_1x_3+x_2x_3$

$E_3 = x_1x_2x_3$

$$\boxed{R \subset R' \subset R''}$$

$R = (R')^{S_3}$ symmetric functions

F - foam, $c \in \mathrm{adm}(F)$ coloring $\rightsquigarrow$ $\langle F, c\rangle \in R''$ evaluation of F at c.

$$\langle F, c\rangle = \frac{x_1^{d_1(c)}\, x_2^{d_2(c)}\, x_3^{d_3(c)}}{(x_1+x_2)^{\chi(F_{12}(c))/2}\,(x_1+x_3)^{\chi(F_{13}(c))/2}\,(x_2+x_3)^{\chi(F_{23}(c))/2}}$$

$d_i(c)$ # of dots on facets colored i

$\chi_{ij}(c) := \chi(F_{ij}(c))$

$$\langle F\rangle = \sum_{c\in \mathrm{adm}(F)} \langle F, c\rangle.$$

Thm $\langle F\rangle \in R$. (symmetric, polynomial)

Easy version of Robert–Wagner integrality theorem for $GL(N)$ foams (over $\mathbb{Z}$)

$\langle F\rangle$ symmetric, since S_3 acts on colorings, $\sigma(\langle F,c\rangle) = \langle F, \sigma(c)\rangle$

$\sigma \in S_3$

Examples 1) $F = S^2$  n S^2

$$\langle F \rangle = \sum_c \frac{\prod_{i=1}^{3} X_i^{d_i(c)}}{\prod_{i<j} (X_i + X_j)^{\chi_{ij}(c)/2}}$$

colouring c_1  n

		$\chi/2$
$d_1(c_1) = n$	$F_{12}(c_1) = S^2$	1
$d_2(c_1) = 0$	$F_{13}(c_1) = S^2$	1
$d_3(c_1) = 0$	$F_{23}(c_1) = \emptyset$	0

$$\langle F, c_1 \rangle = \frac{x_1^n}{(x_1+x_2)(x_1+x_3)}$$

$$\langle F \rangle = \frac{x_1^n}{(x_1+x_2)(x_1+x_3)} + \frac{x_2^n}{(x_1+x_2)(x_2+x_3)} + \frac{x_3^n}{(x_1+x_3)(x_2+x_3)} =$$

$$= \frac{x_1^n(x_2+x_3) + x_2^n(x_1+x_3) + x_3^n(x_1+x_2)}{(x_1+x_2)(x_1+x_3)(x_2+x_3)} = h_{n-2}(x_1, x_2, x_3)$$

complete symmetric function

$$h_k = \sum_{\alpha+\beta+\gamma=k} x_1^\alpha x_2^\beta x_3^\gamma$$

$n=0 \quad \langle F \rangle = 0$

$n=1 \quad \langle F \rangle = 0$

$n=2 \quad \langle F \rangle = 1 \in \mathbb{k}$

$n=3 \quad \langle F \rangle = x_1 + x_2 + x_3$

$n=4 \quad \langle F \rangle = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$

$h_k = \mathrm{char}(S^k V)$

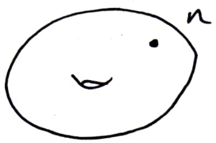
$\uparrow$

fund. $GL(3)$ representation

notice that denominators vanish from $\langle F \rangle$.

$$\langle F\rangle = \sum_{c} \frac{X_1^{d_1(c)} X_2^{d_2(c)} X_3^{d_3(c)}}{(x_1+x_2)^{\chi_{12}(c)/2}(x_1+x_3)^{\chi_{13}(c)/2}(x_2+x_3)^{\chi_{23}(c)/2}}$$

2) $F = T^2_n$



coloring c_1

	$\chi/2$
$F_{12} = T^2$	0
$F_{13} = T^2$	0
$F_{23} = \emptyset$	0

no denominators!

