GALOIS SYMMETRIES OF KNOT SPACES

JOINT WORK WITH GEOFFROY HOREL

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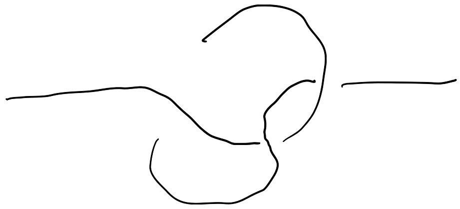
LISBOA.

I Knots and finite-type invariants

$K :=$ space of all smooth knots $\mathbb{R}^1 \longrightarrow \mathbb{R}^3$ standard outside compact.

invariants:

$$\pi_0 K \xrightarrow{\ v\ } A \quad \text{Ab grp}$$



Large class of such — finite-type invariants.

Vassiliev: allow singularities $\times$

$$\begin{tikzcd} \pi_0(K) \arrow[r, "v"] \arrow[d, hook] & A \\ \{\text{singular knots}\} \arrow[ur, dashed, "\bar{v}"'] & \end{tikzcd}$$

(Birman-Lin)

$$\bar{v}\left(\times_{\bullet}\right) = \bar{v}\left(\times_{+}\right) - \bar{v}\left(\times_{-}\right)$$

v type n if $\bar{v}\big(\cdots \underbrace{\times_\bullet \cdots \times_\bullet}_{n+1}\big) = 0$

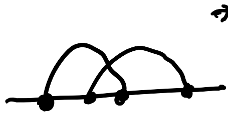
[Birman-Lin, Bar-Natan, Goussarov ...] Conway, Jones, quantum knot invariants, ...

Open: complete invariants?

Theorem [Kontsevich, Bott-Taubes]

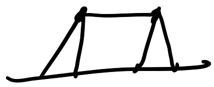
$$\left\{ \begin{array}{c}\text{type} = n \text{ invariants} \\ \text{over } \mathbb{Q}\end{array}\right\} \xrightarrow{\cong} \left(\mathcal{A}_n^{\mathbb{Q}}\right)^\vee$$

algebra of chord diagrams

$\mathbb{Q}\{$  $\}$ / relations

n-chords

$\cong$ Bar-Natan

$\mathbb{Q}\{$  $\}$ / relations

Jacobi or Feynman diagrams

inverse: Kontsevich integral (Bott-Taubes)

$$\pi_0 K \xrightarrow{\int_K} \mathcal{A}_n^{\mathbb{C}}$$

universal

Knots $f \sim_n g$ n-equivalent if $v(f) = v(g)$ $\forall$ type n invariants v

$$\begin{tikzcd} \mathcal{A}_n \arrow[r, "\cong_{\mathbb{Q}}"] & \ker\left[\pi_0 K/\sim_n \to \pi_0 K/\sim_{n-1}\right] \arrow[l, dashed, bend left, "\int_K"] \end{tikzcd}$$

$$\rightsquigarrow \quad \pi_0 K \to \pi_0 K/\sim_n \cong_{\mathbb{Q}} \bigoplus_{s \le n} \mathcal{A}_s^{\mathbb{Q}}$$

Questions

1) What about over $\mathbb{Z}$ or at a prime?

2) What about π_i for $i>0$? H_i?

3) What about higher dimensions?

$$\left(\begin{array}{ll} \mathbb{R}^m \hookrightarrow \mathbb{R}^n & \text{standard outside compact} \\ M \hookrightarrow \mathbb{R}^n & \end{array}\right)$$

This talk: some progress on 1) and 2). (3)?)

Remark: Much is known about 2) and 3) over $\mathbb{Q}$. (codim >2)

Kontsevich, Cattaneo-Ramusino-Longoni, Sakai, Watanabe, ...
Cattaneo-Rossi

classes in H_*, higher m,n

Arone-Lambrechts-Turchin-Volić, Arone-Turchin
$m=1$, $n\geq 4$ $\sim$ $n>2m$

Fresse-Turchin-Willwacher.
$n-m\geq 3$

II A homotopical approach.

