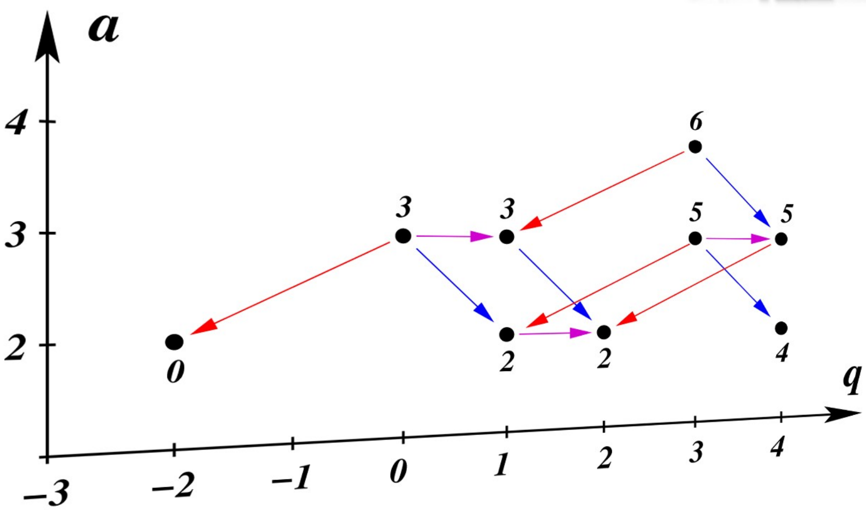
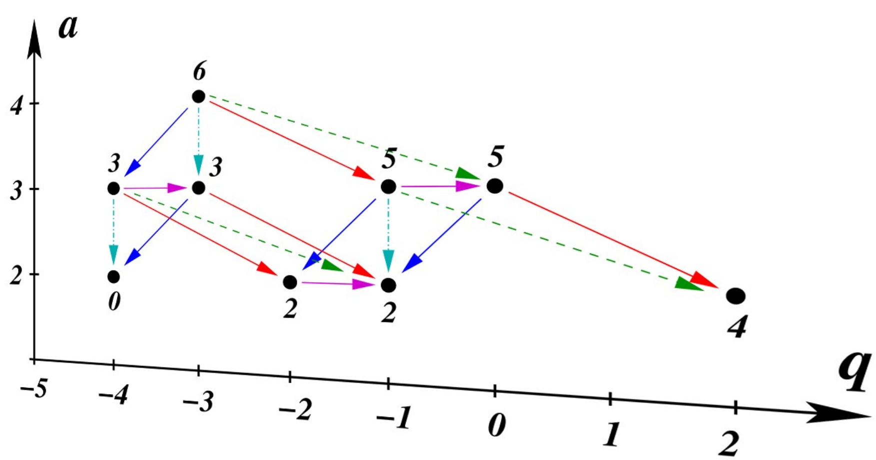
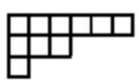
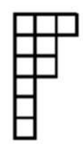
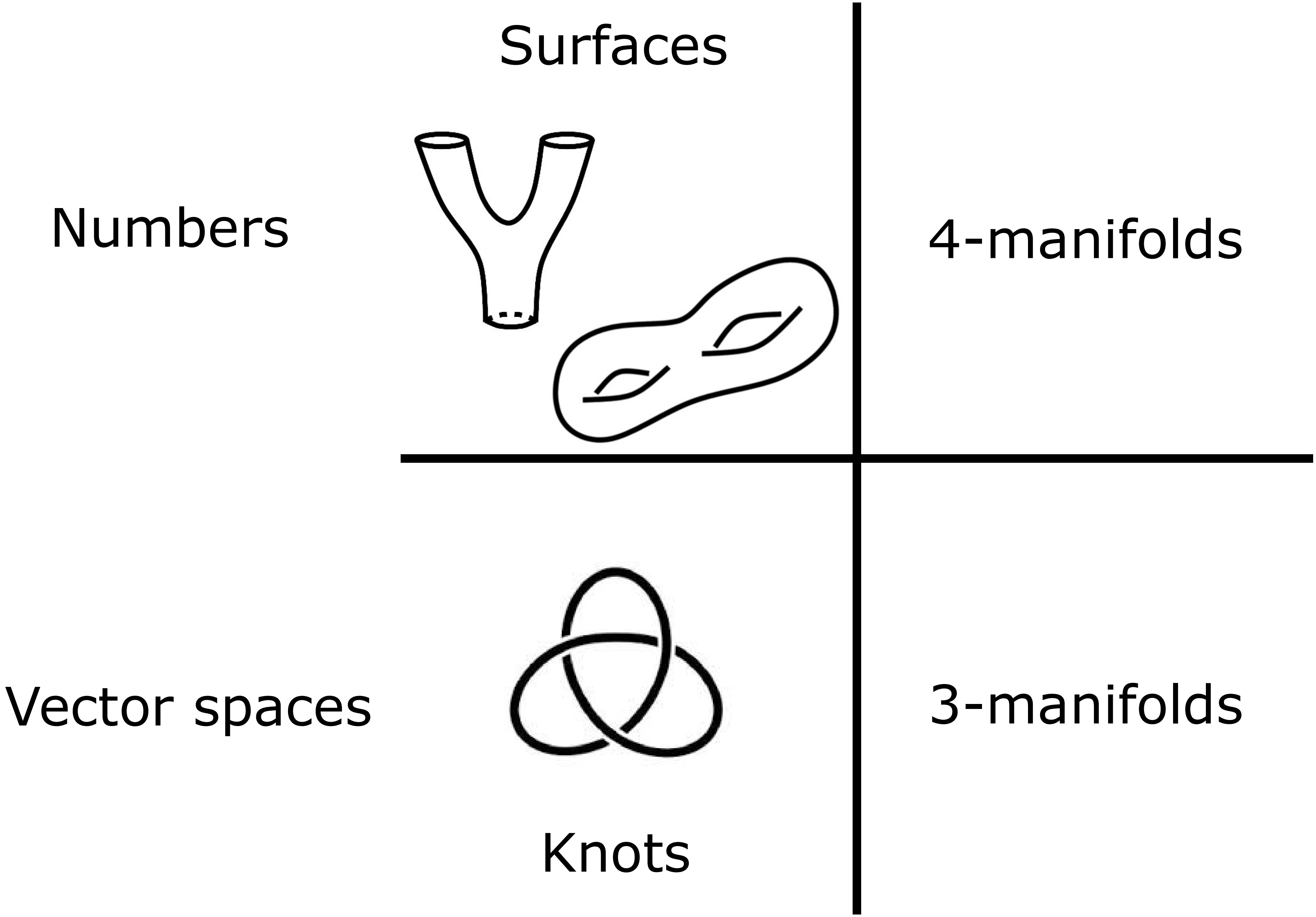
Thank you, Marko!