$\langle F, c_1\rangle = x_1^n$

$\langle F\rangle = x_1^n + x_2^n + x_3^n$

S

no denominators either for higher genus $\chi(S) = 2-2g < 0$, $g > 1$

3) Θ-foam

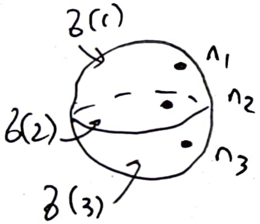
$n_1 \geq n_2 \geq n_3$

F_{12}, F_{13}, F_{23} are

in some order, all S^2

$\chi(F_{ij}(c)) = 2 \quad \forall c, i, j$

colorings $\updownarrow$ $\delta \in S_3$



$$\langle \Theta_{n_1,n_2,n_3}\rangle = \sum_{\delta\in S_3} \frac{x_{\delta(1)}^{n_1} x_{\delta(2)}^{n_2} x_{\delta(3)}^{n_3}}{(x_1+x_2)(x_1+x_3)(x_2+x_3)} = S_\lambda(x_1, x_2, x_3)$$

Schur function (mod 2 coefficients)

$+ = -$

$\lambda = (n_1-2, n_2-1, n_3)$

If $n_1 = n_2$ or $n_2 = n_3$, $\langle \Theta_{n_1,n_2,n_3}\rangle = 0$.

4) if F has no adm. colorings, $\langle F\rangle = 0$

$\underline{\text{Thm}}$ $\langle F \rangle \in R$ $\qquad R = \Bbbk[E_1, E_2, E_3] \subset \Bbbk[x_1, x_2, x_3]$

symmetric polynomials

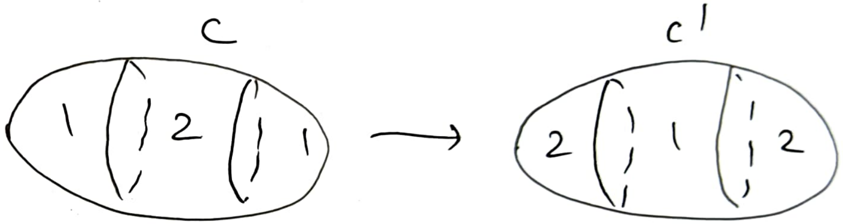
Clearly symmetric.

Need to show denominators cancel out. One at a time, $x_1 + x_2$

fix c, $F_{12}(c) = \Sigma_1 \cup \ldots \cup \Sigma_m$ connected components.

Only 2-spheres contribute to denominator, $x_1 + x_2$ each $\qquad \chi(S^2)/2 = 1$

$\Sigma_i = S^2$ $\qquad \Sigma_i$

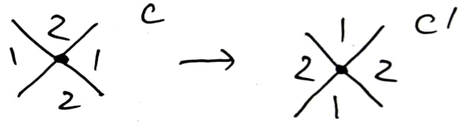


Kempe move

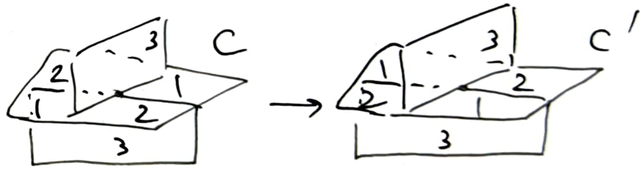
(see Robert-Wagner [RW1])

[extends Kempe moves of graph colorings]

also vertices



$F_{12}(c') = F_{12}(c)$ no change



$\chi(F_{13}(c')) = \chi(F_{13}(c)) + 2\ell$

$\chi(F_{23}(c')) = \chi(F_{23}(c)) - 2\ell$

Remove part of Σ_i colored 1 from $F_{13}(c)$ and add part colored 2 dots on Σ_i reverse color from 1 to 2 $\quad 1 \rightleftarrows 2$

$$\langle F, c\rangle + \langle F, c'\rangle = \alpha \left(x_1^{a_1} x_2^{a_2} (x_2+x_3)^{\ell} + x_2^{a_1} x_1^{a_2} (x_1+x_3)^{\ell} \right)$$

same contribution from c, c'

$$= \alpha \left(f(x_1, x_2, x_3) + f(x_2, x_1, x_3) \right)$$

$\forall$ polynomial $f \in \Bbbk[x_1, x_2, x_3]$ $\quad f(x_1, x_2, x_3) + f(x_2, x_1, x_3)$ is divisible by $x_1 + x_2$

Exercise: Check for monomials. Over $\mathbb{Z}$, add minus signs (divided difference).

