TQFT SEMINARS, IST

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COUNTING  POTENTIALS

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COUNTING MONSTER POTENTIALS

1. THE POTENTIALS. FIX $\alpha>0$, $L\in\mathbb{Q}$, $N\in\mathbb{N}$.

FOR ANY MONIC POLYNOMIAL $P(z)=\prod_{k=1}^{N}(z-z_k)$, $P(0)\neq 0$

$$V_P(x)=\underbrace{x^{2\alpha}+\frac{L}{x^2}}_{\text{GROUND-STATE-POTENTIAL}}-2\frac{\partial^2}{\partial x^2}\log P\left(x^{2\alpha+2}\right),\quad x\in\mathbb{C}_\alpha$$

SINGULARITIES:

$$\left\{\begin{array}{lll} 0 & \text{REGULAR} & \text{(BRANCH POINT)} \\ \infty & \text{IRREGULAR} & \text{(" ")} \end{array}\right\}\ \text{FIXED}$$

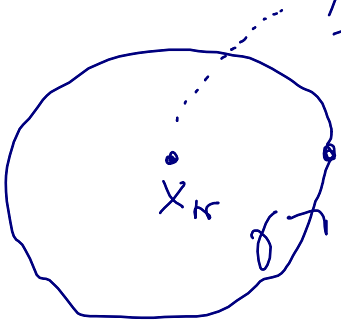
$$\left[\begin{array}{l} x_k \text{ s.t. } x_k^{2\alpha+2}=z_k:\quad V_P(x)=\dfrac{2}{(x-x_k)^2}+O(1) \\ \text{RESONANT SINGULARITY: INDEX: } -1,\ 2 \end{array}\right]\ \text{VARY WITH } P$$

TRIVIAL MONODROMY

(S) $\quad -\psi''(x) + (V_\varphi(x) - E)\,\psi(x) = 0, \qquad x \in \mathbb{C}_\alpha, \quad E \in \mathbb{C}$

$x_k^{2\alpha+2} = z_k$

MONODROMY



$\psi(x) \longrightarrow \psi_\gamma(x) = M_E\,\psi(x)$

x_k RESONANT $\Longleftrightarrow$ $M_E = \begin{pmatrix} 1 & c(E) \\ 0 & 1 \end{pmatrix}$

DEF: V_φ HAS TRIVIAL MONODROMY AT x_k : $M_E = \mathbb{1}, \ \forall E \in \mathbb{C}$

ASSUME P HAS DISTINCT ROOTS. V_φ HAS TRIVIAL MONODROMY AT ALL SINGULARITIES IFF

BLZ SYSTEM

$$\sum_{j \neq k} \frac{z_k\left(z_k^2 + A z_k z_j + B z_j^2\right)}{(z_k - z_j)^3} - \frac{\alpha z_k}{4(\alpha+1)} + \frac{L}{4(\alpha+1)} + \frac{1-4\alpha^2}{16(\alpha+1)} = 0$$

$k = 1, \ldots, N$

ODE/IM CORRESPONDENCE

LET V_p BE A  POTENTIAL, AND E_n, $n \geq 0$ SUCH THAT

$$-\psi''(x) + \left(V_p - E_n^{(p)}\right)\psi(x) = 0, \qquad \lim_{x \to 0} \psi(x) = \lim_{x \to +\infty} \psi(x) = 0$$

THEN $\{E_n^{(p)}\}_{n=0}^{\infty}$ SOLVES THE BETHE ANSATZ EQUATIONS

$L = \ell(\ell+1)$

$\mathrm{Re}\,\ell \geq \frac{1}{2}$

∞ B.A.

$$e^{\frac{i\pi(\ell+1)}{\alpha+1}} \prod_{n=0}^{\infty} \frac{E_m - e^{\frac{i\pi}{\alpha+1}} E_n}{E_m - e^{-\frac{i\pi}{\alpha+1}} E_n} = -1, \qquad m = 0, 1, \ldots$$