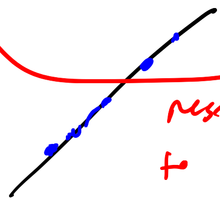
Starting point: [Dwyer–Hess, Turchin, B–Weiss. following Goodwillie–Weiss]

The composite

$$K \xrightarrow{\text{der}} \Omega \mathrm{Inj}(\mathbb{R}^1, \mathbb{R}^3) \xrightarrow{\Omega f} \Omega \mathrm{map}^h_{\mathrm{Fin}}(\mathrm{con}(\mathbb{R}^1), \mathrm{con}(\mathbb{R}^3))$$

$$\boxed{\mathrm{hof}\, \Omega f} = T_\infty$$

restriction to card 2.



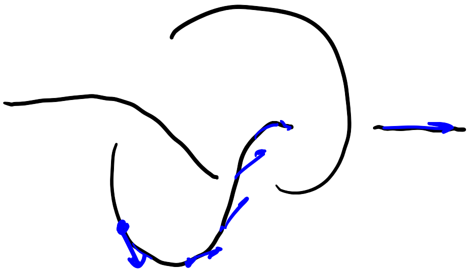
is null-homotopic.

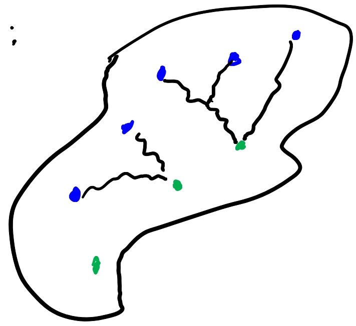
(Andrade) M manifold

$\mathrm{con}(M)$ – obj: $k \hookrightarrow M$, $k \geq 0$

mor:

"Alexander trick"





$5 \to 3$

restricting # points: $\mathrm{con}(M) \rightsquigarrow \mathrm{con}(M; n)$

$$\underset{\substack{\text{is} \\ S^2}}{\mathrm{Inj}(\mathbb{R}^1, \mathbb{R}^3)} \simeq \mathrm{map}^h_{\mathrm{Fin}}(\mathrm{con}(\mathbb{R}^1; 2), \mathrm{con}(\mathbb{R}^3))$$

and $T_\infty \simeq \Omega^2 \,\mathrm{hofib}\left[\mathrm{map}^h(\mathrm{con}(\mathbb{R}^1), \mathrm{con}(\mathbb{R}^3)) \to S^2\right]$

So, get Goodwillie-Weiss tower

$$\begin{tikzcd}
 & T_\infty \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_3 \arrow[d] \\
K \arrow[r, "e_2"] \arrow[ur, "e_3"] \arrow[uur] \arrow[uuur, "e_\infty"] & T_2 \sim *
\end{tikzcd}$$

where $$T_n = \Omega^2 \,\mathrm{hofib}\left[\mathrm{map}^h_{\mathrm{Fin}}\left(\mathrm{Con}(\mathbb{R}^1; n), \mathrm{Con}(\mathbb{R}^3)\right) \to S^2\right]$$

What we know:

1) $\pi_0 T_n$ finitely generated abelian group.

2) Budney-Conant-Scannell-Sinha:

$$K \xrightarrow{e_3} \pi_0 T_3 \cong \mathbb{Z}$$

geometric description in terms of collinearity of three points. — identify with first non-trivial type $\underline{2}$ invariant (2^{nd} coefficient of Conway pol^n)

3) Budney - Conant - Koytcheff - Sinha :

e_n is type $(n-1)$

ie
$$\begin{tikzcd}
\pi_0 K \arrow[rr, "e_n"] \arrow[dr] & & \pi_0 T_n \\
& \pi_0 K/\sim_{n-1} \arrow[ur, dashed] &
\end{tikzcd}$$

alternative proof by Kosanović - Teichner - Shi

4) Kosanović : e_n is surjective on π_0.

