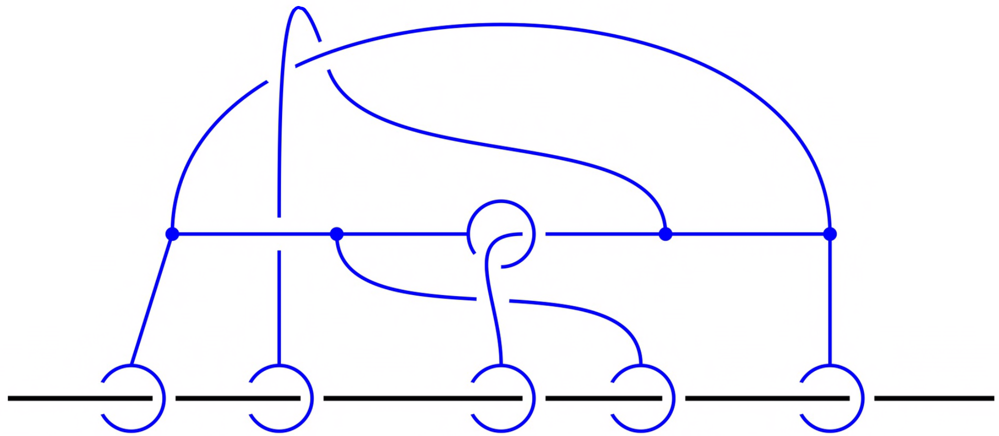


# Knot invariants from homotopy theory

Danica Kosanović

MPIM Bonn

29. 5. 2020.

Online TQFT
IST Lisbon Seminar

# Knot invariants from homotopy theory

Goal:

$$\mathrm{Emb}_\partial(V, M)$$

space (Whitney $C^\infty$ topology)

smooth embeddings with a fixed boundary condition

compact manifolds with boundary

# Knot invariants from homotopy theory

## Goal:

$$\mathrm{Emb}_\partial(V, M)$$

space (Whitney $C^\infty$ topology)

smooth embeddings with a fixed boundary condition

compact manifolds with boundary

## Using:

homotopy theory

homotopy limits

Whitehead / Samelson products

configuration spaces

operads (little discs)
(formality, graph complexes)

'99

Goodwillie - Weiss
EMBEDDING
CALCULUS

# Knot invariants from homotopy theory

**Goal:**

$$\mathrm{Emb}_\partial(V, M)$$

space (Whitney $C^\infty$ topology)

smooth embeddings with a fixed boundary condition

compact manifolds with boundary

**Using:** homotopy theory

- homotopy limits
- Whitehead / Samelson products
- configuration spaces
- operads (little discs) (formality, graph complexes)

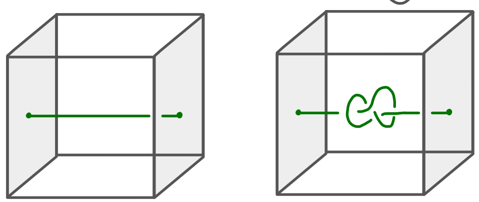
'99

**Goodwillie-Weiss EMBEDDING CALCULUS**

**In this talk:** Give a geometric interpretation of GW calculus for $\mathrm{Emb}_\partial(I, M)$ for $\dim M = 3$.

& Relate to Vassiliev finite type invariants

In particular: $M = I^3$



$$\pi_0\,\mathrm{Emb}(S^1, S^3) \cong \pi_0\,\mathrm{Emb}_\partial(I, I^3) = \text{knots}/\text{isotopy}$$

COMM. MONOID

# Knot invariants from homotopy theory

## Goal:

$\mathrm{Emb}_\partial(V, M)$ space (Whitney $C^\infty$ topology)

smooth embeddings with a fixed boundary condition

compact manifolds with boundary

## Using:

homotopy theory

- homotopy limits
- Whitehead / Samelson products
- configuration spaces
- operads (little discs) (formality, graph complexes)

'99

**Goodwillie - Weiss EMBEDDING CALCULUS**

## Motivation:

All quantum invariants are finite type (Vassiliev) invariants.

$\Longrightarrow$ Any quantum invariant can be calculated as the composite

$$\mathbb{K}(I^3) \xrightarrow{Z} \hat{\mathcal{A}} \xrightarrow{w} \Bbbk$$

- $Z$: a universal Vassiliev invariant
- $\hat{\mathcal{A}}$: Hopf algebra of Feynman diagrams
- $w$: a weight system coming from a semisimple Lie algebra

90's

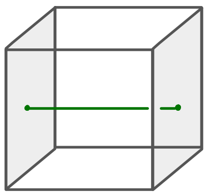
- Kontsevich integral + Drinfeld + Le-Murakami over $\mathbb{Q}$
- Bott - Taubes configuration space integrals

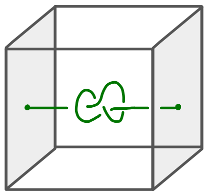
## In this talk:

Give a geometric interpretation of GW calculus for $\mathrm{Emb}_\partial(I, M)$ for $\dim M = 3$.