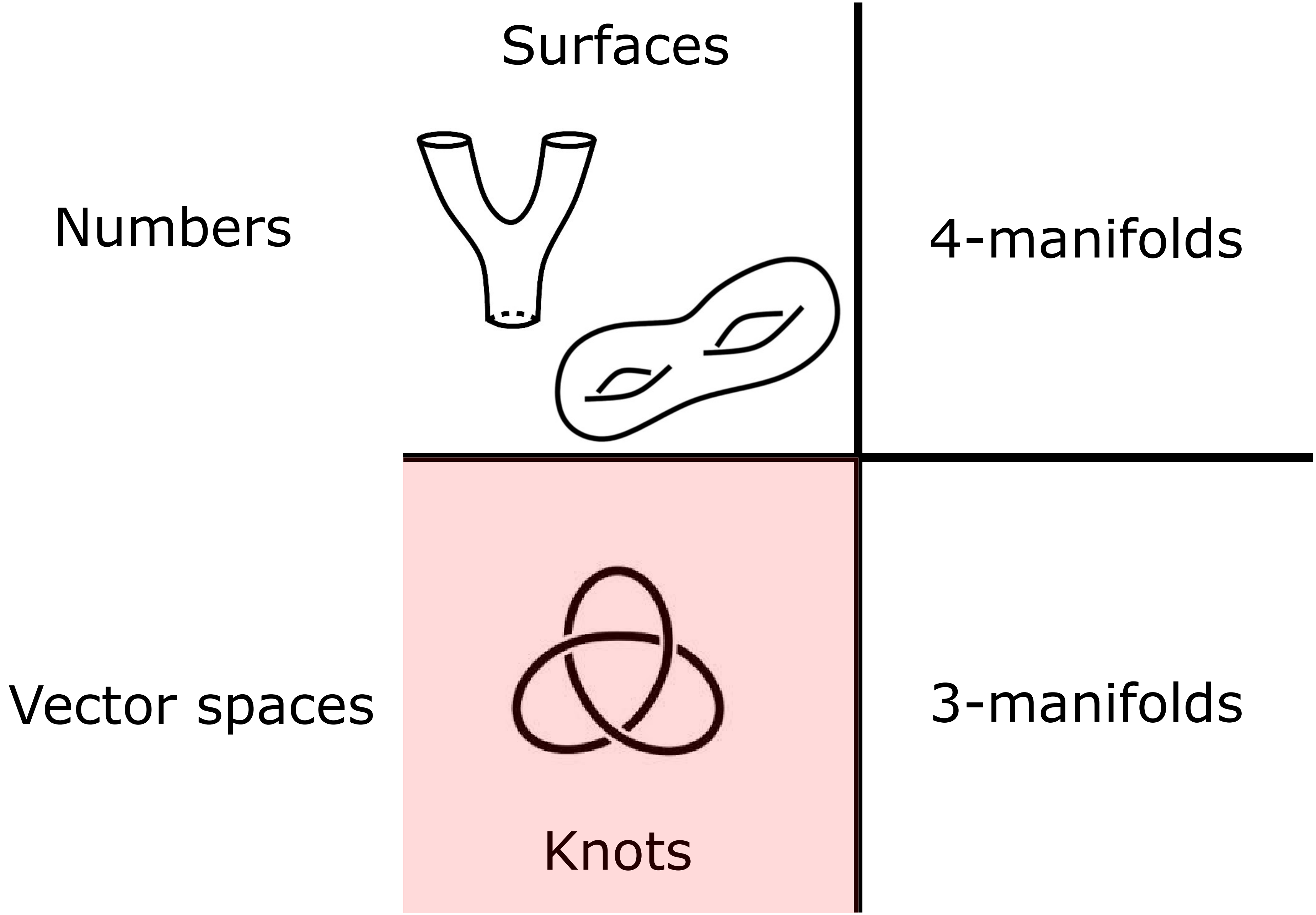




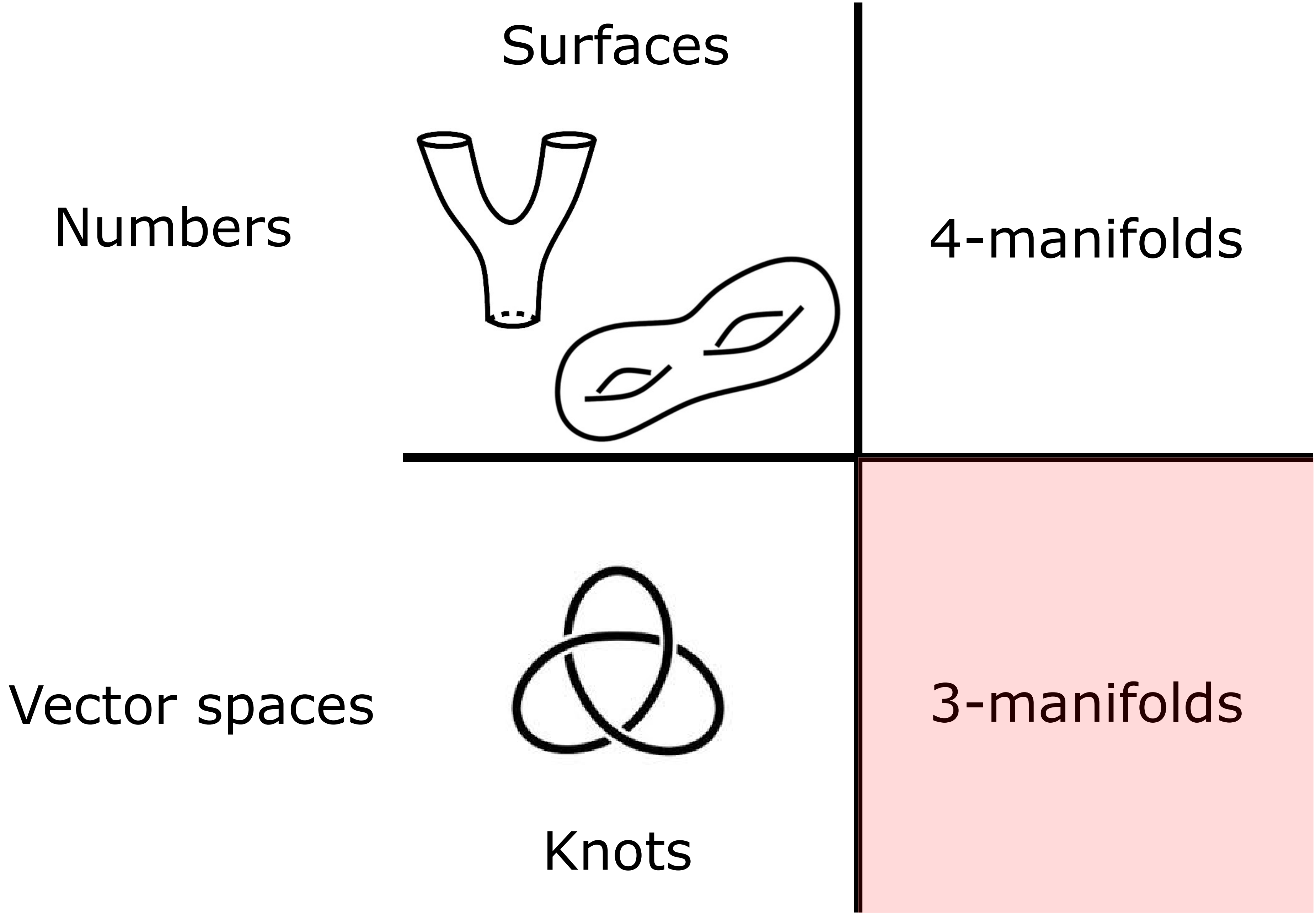
$\lambda =$  $\longleftrightarrow$ $\lambda^t =$ 