$\Rightarrow$ no $(x_1 + x_2)$ in the denominator

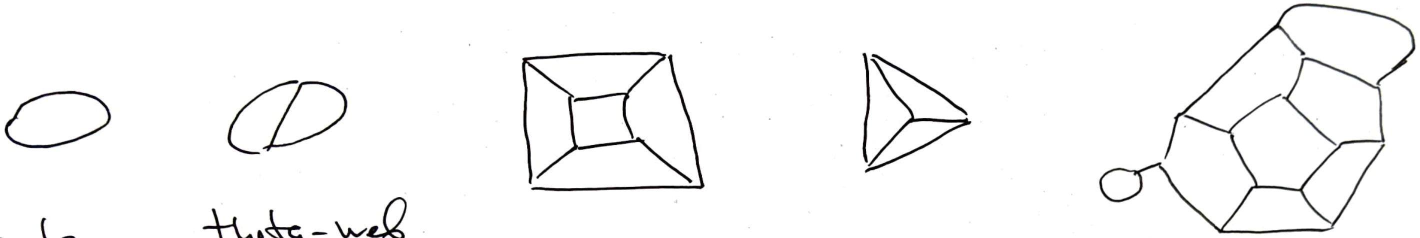
when many components of Σ are 2-spheres, sum over 2^k colorings.

$\square$

Next: homology (state spaces) of $\underbrace{\text{planar graphs}}_{\text{trivalent}}$ or webs Γ.

These appear as generic cross-sections of foams by planes in $\mathbb{R}^3$.

Γ

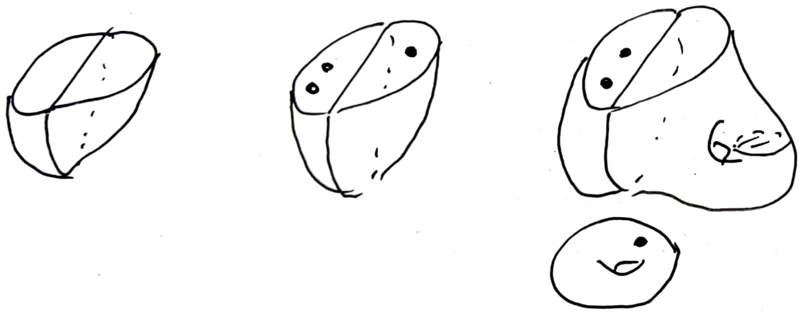


circle   theta-web

$\emptyset$ web

To Γ assign free R-module $Fr(\Gamma)$, basis - all foams F with $\partial F = \Gamma$
(before all foams were closed, $\partial F = \emptyset$)

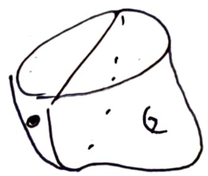
Γ



F

countable basis of $Fr(\Gamma)$

Bilinear form on $Fr(\Gamma)$

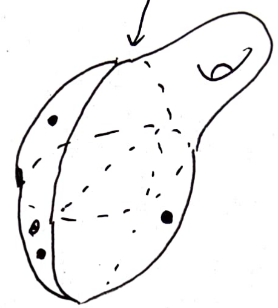


F_0

F_1

singular circle

$\overline{F}_0$



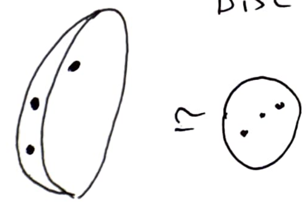
F_1

$\overline{F}_0 F_1$

closed foam, $\langle \overline{F}_0 F_1 \rangle \in R$

facets

Disc



Disc

punctured 2-torus

$$([F_0], [F_1]) = \langle \overline{F}_0 \cdot F_1 \rangle \in R$$

extend to $Fr(\Gamma)$.