& Relate to Vassiliev finite type invariants

In particular: $M = I^3$





$$\pi_0 \mathrm{Emb}(S^1, S^3) \cong \pi_0 \mathrm{Emb}_\partial(I, I^3) = \text{knots}/\text{isotopy}$$

COMM. MONOID

# Knot invariants from homotopy theory

## Goal:

$\mathrm{Emb}_\partial(V, M)$ space (Whitney $C^\infty$ topology)

smooth embeddings with a fixed boundary condition

compact manifolds with boundary

## Using:

homotopy theory

- homotopy limits
- Whitehead / Samelson products
- configuration spaces
- operads (little discs) (formality, graph complexes)

'99 $\Downarrow$

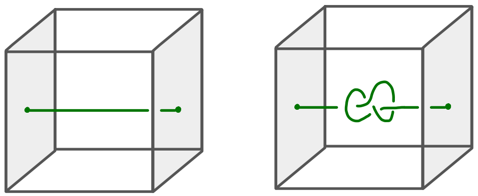
**Goodwillie-Weiss EMBEDDING CALCULUS**

## In this talk:

Give a geometric interpretation of GW calculus for $\mathrm{Emb}_\partial(I, M)$ for $\dim M = 3$.

& Relate to Vassiliev finite type invariants

In particular: $M = I^3$



$$\pi_0 \mathrm{Emb}(S^1, S^3) \cong \pi_0 \mathrm{Emb}_\partial(I, I^3) = \text{knots}/\text{isotopy}$$

COMM. MONOID

## Motivation:

All quantum invariants are finite type (Vassiliev) invariants.

$\Longrightarrow$ Any quantum invariant can be calculated as the composite

$$\mathbb{K}(I^3) \xrightarrow{Z} \hat{\mathcal{A}} \xrightarrow{w} \Bbbk$$

- $Z$: a universal Vassiliev invariant
- $\hat{\mathcal{A}}$: Hopf algebra of Feynman diagrams
- $w$: a weight system coming from a semisimple Lie algebra

90's

- Kontsevich integral + Drinfeld + Le-Murakami over $\mathbb{Q}$
- Bott-Taubes configuration space integrals

## Questions:

- Away from char $\Bbbk = 0$? torsion?
- Computations?
- Geometric meaning?

Yes. '00s: Gusarov, Habiro, Conant-Teichner

# Knot invariants from homotopy theory

**Goal:** $\mathrm{Emb}_\partial(V, M)$ space (Whitney $C^\infty$ topology)

smooth embeddings with a fixed boundary condition

compact manifolds with boundary

**Using:** homotopy theory

homotopy limits

Whitehead / Samelson products

configuration spaces

operads (little disks)

(formality, graph complexes)

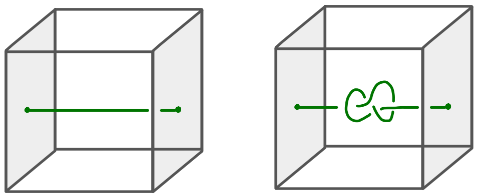
'99

Goodwillie-Weiss
EMBEDDING
CALCULUS

**In this talk:** Give a geometric interpretation of GW calculus for $\mathrm{Emb}_\partial(I, M)$ for $\dim M = 3$.

& Relate to Vassiliev finite type invariants

In particular: $M = I^3$



$\pi_0 \mathrm{Emb}(S^1, S^3) \cong \pi_0 \mathrm{Emb}_\partial(I, I^3) =$ knots/isotopy

COMM. MONOID

**Motivation:**

All quantum invariants are finite type (Vassiliev) invariants.

$\Longrightarrow$ Any quantum invariant can be calculated as the composite

$$\mathbb{K}(I^3) \xrightarrow{Z} \hat{\mathcal{A}} \xrightarrow{w} \mathbb{k}$$

a universal Vassiliev invariant

Hopf algebra of Feynman diagrams

a weight system coming from a semisimple Lie algebra

90's

- Kontsevich integral + Drinfeld + Le-Murakami over $\mathbb{Q}$
- Bott-Taubes configuration space integrals

**Questions:**
- Away from char $k = 0$? torsion?
- Computations?
- Geometric meaning?

Yes. '00s: Gusarov, Habiro, Conant-Teichner

**Outline:**

1° Introduce the main object of GW calculus: the Taylor tower $P_n(M)$.

2° State the main theorem.

3° Explain consequences for Vassiliev theory.

