Rational and algebraic links and knots-quivers correspondence

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Ingredient 1: Knots

Colored HOMFLY–PT polynomials:

Symmetric (S^r)-colored HOMFLY–PT polynomials are 2-variable invariants of knots:

$$P_r(K)(a,q).$$

For $a = q^N$ they are $(sl(N), S^r)$ quantum polynomial invariants:

$$P(a = q^N, q) = P^{sl(N),S^r}(q).$$

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Already interesting is the "bottom row": the coefficient of the lowest nonzero power of a appearing in $P_r(a,q)$

$$P_r^-(q) = \lim_{a \to 0} a^\sharp P_r(a,q)$$

LMOV conjecture

Generating function of all symmetric-colored HOMFLY-PT polynomials of a given knot K is:

$$P(x,a,q) := \sum_{r\geq 0} P_r(a,q)x^r = \exp\left(\sum_{n,r\geq 1}\frac{1}{n}f_r(a^n,q^n)x^{rn}\right),$$

$$f_r(a,q) = \sum_{i,j}\frac{N_{r,i,j}a^iq^j}{q-q^{-1}}.$$

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LMOV conjecture: $N_{r,i,j}\in\mathbb{Z}$!

$N_{r,i,j}$ are BPS numbers. They represent (super)-dimensions of certain homological groups. Physicaly, they "count" particles of certain type (therefore are integers).

Ingredient 2: Quivers (and their representations)

Quivers are oriented graphs, possibly with loops and multiple edges.

$Q_0 = \{1, \ldots, m\}$ – set of vertices.

Q_1 the set of edges $\{\alpha : i \to j\}$.

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Let $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$ be a dimension vector.
We are interested in moduli space of representations of Q with the dimension vector $\mathbf{d}$:

$$M_{\mathbf{d}} = \left\{ R(\alpha) : \mathbb{C}^{d_i} \to \mathbb{C}^{d_j} | \textit{for all } \alpha : i \to j \in Q_1 \right\} // G,$$

where $G = \prod_i GL(d_i, \mathbb{C})$.

Quivers and motivic generating functions

C is a matrix of a quiver with m vertices.

$$P_C(x_1,\ldots,x_m) := \sum_{d_1,\ldots,d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j} d_i d_j}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}} x_1^{d_1}\cdots x_m^{d_m}.$$

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Motivic (quantum) Donaldson-Thomas invariants $\Omega_{d_1,\ldots,d_m;j}$ of a symmetric quiver Q:

$$P_C = \prod_{(d_1,\ldots,d_m)\neq 0} \prod_{j\in\mathbb{Z}} \prod_{k\geq 0} \left(1-\left(x_1^{d_1}\cdots x_m^{d_m}\right) q^{j+2k+1}\right)^{(-1)^{j+1}\Omega_{d_1,\ldots,d_m;j}}.$$

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Theorem (Kontsevich-Soibelman, Efimov)

$\Omega_{d_1,\dots,d_m;j}$ *are nonnegative integers.*

Knots–quivers correspondence

[P. Kucharski, M. Reineke, P. Sulkowski, M.S., Phys. Rev. D 2017]

New relationship between HOMFLY–PT / BPS invariants of knots and motivic Donaldson-Thomas invariants for quivers

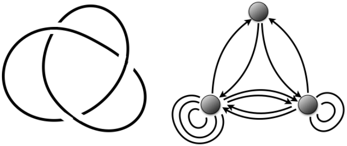

Figure: Trefoil knot and the corresponding quiver.

The generating series of HOMFLY-PT invariants of a knot matches the motivic generating series of a quiver, after setting $x_i \to x$.

Details of the correspondence

Knots	**Quivers**
Generators of HOMFLY homology	Number of vertices
Homological degrees, framing	Number of loops
Colored HOMFLY-PT	Motivic generating series
LMOV invariants	Motivic DT-invariants
Classical LMOV invariants	Numerical DT-invariants
Algebra of BPS states	Cohom. Hall Algebra

Application 1 – LMOV conjecture

BPS/LMOV invariants of knots are refined through motivic DT invariants of a corresponding quiver, and so

Theorem

For all knots for which there exists a corresponding quiver, the LMOV conjecture holds.