$$Fr(\Gamma) \otimes_R Fr(\Gamma) \xrightarrow{(\,,\,)} R$$

Define $\langle \Gamma \rangle = Fr(\Gamma) / \ker((\,,\,))$

state space of Γ.

$\langle \Gamma \rangle$ is a graded R-module

$\forall$ adm. coloring c

$$\deg(F) = -(\chi_{12}(c) + \chi_{13}(c) + \chi_{23}(c)) + 2d(F)$$

$d(F)$: # of dots

$F_{ij}(c) = D^2$

Example

$\deg = -(1+1+1) + 2 \cdot 1 = -1$

<u>Examples</u> 1) $\Gamma = \emptyset$ Closed foams, evaluate to el's of R

$$\langle \Gamma \rangle \simeq R[\emptyset] \simeq H^*_{U(3)}(\cdot, \Bbbk) \qquad R = \Bbbk[E_1, E_2, E_3]$$

equivariant cohomology

2) $\Gamma = \bigcirc$

deg -2 0 2

$\langle \Gamma \rangle$ is a free R-module, basis

1 x x^2

$H^*_{U(3)}(\mathbb{CP}^2, \Bbbk)$

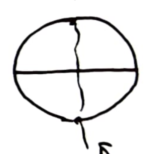
$$A = \langle \bigcirc \rangle \simeq R[x] / (x^3 + E_1 x + E_2 x^2 + E_3)$$

In $R'[x]$, factors $(x+x_1)(x+x_2)(x+x_3)$ $H^*_{U(1)^3}(\mathbb{CP}^2, \Bbbk)$

For graphs Γ with symmetry axis, $\langle \Gamma \rangle$ is a ring,

due to foam from $\Gamma \sqcup \Gamma$ to Γ

$\bigcirc >$



3) $\Gamma =$  $\langle \Gamma \rangle \simeq H^*_{U(3)}(Fl_3, \Bbbk)$

full flags in $\mathbb{C}^3$

For general graphs - no immediate equivariant cohomology interpretation

Thm $\forall \Gamma$, $\langle \Gamma \rangle$ is a finitely-generated graded R-module

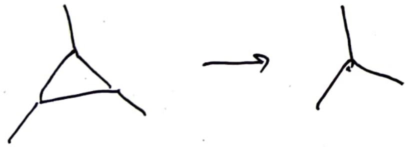
Naive conjecture (no conviction) $\langle \Gamma \rangle$ is a free R-module of rank equal the number of Tait colorings of Γ.

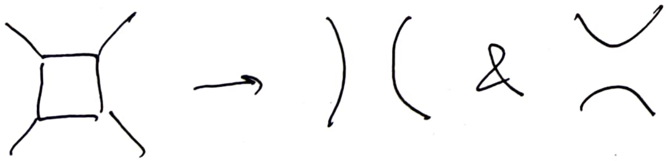
stuck at dodecahedral graph.