## §1. The embedding calculus

$$
\begin{tikzcd}
\mathcal{E}mb_\partial(V,M) \arrow[r, "ev_\infty"] \arrow[dr, bend right, "ev_2"'] \arrow[ddr, bend right, "ev_1"'] & P_\infty(V,M) \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_2(V,M) \arrow[d, "p_2"] \\
 & P_1(V,M) = \mathcal{I}mm_\partial(V,M)
\end{tikzcd}
$$

## §1. The embedding calculus

$$\begin{tikzcd}
\mathcal{E}mb_\partial(V,M) \arrow[r, "ev_\infty"] \arrow[ddr, bend right, "ev_2"] \arrow[dddr, bend right, "ev_1"'] & P_\infty(V,M) \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_2(V,M) \arrow[d, "p_2"] \\
 & P_1(V,M) = \mathcal{I}mm_\partial(V,M)
\end{tikzcd}$$

**Theorem** [Goodwillie - Klein '15]

If $(\dim V, \dim M) \neq (1,3)$ then

$ev_n$ is $\big(3 - \dim M + (n+1)(\dim M - \dim V - 2)\big)$ - connected.

**Corollary**

For $\dim M - \dim V > 2$ $ev_\infty$ is a weak equivalence.

* Recall: a map is $k$-connected if it is an isomorphism on $\pi_i$ $(\forall i < k)$ and a surjection on $\pi_k$

# §1. The embedding calculus

$$\begin{tikzcd}
\mathrm{Emb}_\partial(V,M) \arrow[r, "ev_\infty"] \arrow[ddr, bend right, "ev_2"'] \arrow[dddr, bend right, "ev_1"'] & P_\infty(V,M) \arrow[d] \\
& \vdots \arrow[d] \\
& P_2(V,M) \arrow[d, "p_2"] \\
& P_1(V,M) = \mathrm{Imm}_\partial(V,M)
\end{tikzcd}$$

**Theorem** [Goodwillie - Klein '15]

If $(\dim V, \dim M) \neq (1,3)$ then

$ev_n$ is $\big(3 - \dim M + (n+1)\,(\dim M - \dim V - 2)\big)$ - connected.

**Corollary**

For $\dim M - \dim V > 2$ $\;ev_\infty$ is a weak equivalence.

**Note:** - One can show $ev_\infty$ NOT a w.e. for $\mathrm{Emb}_\partial(I, I^3)$.

- However, the formula predicts

For $\dim M = 3$ $\;ev_n : \mathrm{Emb}_\partial(I, M) \longrightarrow P_n(I,M)$ is 0-connected.

* Recall: a map is k-connected if it is an isomorphism on $\pi_i$ $(\forall\, i<k)$ and a surjection on $\pi_k$

## §1. The embedding calculus

$$
\begin{tikzcd}
\mathrm{Emb}_\partial(V,M) \arrow[r, "ev_\infty"] \arrow[ddr, bend right, "ev_2"'] \arrow[dddr, bend right, "ev_1"'] & P_\infty(V,M) \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_2(V,M) \arrow[d, "p_2"] \\
 & P_1(V,M) = \mathrm{Imm}_\partial(V,M)
\end{tikzcd}
$$

**Theorem** [Goodwillie-Klein '15]

> If $(\dim V, \dim M) \neq (1,3)$ then
>
> $ev_n$ is $\big(3-\dim M + (n+1)(\dim M - \dim V - 2)\big)$-connected.

**Corollary**

For $\dim M - \dim V > 2$ $ev_\infty$ is a weak equivalence.

**Note:**

- One can show $ev_\infty$ NOT a w.e. for $\mathrm{Emb}_\partial(I, I^3)$.
- However, the formula predicts

**Theorem A** [K]

> For $\dim M = 3$ $ev_n : \mathrm{Emb}_\partial(I, M) \longrightarrow P_n(I,M)$ is 0-connected.

* Recall: a map is $k$-connected if it is an isomorphism on $\pi_i$ $(\forall i<k)$ and a surjection on $\pi_k$

- This will follow from a stronger result which considers:

**Outline:**

**Proof of Thm A**

## §1. The embedding calculus

$$\begin{tikzcd}
\mathrm{Emb}_\partial(V,M) \arrow[r, "ev_\infty"] \arrow[rdd, bend right, "ev_2"] \arrow[rddd, bend right, "ev_1"'] & P_\infty(V,M) \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_2(V,M) \arrow[d, "p_2"] \\
 & P_1(V,M) = \mathrm{Imm}_\partial(V,M)
\end{tikzcd}$$

### Theorem [Goodwillie-Klein '15]

If $(\dim V, \dim M) \neq (1,3)$ then

$ev_n$ is $\big(3 - \dim M + (n+1)(\dim M - \dim V - 2)\big)$-connected.

### Corollary

For $\dim M - \dim V > 2$ $ev_\infty$ is a weak equivalence.

**Note:** - One can show $ev_\infty$ NOT a w.e. for $\mathrm{Emb}_\partial(I, I^3)$.

- However, the formula predicts

### Theorem A [K]

For $\dim M = 3$ $ev_n : \mathrm{Emb}_\partial(I, M) \longrightarrow P_n(I,M)$ is 0-connected.