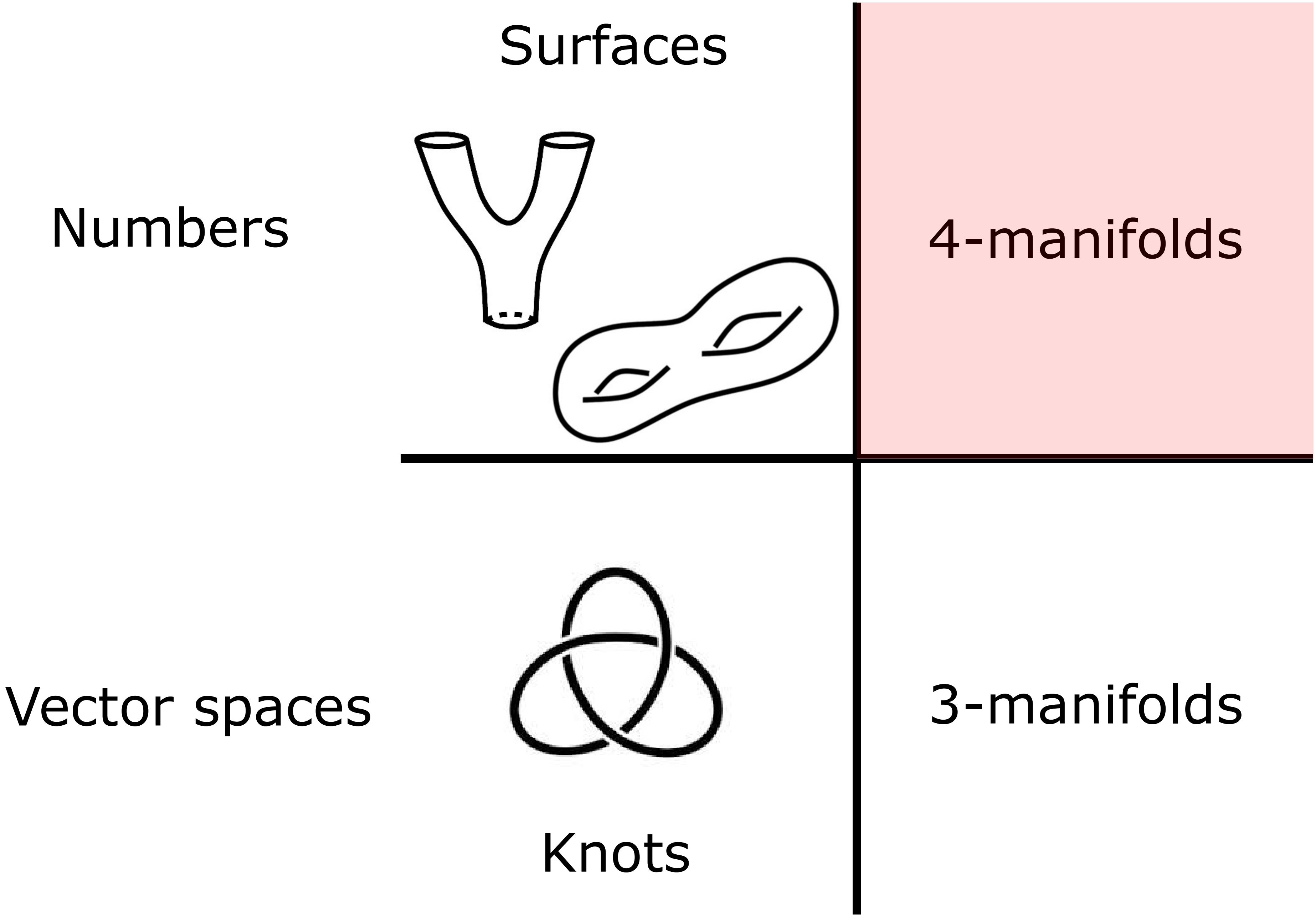
[M.Dedushenko, S.G., P.Putrov]



[M.Dedushenko, S.G., P.Putrov]



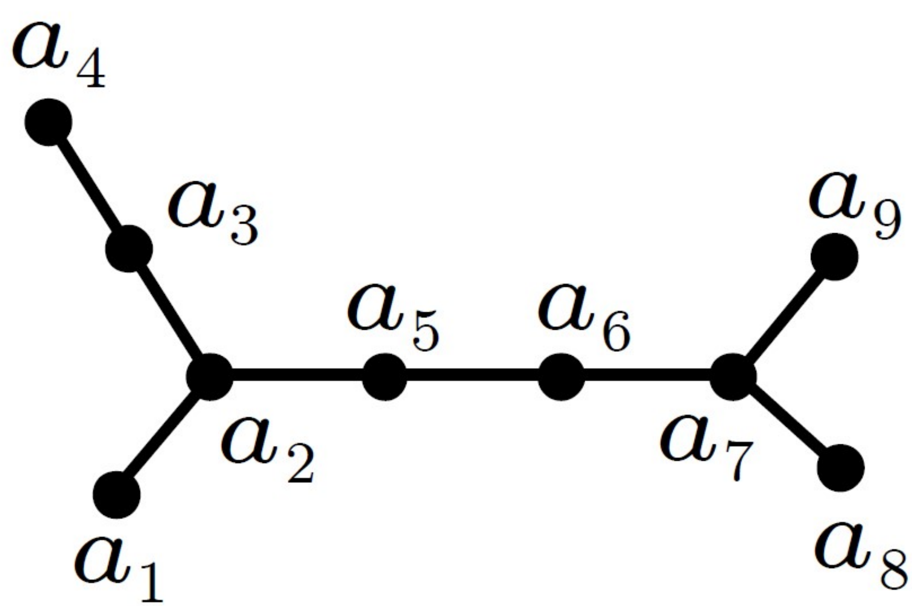
[M.Dedushenko, S.G., P.Putrov]



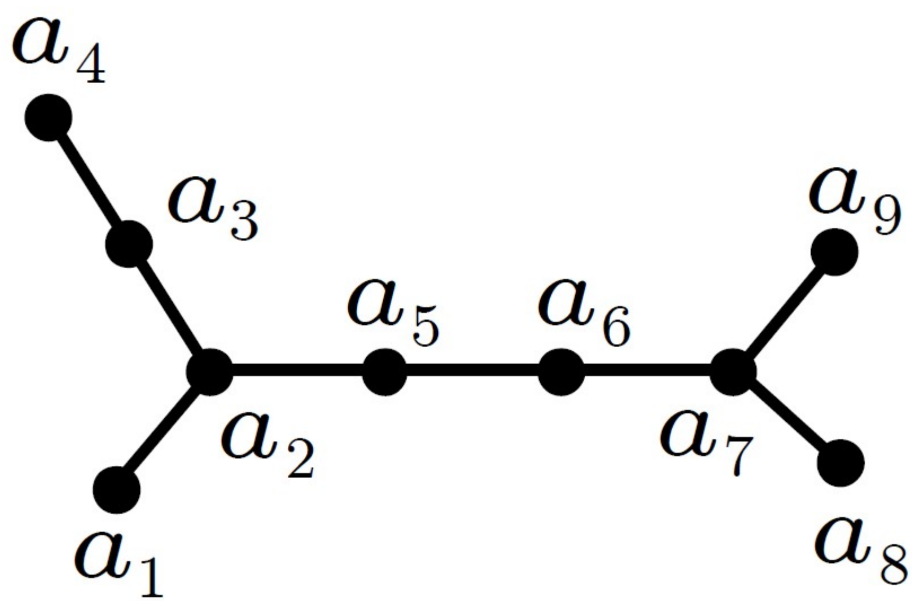
[M.Dedushenko, S.G., P.Putrov]

Definition: A *conformal field theory* is a table of integrals.

- Brian Greene



$$\text{Vertices}, \quad \text{Edges}, \quad a : \text{Vertices} \to \mathbb{Z}$$



$$Q_{ij} = \begin{cases} a_i, & \text{if } i = j \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge} \\ 0, & \text{otherwise} \end{cases}$$

vertex 

$$\longleftrightarrow \quad q^{-\frac{a+3}{4}}\left(x-\frac{1}{x}\right)^2$$

edge

$$\longleftrightarrow \quad \frac{1}{\left(x_1-\frac{1}{x_1}\right)\left(x_2-\frac{1}{x_2}\right)}$$

$$\widehat{Z}_b(q) = \text{v.p.} \int_{|x_j|=1} \prod_{j\in\text{Vertices}} \frac{dx_j}{2\pi i x_j} \cdots \prod_{(i,j)\in\text{Edges}} \cdots \Theta_b^Q(\vec{x})$$

$$\widehat{Z}_b(q) = \text{v.p.} \oint\limits_{|x_j|=1} \prod_{j\in\text{Vertices}} \frac{dx_j}{2\pi i x_j} \left(x_j - \frac{1}{x_j} \right)^{2-\deg(j)} \Theta_b^Q$$

$$\Theta_b^Q = \sum_{\vec{n}\in Q\mathbb{Z}^{|\text{Vert}|}+b} q^{-(n,Q^{-1}n)} \prod_i x_i^{n_i}$$