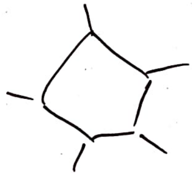
Say that Γ is reducible if can be reduced to empty graph via

$\bigcirc \rightsquigarrow \emptyset$ (remove innermost circle)

 $\rightsquigarrow$ erase





 $\rightarrow$ no reduction available

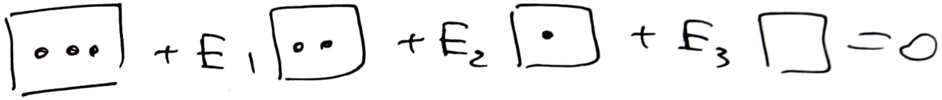
Square and smaller-size regions are reducible

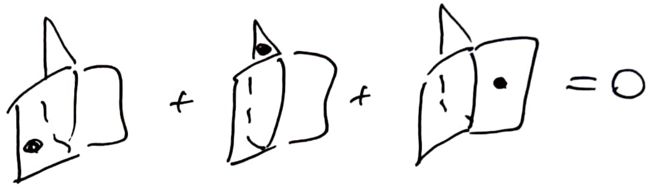
Thm 'Conjecture' holds for reducible graphs.

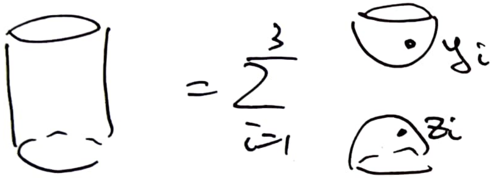
First derive skein relations

(a) evaluations of simplest closed foams (dotted 2-sphere, Θ-foams)

(b) 







$\{y_i\}$, $\{z_i\}$ dual bases of $A = R[x]/(x^3+E_1x^2+E_2x+E_3)$

$(1, x, x^2)$

$(x^2+E_1x+E_2, x+E_1, 1)$

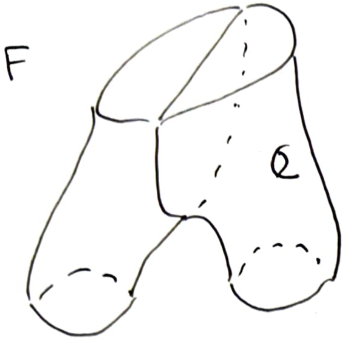
+ more relations

$F \subset \mathbb{R}^2 \times [0,1]$ $\quad\quad$ $\partial_i F = F \cap \mathbb{R}^2 \times \{i\}$, $i=0,1.$

$\partial_1 F$

F

$\partial_0 F$

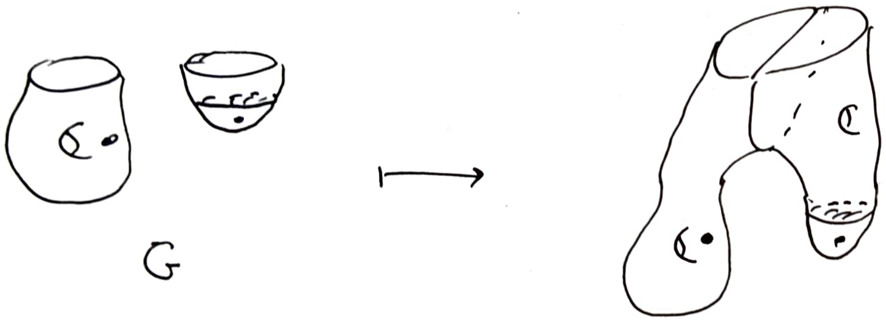


F defines an R-linear map

$$\langle \partial_0 F \rangle \xrightarrow{\langle F \rangle} \langle \partial_1 F \rangle$$