- This will follow from a stronger result which considers:

$$\begin{tikzcd}
\mathrm{hofib}_{ev_{n-1}U}(ev_{n-1}) =: H_{n-1}(M) \arrow[r, "e_n"] \arrow[d] & F_n(M) := \mathrm{fib}_{ev_{n-1}U}(p_n) \arrow[d] \\
\mathrm{Emb}_\partial(I,M) \arrow[r, "ev_n"] \arrow[rd, "ev_{n-1}"'] & P_n(M) \arrow[d, "p_n"] \\
 & P_{n-1}(M)
\end{tikzcd}$$

$\boxed{\text{punctured knots model}}$

$\boxed{\text{surjective fibration}}$

### Outline:

### Proof of Thm A

* Recall: a map is $k$-connected if it is an isomorphism on $\pi_i$ $(\forall i < k)$ and a surjection on $\pi_k$

## §1. The embedding calculus

$$
\begin{tikzcd}
\mathrm{Emb}_\partial(V,M) \arrow[r, "ev_\infty"] \arrow[rdd, bend right, "ev_2"'] \arrow[rddd, bend right=40, "ev_1"'] & P_\infty(V,M) \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_2(V,M) \arrow[d, "p_2"] \\
 & P_1(V,M) = \mathrm{Imm}_\partial(V,M)
\end{tikzcd}
$$

**Theorem** [Goodwillie - Klein '15]

If $(\dim V, \dim M) \neq (1,3)$ then

$ev_n$ is $\big(3 - \dim M + (n+1)(\dim M - \dim V - 2)\big)$-connected.

**Corollary**

For $\dim M - \dim V > 2$ $\quad ev_\infty$ is a weak equivalence.

**Note:** - One can show $ev_\infty$ NOT a w.e. for $\mathrm{Emb}_\partial(I, I^3)$.

- However, the formula predicts

**Theorem A** [K]

For $\dim M = 3$ $\quad ev_n : \mathrm{Emb}_\partial(I,M) \longrightarrow P_n(I,M)$ is 0-connected.

- This will follow from a stronger result which considers:

$$
\begin{tikzcd}
\mathrm{hofib}_{ev_{n-1}U}(ev_{n-1}) =: H_{n-1}(M) \arrow[r, "e_n"] \arrow[d] & F_n(M) := \mathrm{fib}_{ev_{n-1}U}(p_n) \arrow[d] \\
\mathrm{Emb}_\partial(I,M) \arrow[r, "ev_n"] \arrow[dr, "ev_{n-1}"'] & P_n(M) \arrow[d, "p_n"] \\
 & P_{n-1}(M)
\end{tikzcd}
$$

$P_n(M)$: punctured knots model

$p_n$: surjective fibration

**Outline:**

I) compute $\pi_0 F_n(M)$: generated by trees

II) construct explicit points in $H_{n-1}(M)$ using GROPES: modelled on trees

III) MAIN THM:

$e_n$ maps a 'grope point' to its underlying tree

**Proof of Thm A**

* Recall: a map is $k$-connected if it is an isomorphism on $\pi_i$ $(\forall i<k)$ and a surjection on $\pi_k$

## §1. The embedding calculus

$$\begin{tikzcd}
\mathrm{Emb}_\partial(V,M) \arrow[r, "ev_\infty"] \arrow[ddr, bend right, "ev_2"'] \arrow[dddr, bend right, "ev_1"'] & P_\infty(V,M) \arrow[d] \\
& \vdots \arrow[d] \\
& P_2(V,M) \arrow[d, "p_2"] \\
& P_1(V,M) = \mathrm{Imm}_\partial(V,M)
\end{tikzcd}$$

### Theorem [Goodwillie–Klein '15]

If $(\dim V, \dim M) \neq (1,3)$ then

$ev_n$ is $\big(3 - \dim M + (n+1)(\dim M - \dim V - 2)\big)$-connected.

### Corollary

For $\dim M - \dim V > 2$ $ev_\infty$ is a weak equivalence.

**Note:** – One can show $ev_\infty$ NOT a w.e. for $\mathrm{Emb}_\partial(I, I^3)$.

– However, the formula predicts

### Theorem A [K]

For $\dim M = 3$ $ev_n : \mathrm{Emb}_\partial(I,M) \longrightarrow P_n(I,M)$ is 0-connected.