$$b \in \operatorname{coker} Q$$

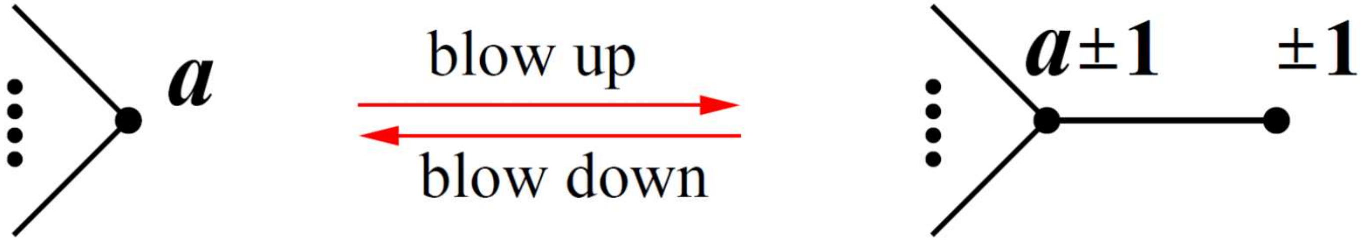
4-Manifolds and Kirby Calculus

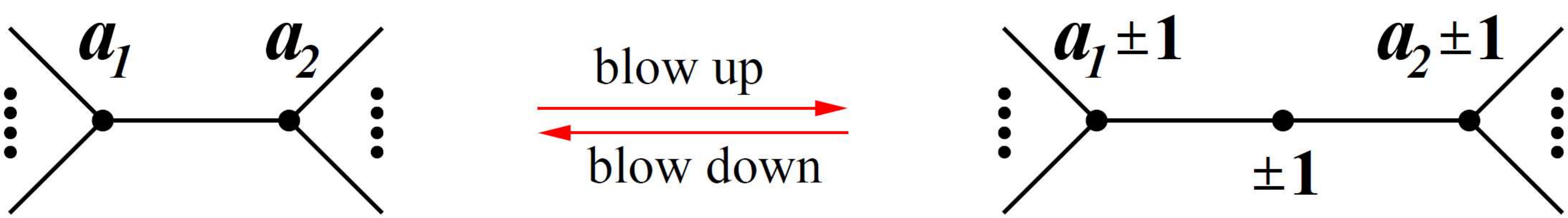
Robert E. Gompf
András I. Stipsicz

Graduate Studies
in Mathematics
Volume 20

American Mathematical Society

Kirby Calculus





new 3d TQFT $\widehat{Z}$

Quantum groups at generic $|q|<1$

complex Chern-Simons theory

new 3d TQFT $\widehat{Z}$

q-deformed $1/\Delta$

associated with Non-semisimple Modular Categories new 4d TQFT $\widehat{Z}$

Quantum groups at generic $|q|<1$

complex Chern-Simons theory

q-deformed $1/\Delta$

associated with Non-semisimple Modular Categories

Theorem [GPPV]:

- $\widehat{Z}_b(M_3; q)$ converges in $|q| < 1$

$$\begin{array}{l} \rotatebox{90}{$\in$} \\ \mathrm{coker}\, Q \;\cong\; H_1(M_3;\mathbb{Z}) \;\cong\; \mathrm{Spin}^c(M_3) \end{array}$$

Theorem [GPPV]:

- $\widehat{Z}_b(M_3; q)$ converges in $|q| < 1$

  $b \in \operatorname{coker} Q \cong H_1(M_3; \mathbb{Z}) \cong \mathrm{Spin}^c(M_3)$

- has integer powers and integer coefficients

$$\widehat{Z}_b = q^{d_b} \left(c_0 + c_1 q + c_2 q^2 + \ldots \right) \in q^{d_b} \mathbb{Z}[[q]]$$

"correction term"
(Heegaard Floer,
Seiberg-Witten theory)

Theorem [GPPV]:

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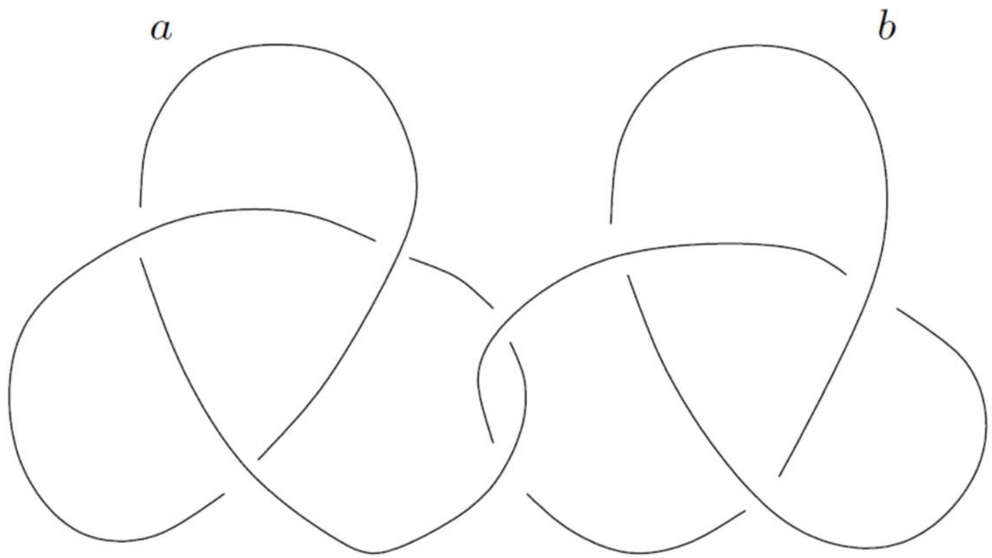
- invariant under Kirby moves
- gives $\mathrm{WRT}(M_3; k)$ as $q \to e^{2\pi i/k}$

"level"

Theorem [Lickorish, Wallace, Kirby]:

Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link $K \hookrightarrow S^3$ (*i.e.* a surgery along a framed link)