3 + 4 + Conant - Teichner imply that

e_n is <u>universal</u> (over R torsion-free (eg $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}_p$))

$\Updownarrow$

spectral sequence collapses at the E^2 page along the diagonal.

$$\begin{array}{c}
\pi_0 \operatorname{hofib}(T_n \to T_{n-1}) \\
\cong
\end{array}$$
$$E^1_{-n,n} \twoheadrightarrow E^2_{-n,n} \xrightarrow{\cong} E^3_{-n,n} \xrightarrow{\cong} \cdots \xrightarrow{\cong} E^\infty_{-n,n} = \ker\left[\begin{array}{c}\pi_0 T_n \\ \downarrow \\ \pi_0 T_{n-1}\end{array}\right]$$
$$E^2_{-n,n} \cong A^I_{n-1} \quad \text{(Conant)}$$

Theorem (B-Hovel)

Many differentials vanish p-locally.
(after $\otimes \mathbb{Z}_{(p)}$)

(More precise statements below.)

Corollary

e_n is universal of type $n-1$ p-locally

for $n \leq p+1$

i.e. $\ker\left(\pi_0 K/_{\sim n-1} \to \pi_0 T_n\right)$

has no p-power torsion for $n \leq p+1$

and $\pi_0 T_n \otimes \mathbb{Z}_{(p)} \cong \bigoplus_{s \leq n} A_s^I \otimes \mathbb{Z}_{(p)}$.

III Galois symmetries

(unordered) configuration space of 2 points in $\mathbb{C}$ is $K(\mathbb{Z},1)$

$\uparrow$ $\mathbb{C}$-points of $\mathbb{Q}$-variety X

$$\prod_p \mathbb{Z}_p \cong \widehat{\mathbb{Z}} \cong \underbrace{\pi_1(X(\mathbb{C}))^\wedge}_{\substack{\text{is} \\ S^1}} \cong \pi_1^{et}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}) \circlearrowleft Gal(\bar{\mathbb{Q}}|\mathbb{Q})$$

$$\left[Gal(\bar{\mathbb{Q}}|\mathbb{Q}) \xrightarrow{\chi} Aut(\widehat{\mathbb{Z}}) \cong \underbrace{\widehat{\mathbb{Z}}^\times}_{\text{units}} \right]$$

cyclotomic character

Similarly, have $Gal(\bar{\mathbb{Q}}|\mathbb{Q}) \circlearrowleft \widehat{Br}_n$ braid group on n strands

$\shortparallel$

π_1(configuration space of n pts in $\mathbb{C}$)

(Belyi: faithful for $n \geq 3$)

$\rightsquigarrow$

Horel, Fresse$_{\mathbb{Q}}$

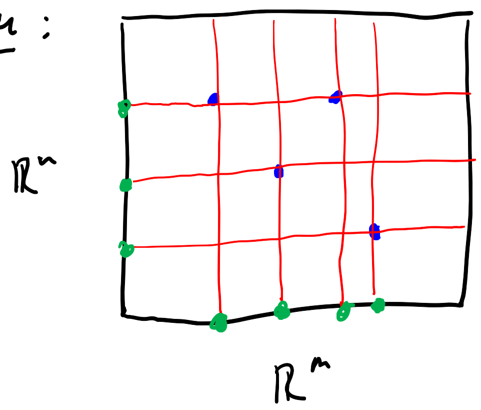
$$\begin{tikzcd}
 & GT \arrow[d, phantom, "\cong"] \\
Gal(\bar{\mathbb{Q}}|\mathbb{Q}) \arrow[ur, hook] \arrow[r] \arrow[d, "\chi"'] & Aut^h_{Fin}(Con(\mathbb{C})^\wedge) \arrow[d] \\
\widehat{\mathbb{Z}} \arrow[r, phantom, "\cong"] & Aut^h((S^1)^\wedge)
\end{tikzcd}$$

How to extend this action to $\mathbb{R}^n$?