take a foam G with $\partial G = \partial_0 F$ & compose with F

$$[G] \longmapsto [FG]$$



G $\quad\quad$ FG

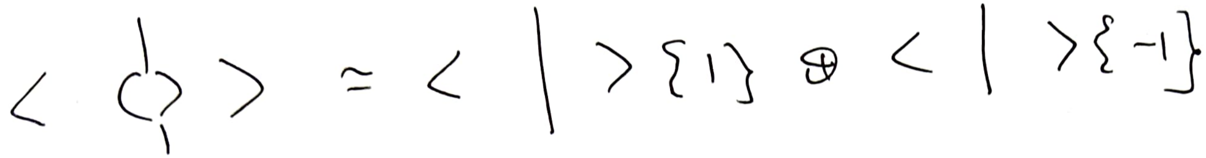
Get a functor

$$\text{Foams} \longrightarrow R\text{-gmod}$$

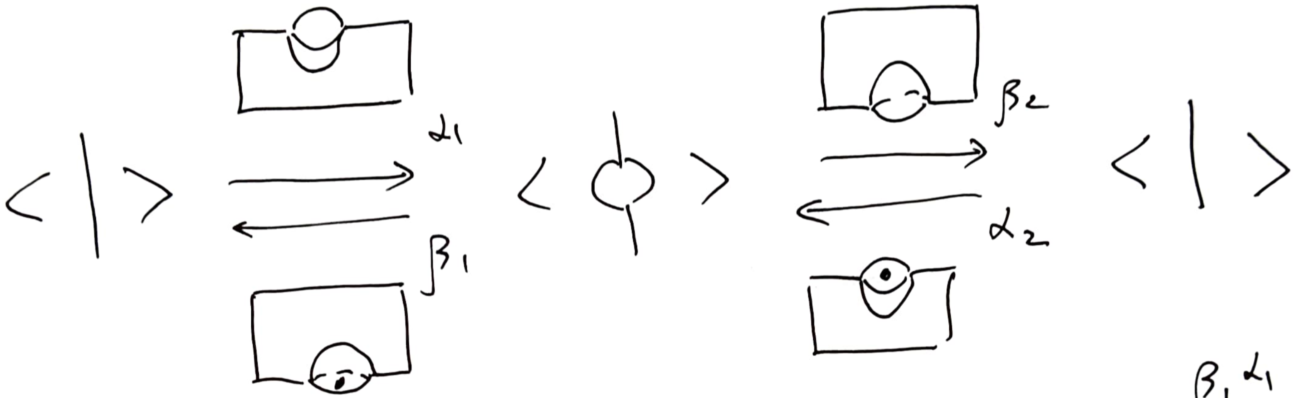
graded R-modules & homogeneous module maps

$$\Gamma \longmapsto \langle \Gamma \rangle$$

$$F \longmapsto \langle F \rangle : \langle \partial_0 F \rangle \longrightarrow \langle \partial_1 F \rangle.$$



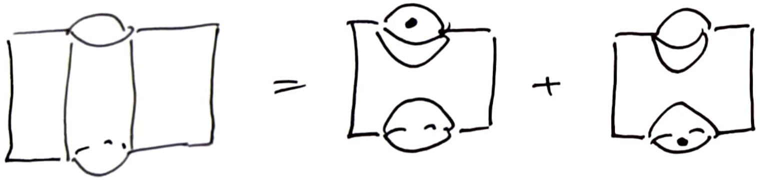
direct sum decomposition



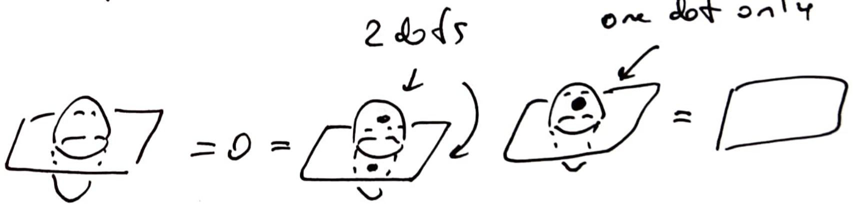
$$\beta_1 \alpha_1 = Id_1$$

$$\beta_2 \alpha_2 = Id_1$$

$$\beta_2 \alpha_1 = 0$$
$$\beta_1 \alpha_2 = 0$$

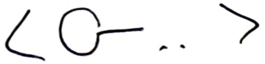


$$id_{\phi} = \alpha_1 \beta_1 + \alpha_2 \beta_2$$

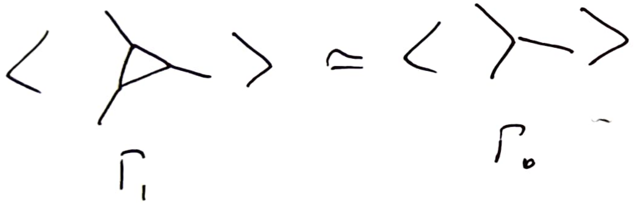


$$\langle \bigcirc \sqcup \Gamma \rangle \simeq \langle\Gamma\rangle\{2\} \oplus \langle\Gamma\rangle \oplus \langle\Gamma\rangle\{-2\} \simeq A \otimes_R \langle\Gamma\rangle$$