Open questions

- Find quivers for (large) classes of knots

- How to find a quiver for a given knot directly (geometrically, topologically...)? Other, better definition?

- The (non)uniqueness of a quiver:

  - What is the smallest possible size of the quiver?

  - Among the ones with the same size – discrete group action?

- What is so special for quivers that correspond to knots? (Combinatorial identities for binomial coefficients, and extended integrality/divisibility hold precisely for them.)

Application 2 – Lattice paths counting

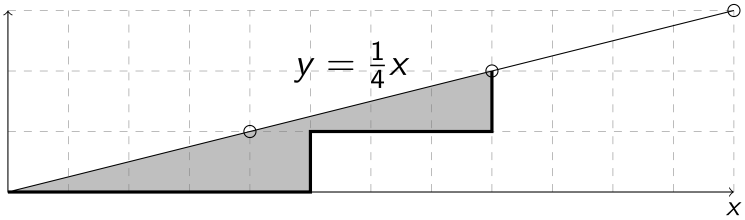

Figure: A lattice path under the line $y = \frac{1}{4}x$, and a shaded area between the path and the line.

$$y_P(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \, k\text{-paths}} x^k = \sum_{k=0}^{\infty} c_k(1) x^k,$$

$$y_{qP}(x) = \sum_{k=0}^{\infty} \sum_{\pi \in \, k\text{-paths}} q^{area(\pi)} x^k = \sum_{k=0}^{\infty} c_k(q) x^k.$$

Counting (rational) lattice paths

Surprisingly, colored HOMFLY–PT polynomials are closely related to the purely combinatorial problem of counting lattice paths under lines with rational slope.

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Proposition (M. Panfil, P. Sulkowski, M.S., 2018)

Let r and s be mutually prime.

Let $K = T_{r,s}^{f=-rs}$ be the rs-framed (r,s)-torus knot.

Then the coefficients a_n of the series governing the growth of the generating series of colored HOMFLY-PT polynomial of K, are equal to the number of directed lattice path from $(0,0)$ to (sn, rn) under the line $y = (r/s)x$.

Knots and quivers – results

Knots-quivers correspondence naturally suggests a particular refinement of the numbers a_n.

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For example, for $2/3$ slope, after computing the relevant invariants (for the bottom row) of the $T_{2,3}$ knot (trefoil), the corresponding quiver is:

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$$\begin{aligned} a_n^{(2/3)} &= \sum_{i+j=n} \frac{1}{7i+5j+1} \binom{7i+5j+1}{i} \binom{5i+5j+1}{j} \\ &= \sum_{i=0}^{n} \frac{1}{5n+i+1} \binom{5n+2i}{i} \binom{5n+1}{n-i}. \end{aligned}$$

(rediscovered Duchon formula)

Consequence 1 – binomial identities

All this also rediscovers some binomial identities, like e.g.

$$\binom{5n}{2n} = \sum_{i=0}^{n} \frac{5n}{5n+2i} \binom{5n+2i}{i} \binom{5n}{n-i}$$

Comes from counting of paths under line with slope 2/3.

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Comes from counting of paths under line with slope $2/3$.

One can obtain such identities precisely for the quivers that correspond to (torus) knots.

Counting of weighted lattice paths

Proposition

The generating function $y_{qP}(x)$ of lattice paths under the line of the slope r/s, weighted by the area between this line and a given path, is equal to

$$y_{qP}(x) = \sum_{k=0}^{\infty} \sum_{\pi \in k\text{-paths}} q^{area(\pi)} x^k = \frac{P_C(q^2x_1, \ldots, q^2x_m)}{P_C(x_1, \ldots, x_m)}\Big|_{x_i = xq^{-1}}.$$

For the line of the slope r/s, the quiver in question is defined by the matrix C that encodes extremal invariants of left-handed (r, s) torus knot in framing rs.