'98 DOREY-TATEO : GROUND-STATE $L = 0$

'98 B-L-Z : " " $L \in \mathbb{C}$

'03 " " " " MONSTER POTENTIALS

MANY GENERALISATIONS

e.g. $\mathfrak{sl}_2^{(1)} \dashrightarrow \mathfrak{g}^{(1)}$

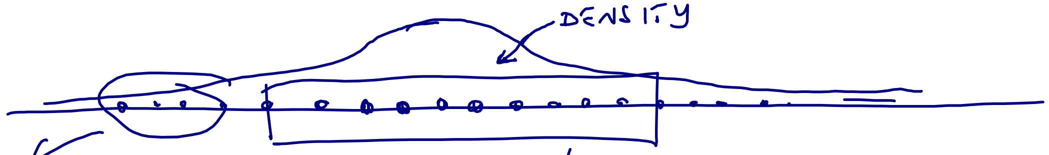
FEIGIN-FRENKEL '11

MO-RAIMONDO-VALERI '16, '18

MO-RAIMONDO, '20

BETHE ANSATZ, 6-VERTEX-MODEL, QUANTUM KdV (CFT $c<1$)

$\boxed{1}$ 6-VERTEX, LATTICE SIZE $\longrightarrow \infty$, BETHE ANSATZ SOLUTION



WHAT ABOUT THE TAIL?

$\hookrightarrow$ BULK: WELL-KNOWN BEHAVIOUR

$\hookrightarrow$ IT SATISFIES THE ∞-B.A. EQUATIONS. KLUMPER, BATCHELOR, PEARCE, '95

$\boxed{2}$ INTEGRABILITY IN C.F.T. $c<1$

$\mathcal{V}_\Delta$ IRREDUCIBLE HIGHEST WEIGHT REPRESENTATION OF Vir, $c<1$

GROUND STATE: $L_0|\Delta\rangle = \Delta|\Delta\rangle$, $L_n|\Delta\rangle = 0 \quad \forall n>0$

HIGHER-STATES: CHOOSE LEVEL N, A PARTITION $\nu = (\nu_1, \ldots, \nu_i)$

$|\nu\rangle := L_{-\nu_1} \cdots L_{-\nu_i}|\Delta\rangle$, $L_0|\nu\rangle = (\Delta+N)|\nu\rangle$

EVERY STATE YIELDS A SOLUTION OF ∞-B.A.

$$\{E_n^{[\nu]}\}_{n=0}^{\infty}$$

BLZ CONJECTURE

WITH $c = 1 - \frac{6\alpha^2}{\alpha+1}$ AND $\Delta = \frac{L}{4(\alpha+1)} + \frac{1-4\alpha^2}{16(\alpha+1)}$:

$\forall$ STATE $|\nu\rangle$ OF LEVEL N, $\exists!$ MONSTER POTENTIAL V_P, $P = \prod_{k=1}^{N}(z - z_k)$ s.t.

$$\left\{ E_n^{[\nu]} \right\}_{n=0}^{\infty} = \left\{ E_n^{(P)} \right\}_{n\geq 0}^{\infty}$$

WEAK BLZ CONJECTURE

FIX $N \in \mathbb{N}$. FOR GENERIC (α, L), THE NUMBER OF MONSTER POTENTIALS WITH "N ROOTS" IS $p(N)$ ← NUMBER OF INTEGER PARTITIONS OF N !

"N ROOTS" $:=$ $P(z) = \prod_{k=1}^{N}(z - z_k)$.

LARGE MOMENTUM LIMIT OF THE BLZ SYSTEM

$L \to \infty$

$$(*) \quad \sum_{j \neq k} \frac{z_k \left(z_k^2 + A z_k z_j + z_j^2\right)}{(z_k - z_j)^3} - \frac{\alpha z_k}{4(1+\alpha)} + \frac{L}{4(\alpha+1)} - \frac{1-4\alpha^2}{16(\alpha+1)} = 0$$

$$B_{N,\alpha} = \left\{ (P, L) \in \mathbb{C}^{N+1},\ P = \prod_{k=1}^{N} (z - z_k),\ z_k \neq z_j,\ z_k \neq 0,\ z_k \text{ SOLVES } (*) \right\}$$

$\overline{B}_{N,\alpha}$ = CLOSURE OF $B_{N,\alpha}$ IN $\mathbb{C}^{N+1}$

(1)

$(P^{(n)}, L^{(n)}) \in \overline{B}_{N,\alpha}$, WITH $L^{(n)} \to \infty$. CALL $z_k^{(n)}$ THE ROOTS OF $P^{(n)}$

$$\lim_{n \to \infty} \frac{z_k^{(n)} \alpha}{L} = 1, \quad k = 1, \dots, N$$

THE PROOF IS SIMPLE, BUT TRICKY! IT TOOK US ALMOST 2 YEARS!