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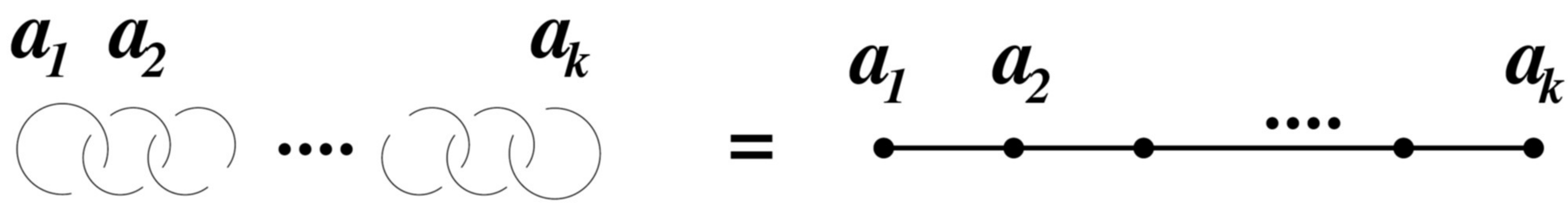
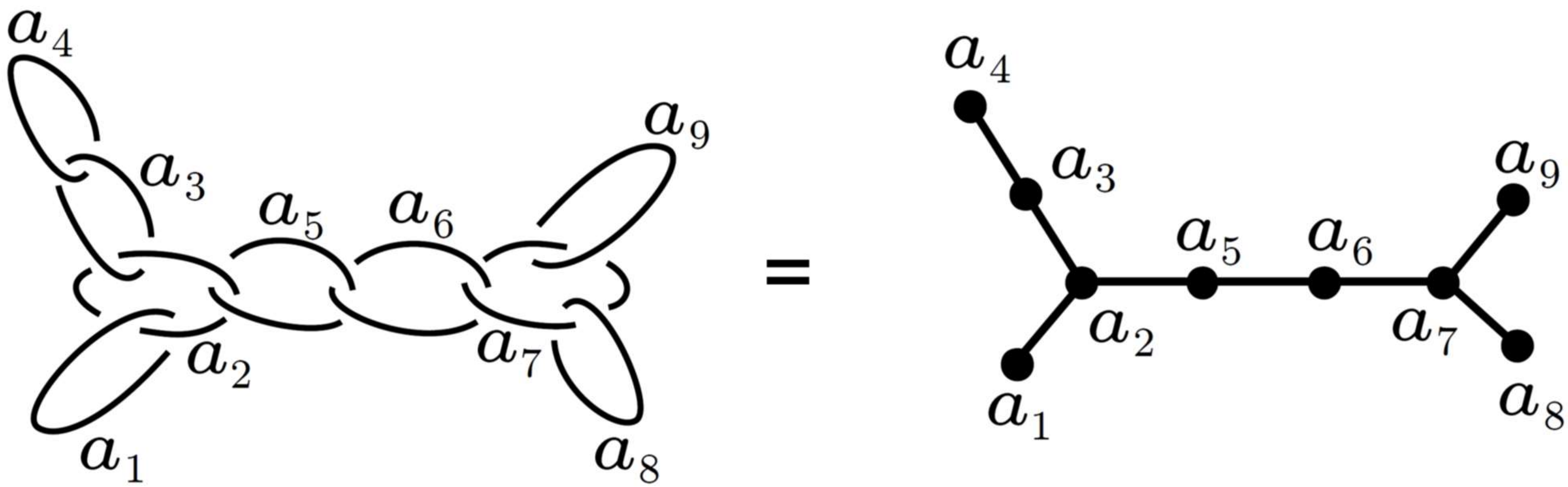
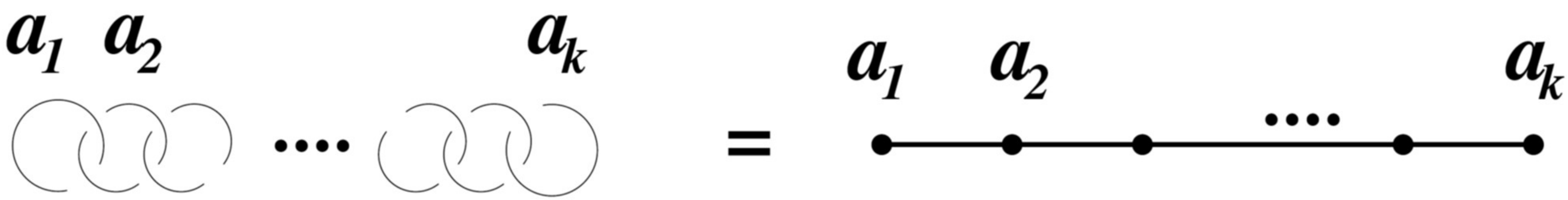
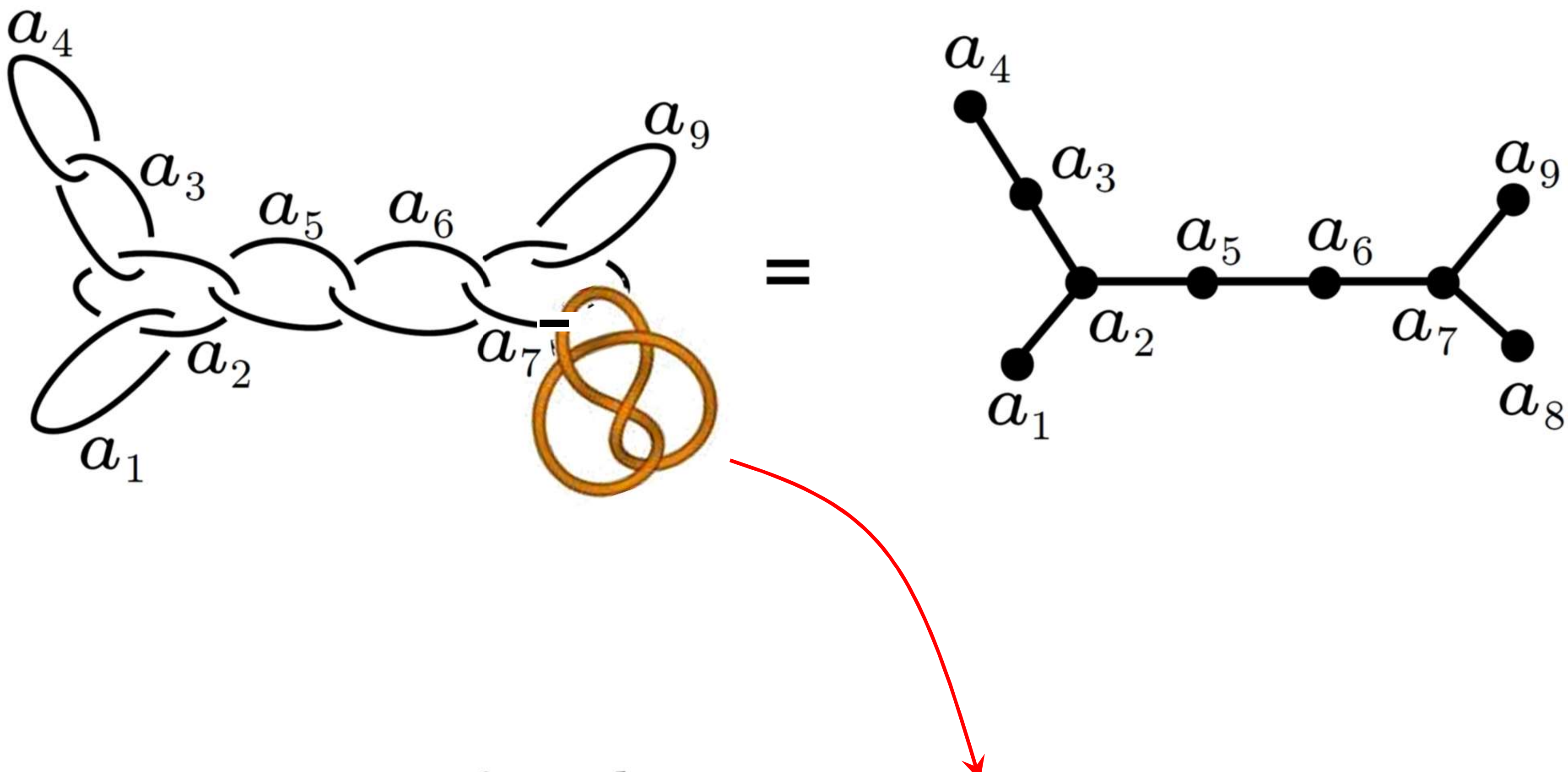




a_4
a_3
a_5
a_6
a_9
a_2
a_7
a_1
a_8
=
a_4
a_3
a_5
a_6
a_9
a_2
a_7
a_1
a_8



a_1 a_2 a_k
....
=
a_1 a_2 a_k
....



$$\widehat{Z} = \sum_{n_v} \oint_{|z_v|=1} \frac{dz_v}{2\pi i z_v} \prod_{\text{vertices}} (\dots) \prod_{\text{edges}} (\dots)$$

$$Z^{(\alpha)}_{\text{pert}}(S^3 \setminus K; \hbar, x) \; = \; \exp\left(\frac{1}{\hbar} S_0^{(\alpha)}(x) + S_1^{(\alpha)}(x) + \hbar S_2^{(\alpha)}(x) + \ldots \right)$$

Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group

Tudor Dimofte,[1] **Sergei Gukov,**[1,2] **Jonatan Lenells,**[3] **and Don Zagier**[4,5]

[1] *California Institute of Technology, Pasadena, CA 91125, USA*
[2] *Department of Physics and Department of Mathematics, University of California, Santa Barbara, CA 93106, USA*
[3] *Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom*
[4] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bo*
[5] *Collège de France, 3 rue d'Ulm, F-75005 Paris, France*

Borel resum

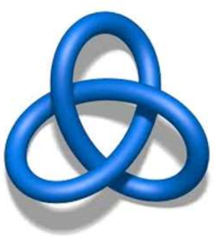
$$\sum_{n=0}^{\infty} q^n \mathrm{DT}_n \qquad \exp \sum_{n=0}^{\infty} S_n \hbar^n$$

$$q = e^{\hbar} \to 1$$

Emile Borel
(1871-1956)

$$
\begin{aligned}
Z^{(\mathrm{ab})}_{\mathrm{pert}}(\mathbf{3_1}) &= (x^{1/2} - x^{-1/2} - x^{5/2} + x^{-5/2} - x^{7/2} + x^{-7/2} + \ldots) \\
&+ \hbar(x^{1/2} - x^{-1/2} - 2x^{5/2} + 2x^{-5/2} - 3x^{7/2} + 3x^{-7/2} + \ldots) \\
&+ \frac{\hbar^2}{2}(x^{1/2} - x^{-1/2} - 4x^{5/2} + 4x^{-5/2} - 9x^{7/2} + 9x^{-7/2} + \ldots) \\
&+ \frac{\hbar^3}{6}(x^{1/2} - x^{-1/2} - 8x^{5/2} + 8x^{-5/2} - 27x^{7/2} + 27x^{-7/2} + \ldots) \\
&+ \frac{\hbar^4}{24}(x^{1/2} - x^{-1/2} - 16x^{5/2} + 16x^{-5/2} - 81x^{7/2} + 81x^{-7/2} + \ldots) \\
&+ \frac{\hbar^5}{120}(x^{1/2} - x^{-1/2} - 32x^{5/2} + 32x^{-5/2} - 243x^{7/2} + 243x^{-7/2} + \ldots) \\
&+ \ldots \\
&= \sum_{m \geq 1} f_m(q) \cdot (x^{\frac{m}{2}} - x^{-\frac{m}{2}})
\end{aligned}
$$

$$f_m = \epsilon_m q^{\frac{m^2+23}{24}}$$



$$f_1 = -q, \quad f_3 = 0, \quad f_5 = q^2, \quad f_7 = q^3, \quad f_9 = 0, \quad \ldots$$