Paths under the line with slope 3/4

$$C^{(3,4)} = \begin{bmatrix} 7 & 7 & 7 & 7 & 7 \\ 7 & 9 & 8 & 9 & 9 \\ 7 & 8 & 9 & 9 & 10 \\ 7 & 9 & 9 & 11 & 11 \\ 7 & 9 & 10 & 11 & 13 \end{bmatrix}$$

$$\sharp paths = \sum_{l_1+\cdots+l_5=n} A_{(3,4)}(l_1,l_2,l_3,l_4,l_5)\times$$
$$\times \frac{1}{7l_1+7l_2+7l_3+7l_4+7l_5+1}\binom{7l_1+7l_2+7l_3+7l_4+7l_5+1}{l_1}\times$$
$$\times \frac{1}{7l_1+9l_2+8l_3+9l_4+9l_5+1}\binom{7l_1+9l_2+8l_3+9l_4+9l_5+1}{l_2}\times$$
$$\times \frac{1}{7l_1+8l_2+9l_3+9l_4+10l_5+1}\binom{7l_1+8l_2+9l_3+9l_4+10l_5+1}{l_3}\times$$
$$\times \frac{1}{7l_1+9l_2+9l_3+11l_4+11l_5+1}\binom{7l_1+9l_2+9l_3+11l_4+11l_5+1}{l_4}\times$$
$$\times \frac{1}{7l_1+9l_2+10l_3+11l_4+13l_5+1}\binom{7l_1+9l_2+10l_3+11l_4+13l_5+1}{l_5}.$$

$$
\begin{aligned}
A_{(3,4)}(l_1,l_2,l_3,l_4,l_5)= \; & 1+28\, l_1+294\, l_1^2+1372\, l_1^3+2401\, l_1^4+33\, l_2+693\, l_1 l_2+4851\, l_1^2 l_2+11319\, l_1^3 l_2+407\, l_2^2+5698\, l_1 l_2^2+ \\
& +19943\, l_1^2 l_2^2+2223\, l_2^3+15561\, l_1 l_2^3+4536\, l_2^4+34\, l_3+714\, l_1 l_3+4998\, l_1^2 l_3+11662\, l_1^3 l_3+838\, l_2 l_3+11732 l_1 l_2 l_3+ \\
& +41062 l_1^2 l_2 l_3+6860 l_2^2 l_3+48020 l_1 l_2^2 l_3+18648 l_2^3 l_3+431 l_3^2+6034 l_1 l_3^2+21119 l_1^2 l_3^2+7051 l_2 l_3^2+49357 l_1 l_2 l_3^2+ \\
& +28728 l_2^2 l_3^2+2414 l_3^3+16898 l_1 l_3^3+19656 l_2 l_3^3+5040 l_3^4+36 l_4+756 l_1 l_4+5292 l_1^2 l_4+12348 l_1^3 l_4+887 l_2 l_4+ \\
& +12418 l_1 l_2 l_4+43463 l_1^2 l_2 l_4+7258 l_2^2 l_4+50806 l_1 l_2^2 l_4+19719 l_2^3 l_4+21294 l_3^3 l_4+482 l_4^2+6748 l_1 l_4^2+23618 l_1^2 l_4^2+ \\
& +912 l_3 l_4+12768 l_1 l_3 l_4+44688 l_1^2 l_3 l_4+14914 l_2 l_3 l_4+104398 l_1 l_2 l_3 l_4+60732 l_2^2 l_3 l_4+7656 l_3^2 l_4+53592 l_1 l_3^2 l_4+62307 l_2 l_3^2 l_4+ \\
& +7879 l_2 l_4^2+55153 l_1 l_2 l_4^2+32067 l_2^2 l_4^2+8086 l_3 l_4^2+56602 l_1 l_3 l_4^2+23688 l_3 l_4^3+6237 l_4^4+65772 l_2 l_3 l_4^2+37 l_5+777 l_1 l_5+ \\
& +33705 l_3^2 l_4^2+2844 l_4^3+19908 l_1 l_4^3+23121 l_2 l_4^3+5439 l_1^2 l_5+12691 l_1^3 l_5+912 l_2 l_5+12768 l_1 l_2 l_5+44688 l_1^2 l_2 l_5+ \\
& +7465 l_2^2 l_5+52255 l_1 l_2^2 l_5+20286 l_2^3 l_5+69524 l_2 l_3 l_5^2+35630 l_3^2 l_5^2+9010 l_4 l_5^2+63070 l_1 l_4 l_5^2+39501 l_4^2 l_5^2+25074 l_2 l_5^3+ \\
& +938 l_3 l_5+13132 l_1 l_3 l_5+45962 l_1^2 l_3 l_5+15342 l_2 l_3 l_5+107394 l_1 l_2 l_3 l_5+62482 l_2^2 l_3 l_5+7877 l_3^2 l_5+3083 l_5^3+21581 l_1 l_5^3+ \\
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& +113428 l_1 l_2 l_4 l_5+65961 l_2^2 l_4 l_5+16632 l_3 l_4 l_5+116424 l_1 l_3 l_4 l_5+135296 l_2 l_3 l_4 l_5+69335 l_3^2 l_4 l_5+25690 l_3 l_5^3+ \\
& +8771 l_4^2 l_5+61397 l_1 l_4^2 l_5+71316 l_2 l_4^2 l_5+73066 l_3 l_4^2 l_5+25641 l_4^3 l_5+509 l_5^2+7126 l_1 l_5^2+27027 l_4 l_5^3+ \\
& +24941 l_1^2 l_5^2+8325 l_2 l_5^2+58275 l_1 l_2 l_5^2+33894 l_2^2 l_5^2+8546 l_3 l_5^2+59822 l_1 l_3 l_5^2+6930 l_5^4 .
\end{aligned}
$$