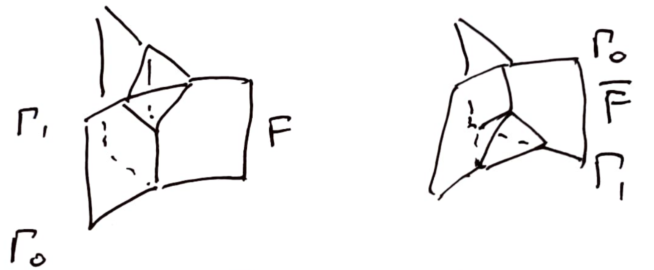
3 copies of Γ



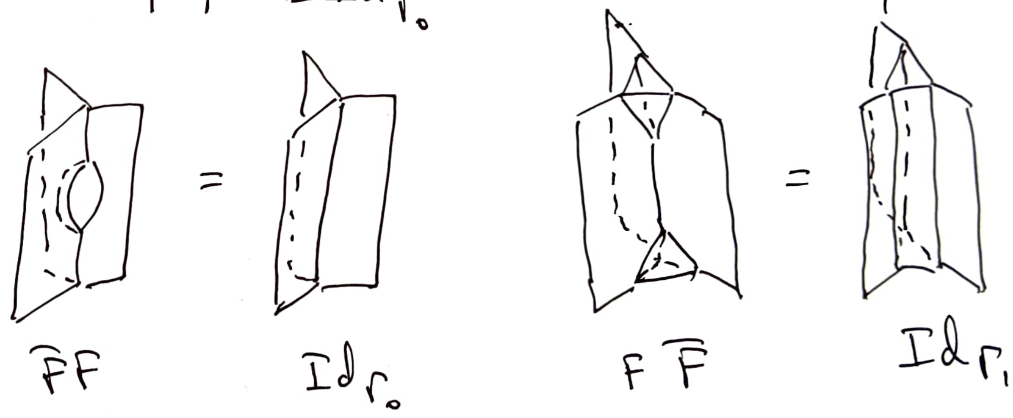
$\simeq 0$ no admissible colorings on a foam GF, $\partial F = \Gamma$

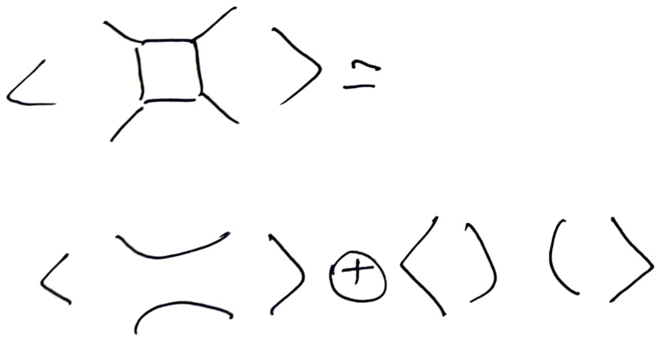


via mutually-inverse isomorphism foams

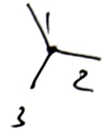


$\bar F F = \mathrm{Id}_{\Gamma_0}$ $\qquad$ $F \bar F = \mathrm{Id}_{\Gamma_1}$





via suitable foams.