Additivity theorem [Dunn, Lurie, B-Weiss]

$$\mathrm{con}(\mathbb{R}^n) \boxtimes \mathrm{con}(\mathbb{R}^m) \xrightarrow{\simeq} \mathrm{con}(\mathbb{R}^{n+m})$$

Idea:



describe configurations in $\mathbb{R}^{n+m}$ "homotopically" in terms of configurations in factors.

B-Horel: this interacts well with completions

$$\mathrm{con}(\mathbb{R}^n)^\wedge_p \boxtimes \mathrm{con}(\mathbb{R}^m)^\wedge_p \simeq \mathrm{con}(\mathbb{R}^{n+m})^\wedge_p$$

$\Longrightarrow$ can use action on $\mathrm{con}(\mathbb{R}^2)^\wedge_p$ to get action on $\mathrm{con}(\mathbb{R}^{n+2})^\wedge_p$ $\quad \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$

$\therefore$ Tower $T_n^p = \Omega^2 \,\mathrm{hofib}\left[\begin{array}{c} \mathrm{map}^h(\mathrm{con}(\mathbb{R}^1; n), \mathrm{con}(\mathbb{R}^3)^\wedge_p) \\ \downarrow \\ (S^2)^\wedge_p \end{array} \right]$

$\circlearrowleft$

$\mathrm{Gal}(\bar{\mathbb{Q}} | \mathbb{Q})$.

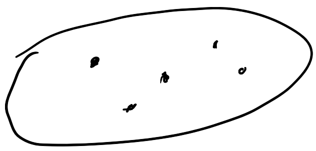
IV The collapse

$$E^1_{-s,t} = \pi_{t-s}\, \mathrm{hof}\left(T_s \to T_{s-1}\right) \Rightarrow \pi_{t-s} T_n$$

$$\cong$$

$$\pi_{t-s}\,\Omega^2\, \mathrm{hof}\left[\begin{array}{c} \mathrm{map}^h(\mathrm{con}(\mathbb{R}^1; s), \mathrm{con}(\mathbb{R}^3)) \\ \downarrow \\ \ldots\ldots \quad s-1 \end{array}\right]$$

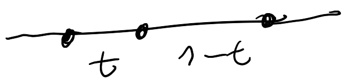
$$\boxed{\cong \pi_{t-s}\,\Omega^2\,\underbrace{\Omega^{s-2}}_{\overline{\mathrm{emb}}(s,\mathbb{R}^1)/\partial}\,\tilde{\bigvee_{s-1}} S^2}$$