[S.G., C.Manolescu]

$$\widehat{A}(\widehat{x}, \widehat{y}; K, q)\, F_K(x,q) \;=\; 0$$

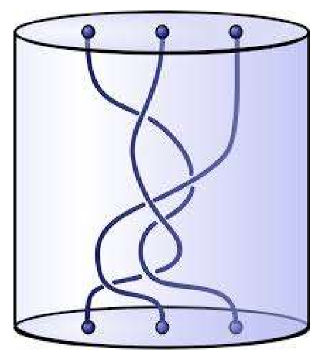
$$F_K(x,q) = \sum_{m=1}^{\infty} f_m(q)\left(x^m - x^{-m}\right)$$

$$f_1 = 1,$$
$$f_3 = 2,$$
$$f_5 = 1/q + 3 + q,$$
$$f_7 = 2/q^2 + 2/q + 5 + 2q + 2q^2,$$
$$f_9 = 1/q^4 + 3/q^3 + 4/q^2 + 5/q + 8 + 5q + 4q^2 + 3q^3 + q^4,$$
$$f_{11} = 2/q^6 + 2/q^5 + 6/q^4 + 7/q^3 + 10/q^2 + 10/q + 15 + 10q + 10q^2 + 7q^3 + 6q^4 + 2q^5 + 2q^6,$$
$$f_{13} = 1/q^9 + 3/q^8 + 4/q^7 + 7/q^6 + 11/q^5 + 15/q^4 + 18/q^3 + 21/q^2 + 23/q + 27 + 23q + 21q^2 + 18q^3 + 15q^4 + 11q^5 + 7q^6 + 4q^7 + 3q^8 + q^9,$$

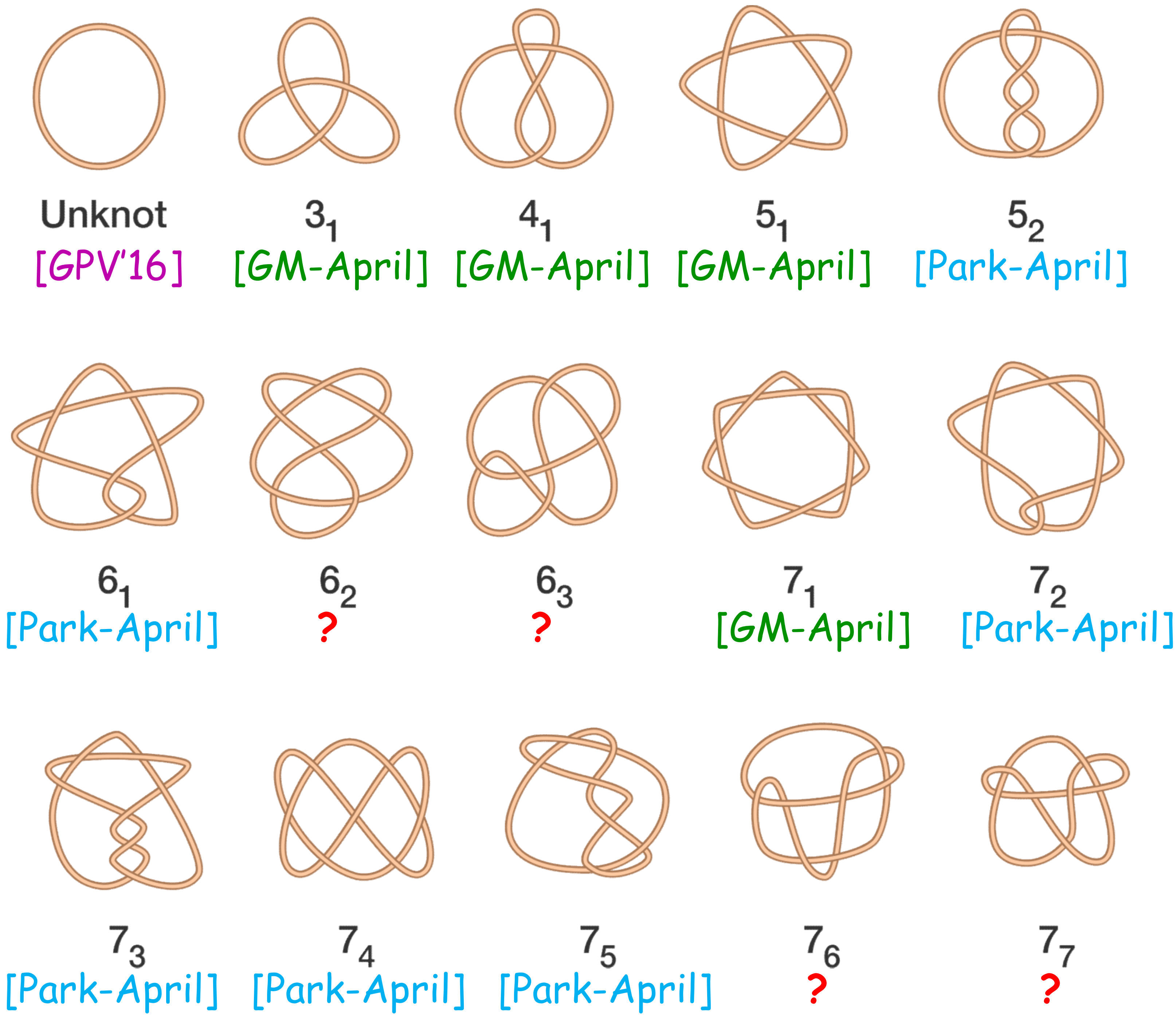
[S.G., C.Manolescu]

LARGE COLOR R-MATRIX FOR KNOT COMPLEMENTS AND STRANGE IDENTITIES

SUNGHYUK PARK



$$KEK^{-1} = q^2 E\,, \qquad KFK^{-1} = q^{-2} F\,, \qquad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$


Unknot
[GPV'16]
3_1
[GM-April]
4_1
[GM-April]
5_1
[GM-April]
5_2
[Park-April]
6_1
[Park-April]
6_2
?
6_3
?
7_1
[GM-April]
7_2
[Park-April]
7_3
[Park-April]
7_4
[Park-April]
7_5
[Park-April]
7_6
?
7_7
?

Theorem ("surgery formula"):

$$\widehat{Z}_b\big(S^3_{-p/r}(K)\big) = \oint_{|x|=1} \frac{dx}{2\pi i x}\big(x^{\frac{1}{r}} - x^{-\frac{1}{r}}\big) F_K(x,q) \sum_{rn\, =\, b \bmod p} q^{\frac{r}{p}n^2} x^n$$

$$F_K(x,q) = \sum_{m=1}^{\infty} f_m(q)\left(x^m - x^{-m}\right)$$

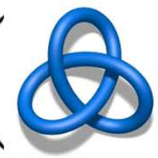
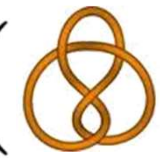
[S.G., C.Manolescu]

Theorem ("surgery formula"):

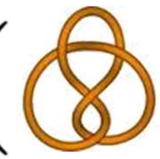
$$\widehat{Z}_b\big(S^3_{-p/r}(K)\big) = \oint_{|x|=1} \frac{dx}{2\pi i x}\left(x^{\frac{1}{r}} - x^{-\frac{1}{r}}\right) F_K(x,q) \sum_{rn \,=\, b \bmod p} q^{\frac{r}{p}n^2} x^n$$

$$F_{\mathbf{3_1}}(x,q) = \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}} \left(x^m - x^{-m}\right)$$