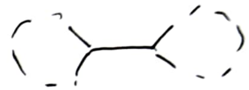
Tait colorings Tait(Γ). Maps edges(Γ) ⟶ {1, 2, 3} s.t. at each vertex 

$\bigcirc \rightsquigarrow$ 3 colorings

$\ominus \rightsquigarrow$ 6 colorings

$t(\Gamma) = |\mathrm{Tait}(\Gamma)|$.

4-color theorem: Γ connected, no bridge ⇒ $t(\Gamma) \neq 0$

Color regions by el's of $\mathbb{Z}/2 \times \mathbb{Z}/2$ 

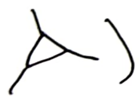
↕

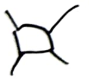
Color edges by el's of $(\mathbb{Z}/2 \times \mathbb{Z}/2)^*$

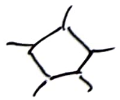
$t(\bigcirc \sqcup \Gamma) = 3\, t(\Gamma)$

$t(\bigcirc\!\!- \,..) = 0$

$t(-\!\bigcirc\!-) = 2\, t(-)$

$t($  $) = t(\succ\!-)$

$t($  $) = t(\,)(\,) + t(\asymp)$

$t($  $)$ no simplification

Naive conj: $\langle \Gamma \rangle$ is a free R-module of rank $t(\Gamma)$.

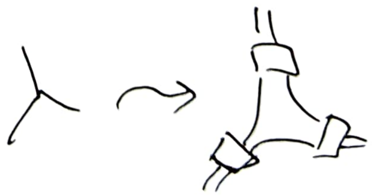
True for reducible graphs.

Also get q-deformation of $t(\Gamma)$ for reducible graphs, via graded rank

$t_q(\bigcirc) = q^2 + 1 + q^{-2}$

$t_q(\phi) = (q+q^{-1})\, t(\,|\,)$

...



(confirm?)

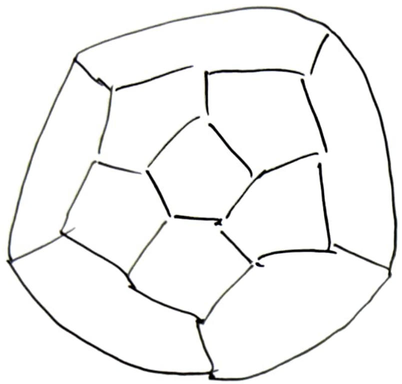
Not the Yamada polynomial, a.k.a. $U_q(sl_2)$, V_2 invariant

$V_2 = S_q^2 V_1$

3-dim irrep

Dodecahedral graph

Γ_d



$t(\Gamma_d) = 60$

60 Tait colorings

Not known if $\langle \Gamma_d \rangle$ a free rank 60 R-module

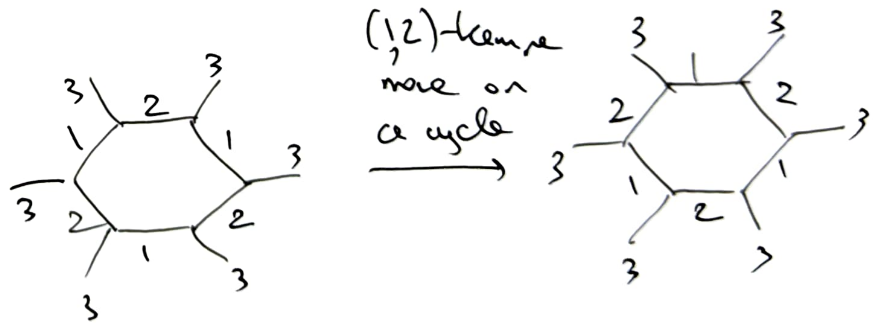
David Boozer (arXiv)

computations $\Rightarrow$ rank (over a field, $x_i = 0$) ≥ 58

Also compare with

Kronheimer - Mrowka, $SO(3)$ gauge theory for 3-orbifolds.

Kempe moves of Tait colorings are 1D analogues of Kempe moves of foam colorings



THANK YOU!