– This will follow from a stronger result which considers:

$$\begin{tikzcd}
\mathrm{hofib}_{ev_{n-1}U}(ev_{n-1}) =: H_{n-1}(M) \arrow[r, "e_n"] \arrow[d] & F_n(M) := \mathrm{fib}_{ev_{n-1}U}(p_n) \arrow[d] \\
\mathrm{Emb}_\partial(I,M) \arrow[r, "ev_n"] \arrow[dr, "ev_{n-1}"'] & P_n(M) \arrow[d, "p_n"] \\
& P_{n-1}(M)
\end{tikzcd}$$

punctured knots model

surjective fibration

### Outline:

I) compute $\pi_0 F_n(M)$: generated by trees

II) construct explicit points in $H_{n-1}(M)$ using GROPES: modelled on trees

III) MAIN THM:

$e_n$ maps a 'grope point' to its underlying tree

### Proof of Thm A

Any tree can be realized by a grope

$\Longrightarrow \pi_0 e_n$ is surjective

Diagram + induction. $\square$

* Recall: a map is $k$-connected if it is an isomorphism on $\pi_i$ $(\forall i < k)$ and a surjection on $\pi_k$

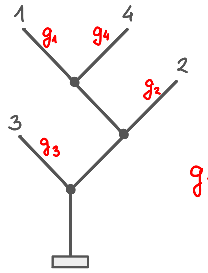
1) **Theorem B** [K]

$$\pi_0 F_n(M) \cong \frac{\mathbb{Z}[\mathrm{Tree}_{\pi,M}(n-1)]}{AS.IHX}$$

## §2. Results

1) **Theorem B** [K]

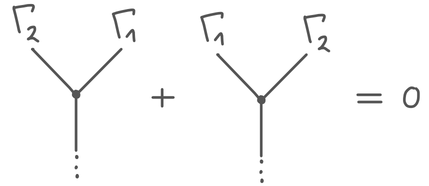
$$\pi_0 F_n(M) \cong \frac{\mathbb{Z}[\mathrm{Tree}_{\pi_1 M}(n-1)]}{AS, IHX}$$



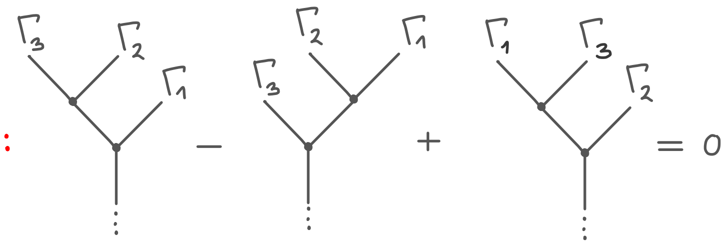

$g_i \in \pi_1 M$
$n-1=4$

$\pi_1 M$-decorated rooted planar binary trees

AS:



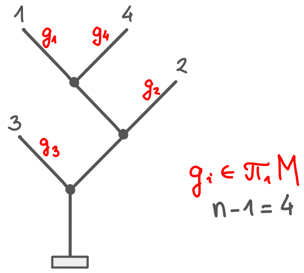

IHX:



## §2. Results

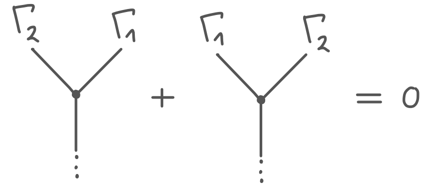
I) **Theorem B** [K]

$$\pi_0 F_n(M) \cong \frac{\mathbb{Z}[\mathrm{Tree}_{\pi_1 M}(n-1)]}{AS,\ IHX}$$

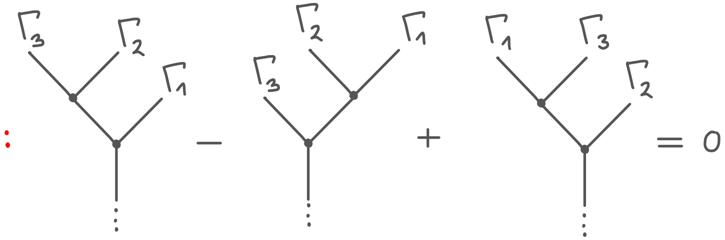



$\pi_1 M$ - decorated rooted planar binary trees

AS:




IHX:




II) **Theorem C** [Joint w/ Y. Shi & P. Teichner]

$$\exists \quad \mathrm{Grop}_{n-1}(M;U) \xrightarrow{\ \psi\ } H_{n-1}(M)$$

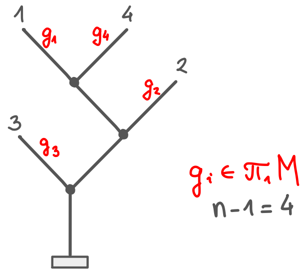
a space of thick gropes on $U$ in $M$

(3D) gropes OR claspers are objects used in the geometric approach to Vassiliev's theory of finite type knot invariants