Schröder paths

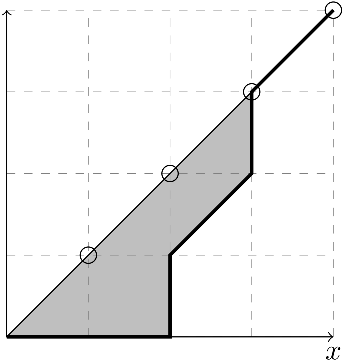

Figure: An example of a Schröder path of length 6.

Schröder paths and full colored HOMFLY-PT

Quiver corresponding to the full colored HOMFLY-PT invariants of knots in framing $f = 1$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

This corresponds to counting paths under the diagonal line $y = x$.

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Then from the quiver generating function of C we get

$$y(x, a, q) = 1 + (q + a)x + \left(q^2 + q^4 + (2q + q^3)a + a^2\right)x^2 + \dots$$

with the height of a path measured by the power of x and the number of diagonal steps measured by the power of a.

Schröder paths and full colored HOMFLY-PT

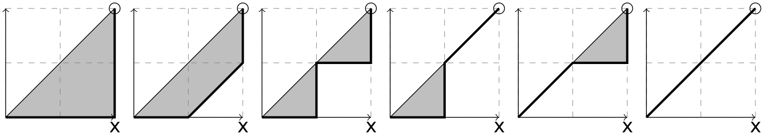

Figure: All 6 Schröder paths of height 2 represented by the quadratic term $q^2 + q^4 + (2q + q^3)a + a^2$ of the generating function.

Consequence 2 – Some divisibilities (integrality)

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If $r \in \mathbb{N}$, then $\quad r^2 \mid \sum_{d|r} \mu\left(\frac{r}{d}\right)\binom{3d-1}{d-1}.$

$$\mu(n) = \begin{cases} (-1)^k, & n = p_1 p_2 \cdots p_k, \\ 0, & p^2 | n \end{cases}$$

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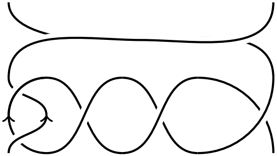
Corresponds to the fact that DT invariants are non-negative integers (in this case of the quiver of the framed unknot — one vertex, m-loop quiver)

Quivers for rational knots

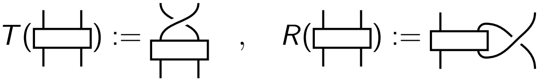
[M.S., P. Wedrich, IMRN 2019].