$\mathbb{R}^3$

$s-1$ points fixed

eg $s=3$



$$\cong \pi_t \tilde{\bigvee_{s-1}} S^2$$

$\Big($ dim 3 $\rightsquigarrow$ dim $d \geq 3$

$S^2 \rightsquigarrow S^{d-1}$ $\Big)$

Theorem (B–Horel) p prime, $r > 1$

$$d^r_{-s,t} : E^r_{-s,t} \longrightarrow E^r_{-s-r,\, t+r-1}$$

vanishes p-locally if $r-1$ is not a multiple of $\underbrace{(p-1)(d-2)}_{p-1}$

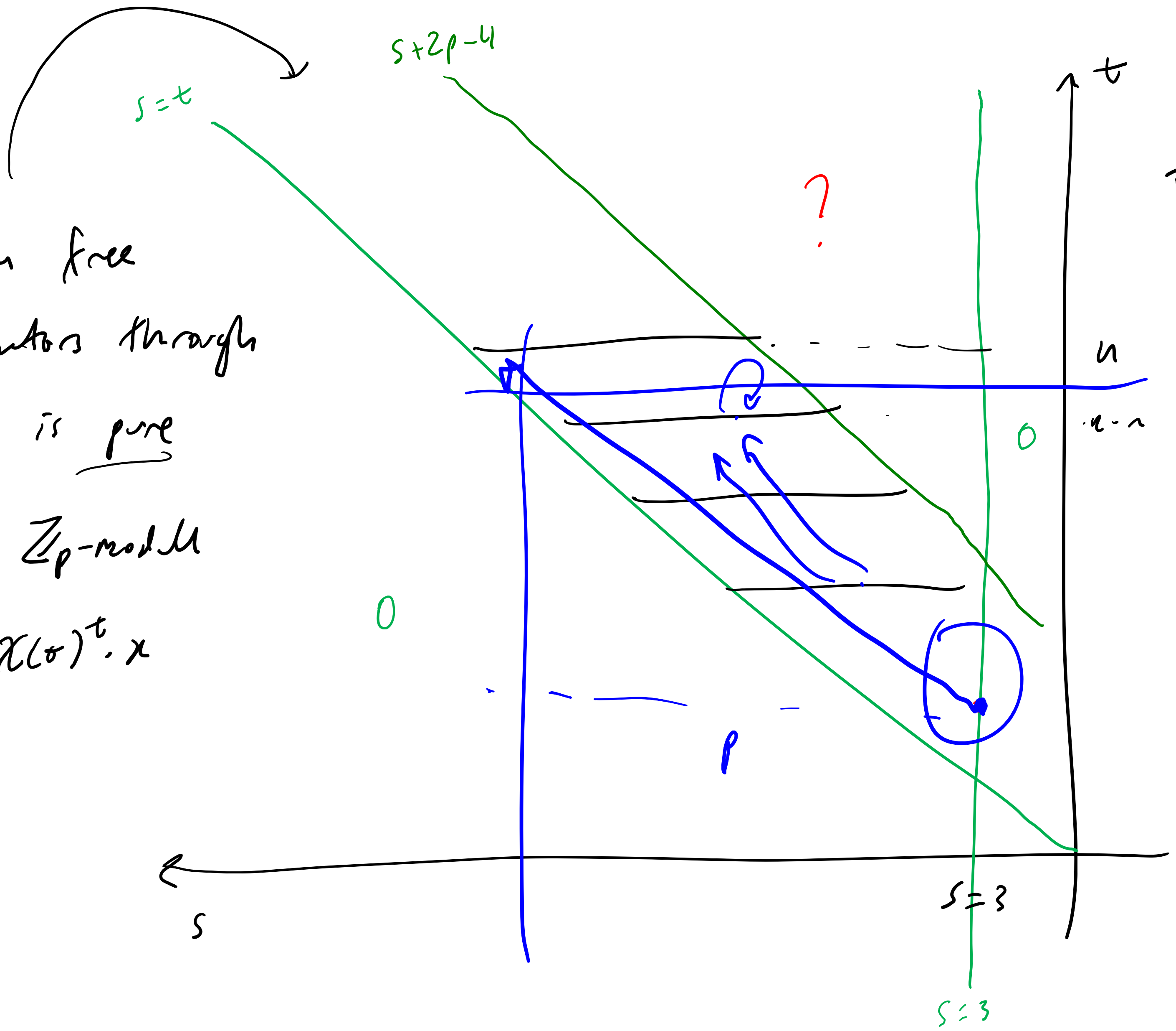
and $\quad t < 2p-2+(s-1)(d-2)$.

$d=3$

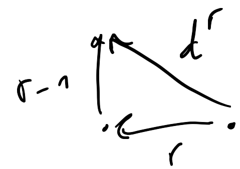
p-torsion free action factors through χ and is pure