# §2. Results

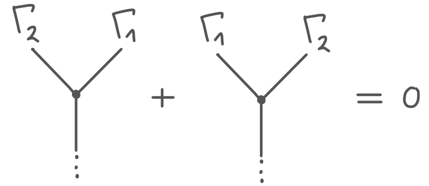
I) **Theorem B** [K]

$$\pi_0 F_n(M) \cong \frac{\mathbb{Z}[\mathrm{Tree}_{\pi_1 M}(n-1)]}{AS, IHX}$$

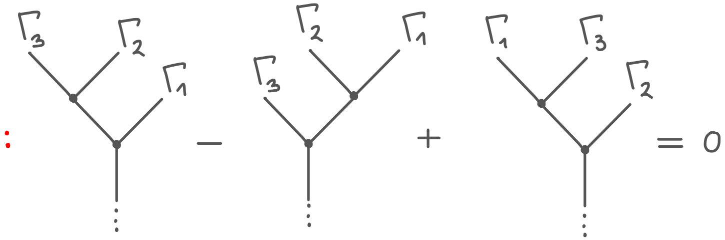



$\pi_1 M$-decorated rooted planar binary trees

AS :




IHX :




II) **Theorem C** [Joint w/ Y. Shi & P. Teichner]

$$\exists \quad \mathrm{Grop}_{n-1}(M;U) \xrightarrow{\psi} H_{n-1}(M)$$

a space of thick gropes on U in M

(3D) gropes OR claspers are objects used in the geometric approach to Vassiliev's theory of finite type knot invariants

**Definition** Two knots $K, K' \in \mathrm{Emb}_\partial(I, M)$ are n-equivalent if $\exists\, G \in \mathrm{Grop}_n(M;K)$ whose output is knot $K'$.

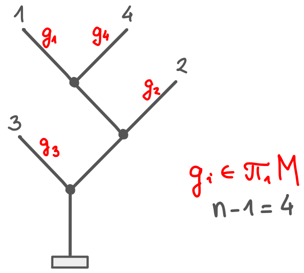
$K \sim_n K'$

⇝ see example on the last slide

## §2. Results

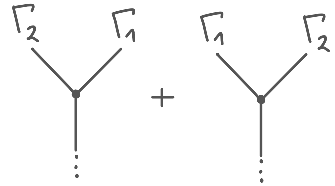
I) **Theorem B** [K]

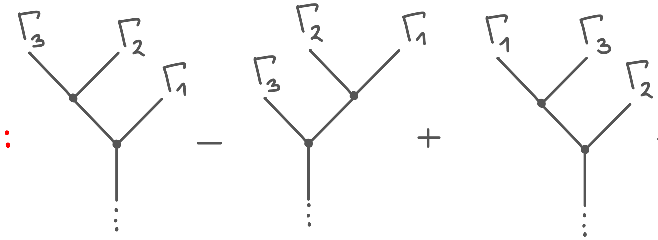
$$\pi_0 F_n(M) \cong \frac{\mathbb{Z}[\mathrm{Tree}_{\pi_1 M}(n-1)]}{AS,\ IHX}$$




$g_i \in \pi_1 M$
$n-1=4$

$\pi_1 M$-decorated rooted planar binary trees

AS:  $= 0$


IHX:  $= 0$


II) **Theorem C** [Joint w/ Y. Shi & P. Teichner]

$$\exists \ \mathrm{Grop}_{n-1}(M;U) \xrightarrow{\psi} H_{n-1}(M)$$

a space of thick gropes on $U$ in $M$

(3D) gropes OR claspers are objects used in the geometric approach to Vassiliev's theory of finite type knot invariants

**Definition** Two knots $K, K' \in \mathrm{Emb}_\partial(I, M)$ are n-equivalent if $\exists\, \mathcal{G} \in \mathrm{Grop}_n(M;K)$ whose output is knot $K'$.
$K \sim_n K'$

⇝ see example on the last slide

III) **Theorem D (main)** [K]

$$\begin{tikzcd}
 & & \mathbb{Z}[\mathrm{Tree}_{\pi_1 M}(n-1)] \arrow[d, two heads, "\text{mod } AS,\, IHX"] \\
\pi_0 \mathrm{Grop}_{n-1}(M;U) \arrow[urr, two heads, "uf_{n-1}"] \arrow[r, "\pi_0\psi"] \arrow[ur, phantom, "\circlearrowright"] & \pi_0 H_{n-1}(M) \arrow[r, "\pi_0 e_n"] & \pi_0 F_n(M)
\end{tikzcd}$$

$uf_{n-1}$

maps a thick grope to its underlying tree

## Corollary of Thm C

$\pi_0 ev_n$ is an invariant of geometric n-equivalence, i.e. it factors through

$$\pi_0 \mathcal{E}mb_\partial(I, M)/\sim_n \xrightarrow{\overline{ev}_n} \pi_0 P_n(M)$$

## Corollary of Thm C

$\pi_0 ev_n$ is an invariant of geometric $n$-equivalence, i.e. it factors through

$$\pi_0 \operatorname{Emb}_\partial(I, M)/\sim_n \xrightarrow{\overline{ev}_n} \pi_0 P_n(M)$$

## Corollary of Corollary using [Habiro'00]

For $M = I^3$ $\pi_0 ev_n$ is a Vassiliev invariant of type $\leq n-1$.