$$p/q = [a_1, \dots, a_r] = a_r + \frac{1}{a_{r-1} + \frac{1}{a_{r-2}+\dots}}$$

Rational tangle encoded by $[2, 3, 1]$



Rational knots



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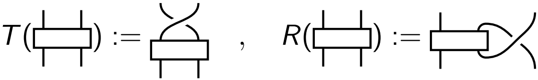
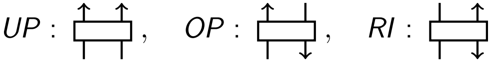
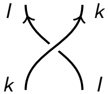
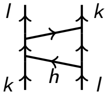


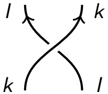


Skein theory

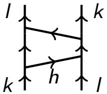


$$\overset{k\geq l}{\equiv} \sum_{h=0}^{l} (-q)^{h-l}$$

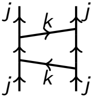
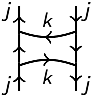
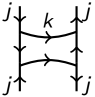




$$\overset{k\leq l}{\equiv} \sum_{h=0}^{k} (-q)^{h-k}$$



Basic webs and twist rules

$UP[j,k] =$  , $OP[j,k] =$  , $RI[j,k] =$ 

1. $TUP[j,k] = \sum_{h=k}^{j} (-q)^{h-j} q^{k^2} \begin{bmatrix} h \\ k \end{bmatrix}_+ UP[j,h]$
2. $RUP[j,k] = \sum_{h=0}^{k} (-q)^{h-j} a^{h-j} q^{-2kh+k^2+j^2} \begin{bmatrix} j-h \\ k-h \end{bmatrix}_+ OP[j,h]$
3. $TOP[j,k] = \sum_{h=k}^{j} (-q)^{h} a^{k} q^{k^2-2jk} \begin{bmatrix} h \\ k \end{bmatrix}_+ RI[j,h]$
4. $ROP[j,k] = \sum_{h=0}^{k} (-q)^{h-j} a^{k-j} q^{2h(j-k)+(k-j)^2} \begin{bmatrix} j-h \\ k-h \end{bmatrix}_+ UP[j,h]$
5. $TRI[j,k] = \sum_{h=k}^{j} (-q)^{h} a^{h} q^{k^2-2jh} \begin{bmatrix} h \\ k \end{bmatrix}_+ OP[j,h]$
6. $RRI[j,k] = \sum_{h=0}^{k} (-q)^{h} q^{h(2j-2k)+k^2-j^2} \begin{bmatrix} j-h \\ k-h \end{bmatrix}_+ RI[j,h]$

Rational knots: Knots-quivers correspondence works!

Theorem

Let K be a rational knot and let Q_K be the corresponding quiver. Then, the vertices of Q_K are in bijection with generators of the reduced HOMFLY-PT homology of K, such that the (a, q, t)-trigrading of the i^{th} generator is given by $(a_i, -Q_{i,i} - q_i, -Q_{i,i})$ where $Q_{i,i}$ denotes the number of loops at the i^{th} vertex of Q_K.

Rational knots: quiver size and continued fraction

$K_{p/q}$ rational knot can be presented as the closure of

$$T^{a_r} R^{a_{r-1}} T^{a_{r-2}} \cdots T^{a_3} R^{a_2} T^{a_1} K_0$$

$$T^{a_1} K_0$$

$$p/q = a_r + \cfrac{1}{a_{r-1} + \cfrac{1}{a_{r-2} + \cfrac{1}{\cdots + \cfrac{1}{a_3 + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}}}$$

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Tangle-quivers correspondence

Extend the correspondence to 4-ended tangles:

Ask that knots-quivers correspondence hold for any link obtained by closing off the four ends of a tangle.

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Ask that knots-quivers correspondence hold for any link obtained by closing off the four ends of a tangle.

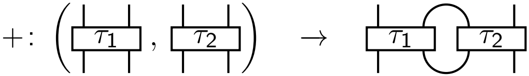
The rational tangles satisfy tangle-quiver correspondence.