[S.G., C.Manolescu]

$M_3 = S^3_{-1}($  $) = S^3_{+1}($  $)$

$$\widehat{Z}(q) = q^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^{n+1};q)_n}$$

$M_3 = S^3_{-1}($  $) = S^3_{+1}($  $)$

$$\widehat{Z}(q) = q^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^{n+1};q)_n}$$

$=$ character of (1,p) "singlet" log-VOA with p=42

"3d Modularity"
[M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison]

Dear Hardy,

I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call "Mock" theta-functions. Unlike the "False" theta-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as ordinary theta-functions. I am sending you with this letter some examples …

Srinivasa Ramanujan
(January 12, 1920)

What kind of function is $\hat{Z}(q)$?

cf. [K.Bringmann, K.Mahlburg, A.Milas]

$S^3_{-1/r}(\mathbf{4_1})$	$\widehat{Z}_a(q)$
$r=2$	$-q^{-1/2}(1-q+2q^3-2q^6+q^9+3q^{10}+q^{11}-q^{14}-3q^{15}-3q^{15}-q^{16}+2q^{19}$ $+2q^{20}+5q^{21}+2q^{22}+2q^{23}-2q^{26}-2q^{27}-5q^{28}-2q^{29}-2q^{30}+\cdots)$
$r=3$	$-q^{-1/2}(1-q+2q^5-2q^8+q^{15}+3q^{16}+q^{17}-q^{20}-3q^{21}-q^{22}+2q^{31}$ $+2q^{32}+5q^{33}+2q^{34}+2q^{35}-2q^{38}-2q^{39}-5q^{40}-2q^{41}-2q^{42}+\cdots)$
$r=4$	$-q^{-1/2}(1-q+2q^7-2q^{10}+q^{21}+3q^{22}+q^{23}-q^{26}-3q^{27}-q^{28}+2q^{43}$ $+2q^{44}+5q^{45}+2q^{46}+2q^{47}-2q^{50}-2q^{51}-5q^{52}-2q^{53}-2q^{54}+\cdots)$
$r=5$	$-q^{-1/2}(1-q+2q^9-2q^{12}+q^{27}+3q^{28}+q^{29}-q^{32}-3q^{33}-q^{34}+2q^{55}$ $+2q^{56}+5q^{57}+2q^{58}+2q^{59}-2q^{62}-2q^{63}-5q^{64}-2q^{65}-2q^{66}+\cdots)$
$r=6$	$-q^{-1/2}(1-q+2q^{11}-2q^{14}+q^{33}+3q^{34}+q^{35}-q^{38}-3q^{39}-q^{40}+2q^{67}$ $+2q^{68}+5q^{69}+2q^{70}+2q^{71}-2q^{74}-2q^{75}-5q^{76}-2q^{77}-2q^{78}+q^{112}+\cdots)$
$r=7$	$-q^{-1/2}(1-q+2q^{13}-2q^{16}+q^{39}+3q^{40}+q^{41}-q^{44}-3q^{45}-q^{46}+2q^{79}$ $+2q^{80}+5q^{81}+2q^{82}+2q^{83}-2q^{86}-2q^{87}-5q^{88}-2q^{89}-2q^{90}+\cdots)$

[S.G., C.Manolescu]

Theorem:

Frobenius algebra → 2d TQFT

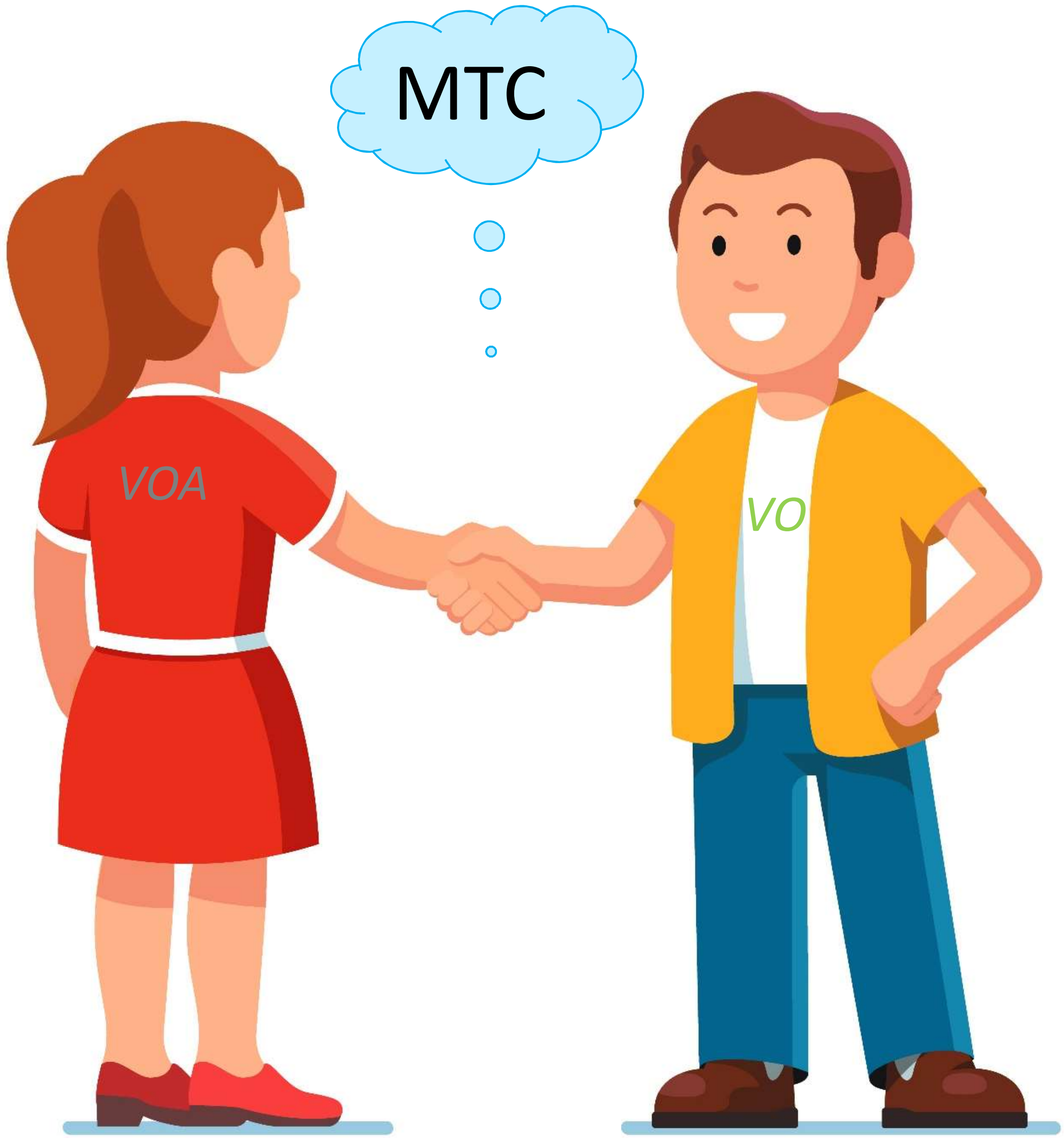
Theorem:

MTC → 3d TQFT

Reshetikhin-Turaev construction

3-manifold → { MTC[M_3], Log-VOA[M_3] }

4-manifold → { VOA[M_4], TMF class [M_4] }


MTC
VOA
VO

$$\mathrm{TQFT}_d : \quad \mathrm{Bord}_d \longrightarrow d\text{-category}$$

Fiber Integration

$$\mathrm{TQFT}_d : \quad \mathrm{Bord}_d \longrightarrow d\text{-category}$$

$$\mathrm{TQFT}_{d-n}(\ \ldots\ ) \quad := \quad \mathrm{TQFT}_d(\ \ldots \times M_n)$$

labeled by M_n

EFT-valued Topological Invariants

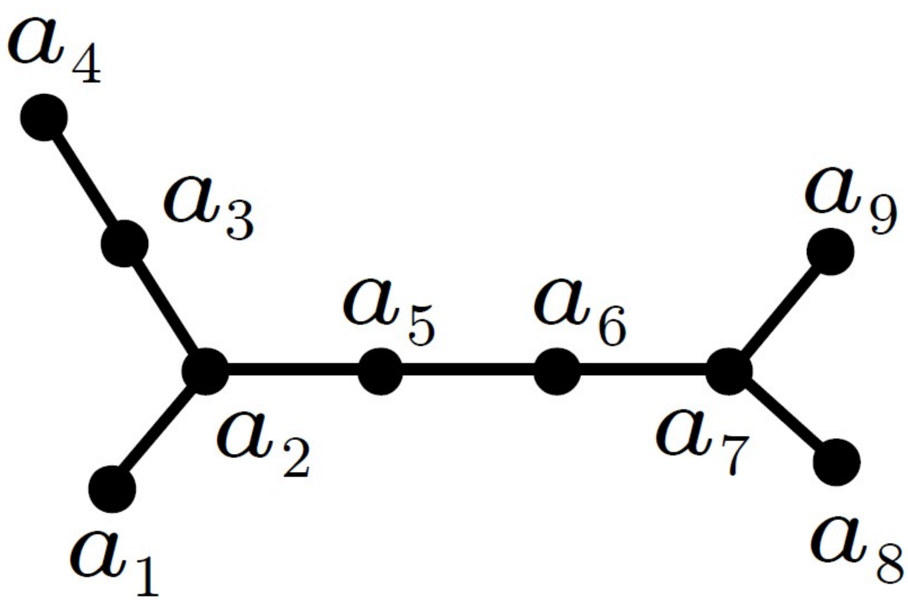
$$\mathrm{EFT}_{d-n}(\ \dots\ ) \ := \ \mathrm{EFT}_{d}(\ \dots \times M_n\ )$$

$$\mathrm{TQFT}_{d-n}(\ \dots\ ) \ := \ \mathrm{TQFT}_{d}(\ \dots \times M_n\ )$$

labeled by M_n

CFT-valued Topological Invariants

$3|4$-CFT(…) = $6|16$-CFT(… x M_3)



CFT-valued Topological Invariants

$2|2$-CFT(…) = $6|16$-CFT(… x M_4)

VOA[M_4]

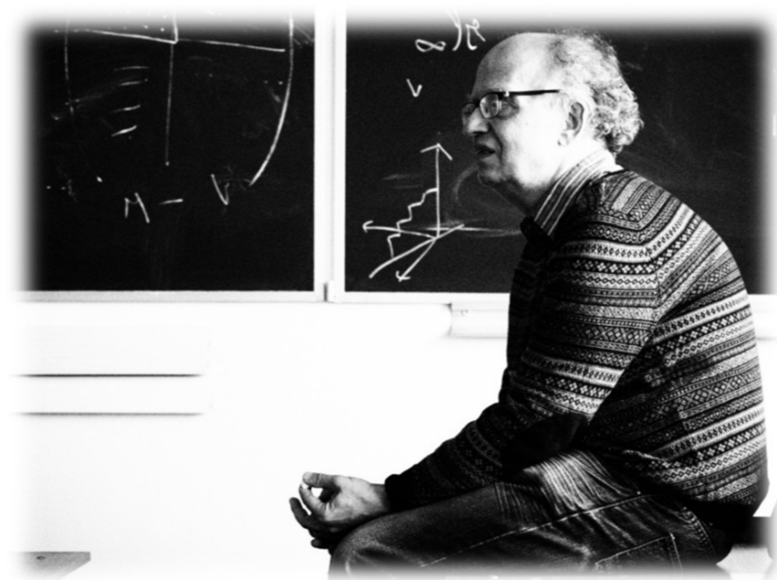
[B.Feigin, S.G.]
[A.Gadde, S.G., P.Putrov]

M_4	c_L	c_R
S^4	$26 = 2 + 24$	$27 = 3 + 24$
$\mathbb{C}\mathbf{P}^2$	57	60
$\mathbb{C}\mathbf{P}^1 \times \Sigma_{g,n}$	$2g + 4n + 4$	$6n + 6$
$m\mathbb{C}\mathbf{P}^2 \# n\overline{\mathbb{C}\mathbf{P}}^2$	$26 + 31m - 5n$	$27 + 33m - 6n$

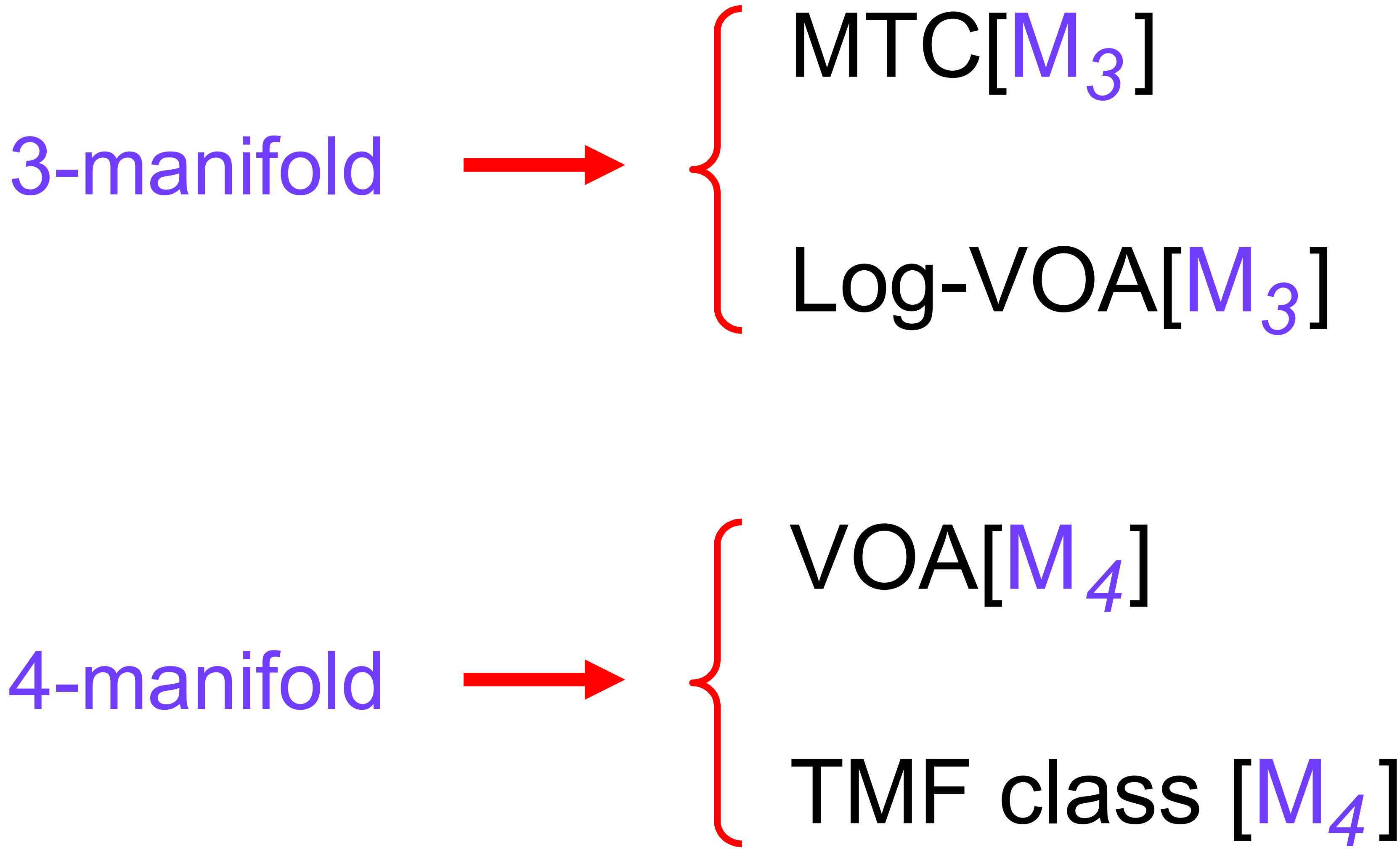
Equivalences (e.g. trialities)

$\cong$

Kirby moves



[B.Feigin, S.G.]
[A.Gadde, S.G., P.Putrov]


3-manifold
MTC[M_3]
Log-VOA[M_3]
4-manifold
VOA[M_4]
TMF class [M_4]