**Remark:** Shown by Budney-Conant-Koytcheff-Sinha '17

They also show: $\pi_0 ev_n : \pi_0 \mathcal{K}(I^3) \longrightarrow \pi_0 P_n(I^3)$ is a monoid map.

## Corollary of Thm C

$\pi_0 ev_n$ is an invariant of geometric $n$-equivalence, i.e. it factors through

$$\pi_0 \mathrm{Emb}_\partial(I, M)/\sim_n \xrightarrow{\overline{ev}_n} \pi_0 P_n(M)$$

## Corollary of Corollary using [Habiro '00]

For $M = I^3$ $\pi_0 ev_n$ is a Vassiliev invariant of type $\leq n-1$.

**Remark:** Shown by Budney-Conant-Koytcheff-Sinha '17

They also show: $\pi_0 ev_n : \pi_0 \mathcal{K}(I^3) \longrightarrow \pi_0 P_n(I^3)$ is a monoid map.

**Conjecture** For $M = I^3$ $\pi_0 ev_n$ is a universal additive invariant of type $\leq n-1$ over $\mathbb{Z}$. $\iff$ $\pi_0 ev_n$ is a monoid map which factors through $\overline{ev}_n$ and $\overline{ev}_n$ is an isomorphism (of fin.gen. ab. gps).

**Remark:** Such invariants constructed so far only over $\mathbb{Q}$: Kontsevich / Bott-Taubes integrals.

## Corollary of Thm C

$\pi_0 ev_n$ is an invariant of geometric $n$-equivalence, i.e. it factors through

$$\pi_0 \mathrm{Emb}_\partial(I, M)/\sim_n \xrightarrow{\overline{ev}_n} \pi_0 P_n(M)$$

## Corollary of Corollary using [Habiro'00]

For $M = I^3$ $\pi_0 ev_n$ is a Vassiliev invariant of type $\leq n-1$.

**Remark:** Shown by Budney-Conant-Koytcheff-Sinha '17

They also show: $\pi_0 ev_n : \pi_0 \mathcal{K}(I^3) \longrightarrow \pi_0 P_n(I^3)$ is a monoid map.

**Conjecture** For $M = I^3$ $\pi_0 ev_n$ is a universal additive invariant of type $\leq n-1$ over $\mathbb{Z}$. $\Big\}$ $\Leftrightarrow$ $\pi_0 ev_n$ is a monoid map which factors through $\overline{ev}_n$ and $\overline{ev}_n$ is an isomorphism (of fin.gen. ab.gps).