For p/q rational tangle we can write the generating series of the colored HOMFLY-PT invariants as a summation with $p+q$ variables (p "active", q "inactive").

Tangle addition

For rational tangles, we had Top and Right twists.

Tangle addition operation:



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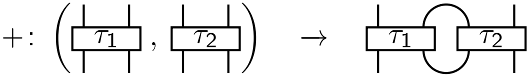

Addition of rational tangles give algebraic tangles.
Closures of such tangles are called arborescent links, or algebraic links (in the sense of Conway).

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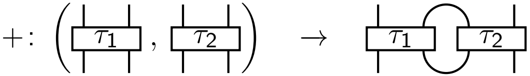

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IN MEMORIAM: John H. Conway (1937-2020)

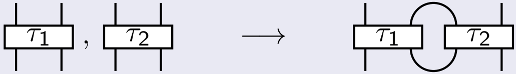
Tangle-quivers correspondence

4-ended tangle family QT_4

Theorem (M.S., P. Wedrich, 2020)

There exists a family QT_4 *of* 4*-ended framed oriented tangles with the following properties:*

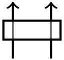
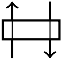
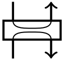
- QT_4 *contains the trivial 2-strand tangle.*
- QT_4 *is closed under diffeomorphisms of* $(B^3, \partial B^3, \{4 \text{ pts}\})$.
- QT_4 *is closed under Conway's tangle addition, the binary operation of gluing two 4-ended tangles at pairs of boundary points as follows:*

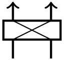
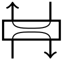
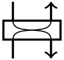


- *The appropriate analogue of the knots-quivers correspondence holds for any link obtained by closing off a tangle in* QT_4.

Tangle addition and quivers

We will consider six refined types of tangles $\tau \in {}_\bullet\mathrm{T}_4$, which encode boundary types and connectivity between boundary points:

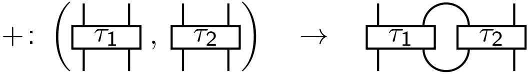
UP_{par} : , OP_{ud} : , RI_{par} : ,

UP_{cr} : , OP_{lr} : , RI_{cr} : .

For example, an UP_{par} tangle has one strand directed from the *SW* to the *NW* boundary point and the other strand directed from the *SE* to the *NE* boundary point, and possibly additional closed components.

Tangle addition and quivers

Tangle addition: binary addition operation on 4-ended tangles, which is given by gluing along pairs of adjacent boundary points, provided the orientations are compatible there.



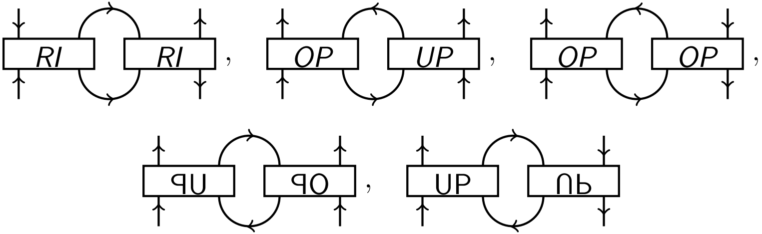
Main result is the following theorem:

Theorem

Let $\tau_1, \tau_2 \in \mathrm{QT}_4$ *with orientations such that* $\tau_1 + \tau_2$ *is defined. Then* $\tau_1 + \tau_2 \in \mathrm{QT}_4$.

Tangle addition and quivers

Due to different types of tangles and orientations, there are five cases to check. In each of them the gluing formula for the HOMFLY-PT skein generating functions is established.



Conclusion: Algebraic links satisfy knots-quivers correspondence.
This includes Montesinos links, pretzel knots, etc..

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What is the minimal size of the corresponding quiver?

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What is the minimal size of the corresponding quiver?

The size of the quiver produced by algorithm has an upper bound (for addition of two tangles the bound is bilinear in the sizes of the quivers of individual tangles), but in particular cases it seems that it can be lowered even further.

Thank you for your attention !