$\pi_t(\dots)$ $\mathbb{Z}_p$-module

$\sigma . x = \chi(\sigma)^t . x$



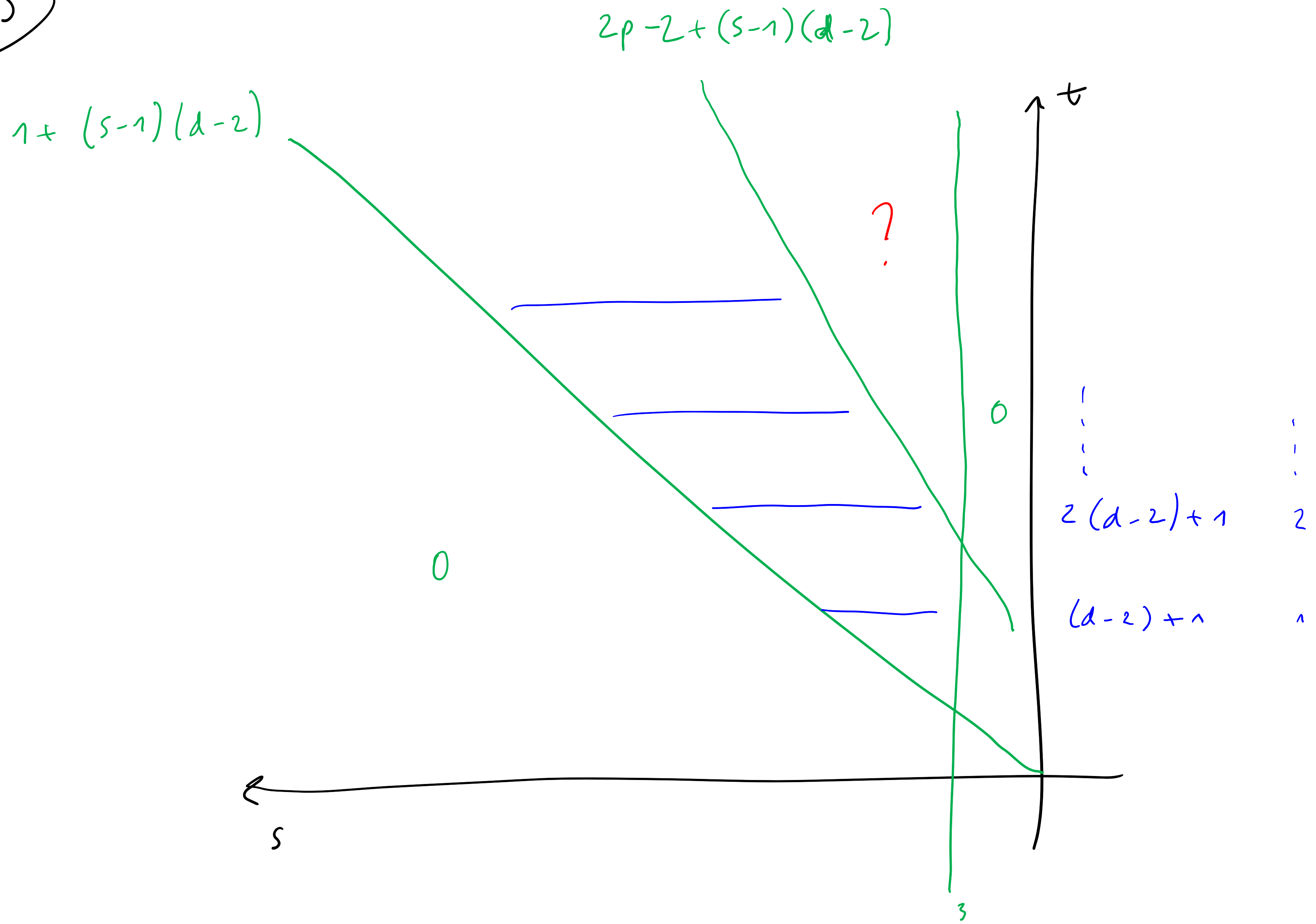
$\pi_t(\tilde{\bigvee}_{S^1} S^2) \otimes \mathbb{Z}_{(p)}$



$\overset{m}{\circlearrowright} M \xrightarrow{f} N \overset{n}{\circlearrowleft}$

f $Gal(\bar{\mathbb{Q}}|\mathbb{Q})$-equivariant and $\left\{ \begin{array}{l} m-n \ \underline{\text{not}} \text{ a} \\ \text{multiple of } p-1 \end{array} \right.$ $\Rightarrow f = 0$.

$d \geq 3$



Eg. Cor (using Goodwillie–Klein) $\pi_i \, emb_c(\mathbb{R}^1, \mathbb{R}^4) \cong \bigoplus_{t-s=i} E^2_{-s,t}$ p-locally for $\sim$ $i \leq 2(p-1)$ $(p>2)$

Thanks !