**Remark:** Such invariants constructed so far only over $\mathbb{Q}$: Kontsevich / Bott-Taubes integrals.

## Corollary of Thm D

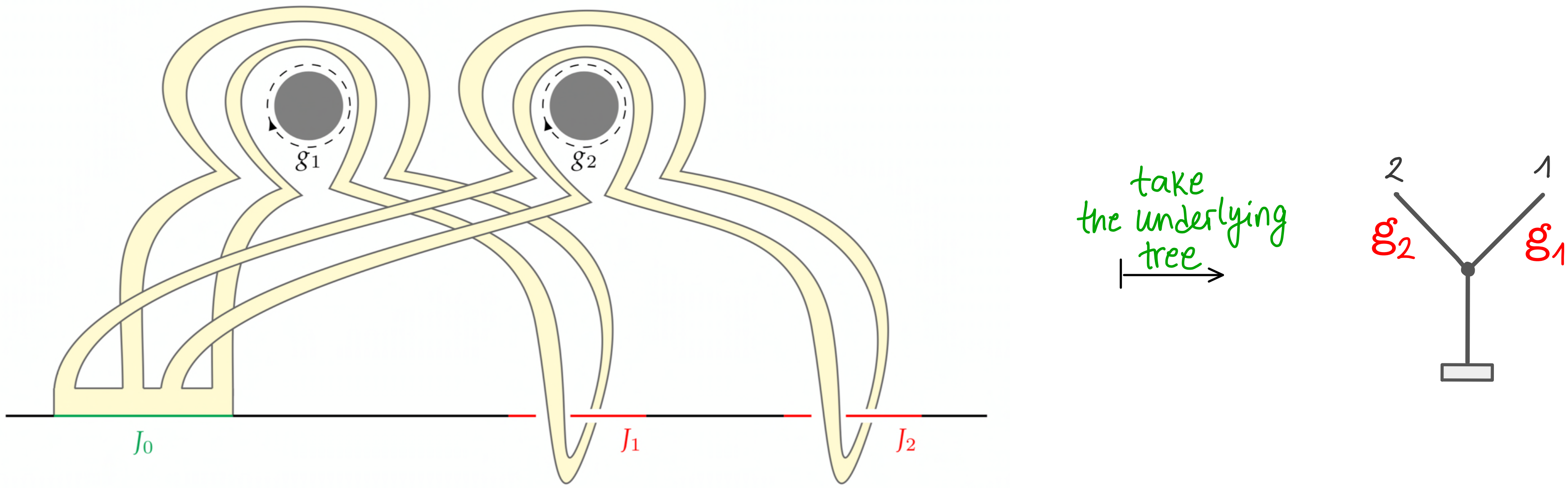
For $M = I^3$

1° $\overline{ev}_n$ is surjective

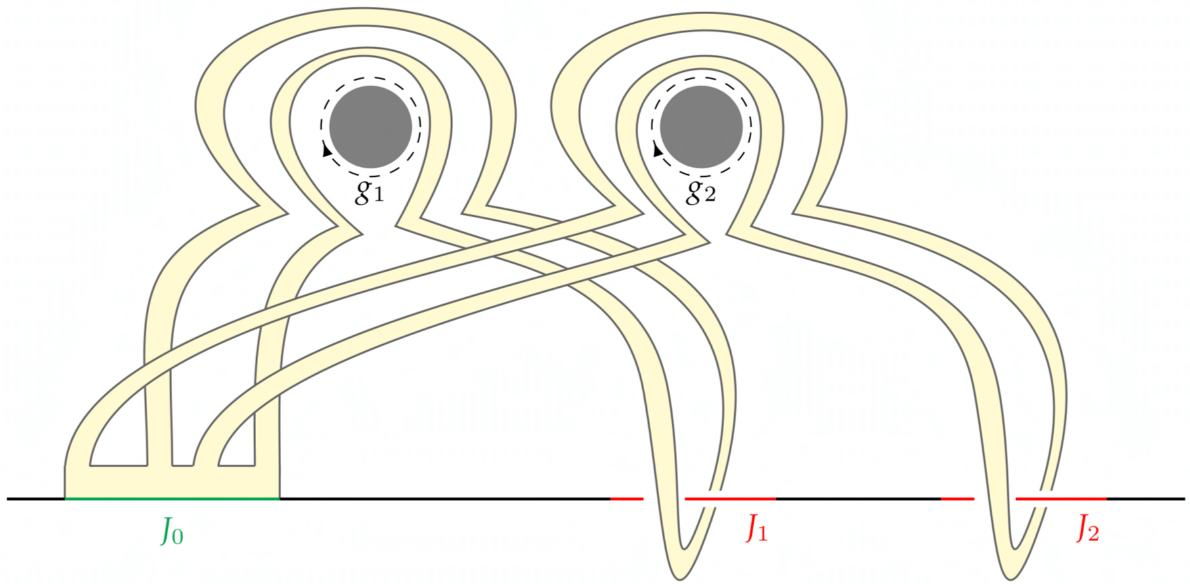
2° CONJECTURE is TRUE over $\mathbb{Q}$: $\pi_0 \mathrm{Emb}_\partial(I, I^3)/\sim_n \otimes \mathbb{Q} \xrightarrow[\cong]{\overline{ev}_n} \pi_0 P_n(I^3) \otimes \mathbb{Q}$

3° CONJECTURE is TRUE over $\mathbb{Z}_p$ in a RANGE: $n \leq p+2$.

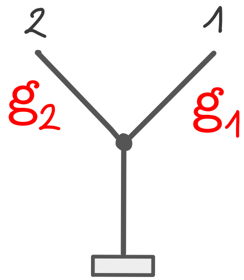
[Boavida de Brito - Horel]



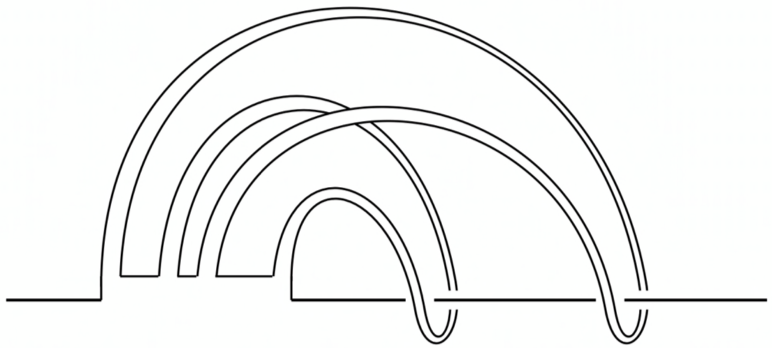
$g_1$
$g_2$
$J_0$
$J_1$
$J_2$
take the underlying tree
2
1
$g_2$
$g_1$



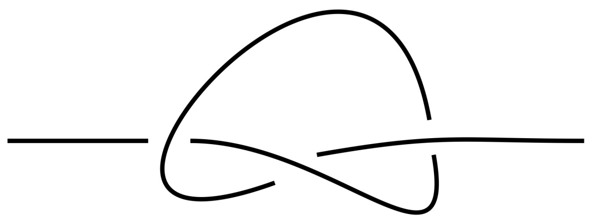

take the underlying tree $\longmapsto$




If group elements trivial get:



This is isotopic to:



<u>Hence: trefoil is 2-equivalent to the unknot</u>