② CONVERGENCE TO A RATIONAL EXTENSION OF THE HARMONIC OSCILLATOR

- AS $L \to \infty$, POLES OF V_p "CONDENSATE" ABOUT THE POINTS X s.t. $X^{2\alpha+2} = \frac{L}{\alpha}$
- THESE ARE THE (COMPLEX) EQUILIBRIA OF THE GROUND STATE POTENTIAL
- WE EXPAND THE SCHRÖDINGER EQUATION $-\psi''(x) + (V_p(x) - E)\psi(x) = 0$
- CHOOSING $x = \left(\cdots \varepsilon^{\frac{\alpha-1}{\alpha+1}} t + \left(\frac{L}{\alpha}\right)^{\frac{1}{2\alpha+2}} \right)$, $z_k = \frac{L}{\alpha}\left(1 + (2\alpha+2)^{-1/4} \varepsilon t_k\right)^{2\alpha+2}$, $\varepsilon = L^{-1/4}$

WE GET

$$-\psi''(t) + \left(t^2 + O(\varepsilon) - \sum_{k=1}^{N} \frac{2}{(t - t_k + O(\varepsilon))^2} - \tilde{E} \right)\psi(t) = 0$$

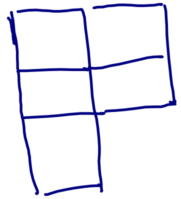
WHERE THE MONODROMY AT $(t_1, \ldots, t_N)$ IS TRIVIAL

$\underline{\varepsilon = 0}$ RATIONAL EXTENSION OF THE HARMONIC OSCILLATOR!

OBLOMKOV'S THEOREM

LET $\nu = (\nu_1, \dots, \nu_j)$, $\nu_i \ge \nu_{i+1}$, $\nu_i > 0$, $\sum_{i=1}^{j} \nu_i = N$

BE A PARTITION OF N



$= (3,2)$

$$P^{[\nu]}(t) = c_\nu^{-1}\, \mathrm{Wr}\left[H_{\nu_j}(t), H_{\nu_{j-1}+1}(t), \dots, H_{\nu_1+(j-1)}(t) \right], \quad H_n = \text{Hermite}$$

$$= \prod_{k=1}^{N} \left(t - \sigma_k^{[\nu]} \right).$$

E.G. $\nu = (N)$, $P^{[\nu]} = c^{-1} H_N(t)$

$$P^{[\nu^*]}(t) = i^N P^{[\nu]}(i t)$$

THM IF $P = \prod_{k=1}^{N} (t - t_k)$ IS SUCH THAT

$$U_P = t^2 - 2 \frac{\partial^2}{\partial t^2} \ln P(t) = t^2 - \sum_{k=1}^{N} \frac{2}{(t - t_k)^2}$$

HAS TRIVIAL MONODROMY AT $(t_1, \dots, t_N)$, THEN

$P(t) = P^{[\nu]}(t)$ FOR SOME PARTITION ν OF N

THEOREM M-CONTI 2020

FIX $\alpha > 0$, $N \in \mathbb{N}$.

LET $(P^{(n)}, L^{(n)}) \in \overline{B}_{N,\alpha}$ WITH $L^{(n)} \xrightarrow[n\to\infty]{} \infty$

THE SEQUENCE $P^{(n)}$ CAN BE SPLIT INTO J, $1 \leq J \leq P(N)$, SUBSEQUENCES SUCH THAT, EACH SUBSEQUENCE ASSOCIATED TO A UNIQUE PARTITION ν OF N SUCH THAT

$$z_k^{(n)} = \frac{L^{(n)}}{\alpha} + \frac{(2\alpha+2)^{3/4}}{\alpha} N_k^{[\nu]} \left(L^{(n)}\right)^{3/4} + O\left(L^{(n)}\right)^{1/2}$$

WHERE $z_k^{(n)}$ ARE THE, APPROPRIATELY ORDERED, ROOTS OF $P^{(n)}$.

SPECTRUM

LET $L \to +\infty$ (L REAL) AND V_P, $P := P(z; L)$

SATISFY THE ν-ASYMPTOTICS FOR SOME PARTITION

$\nu = (\nu_1, \ldots, \nu_j)$ OF N, THEN

$$E_n^{[\nu]} = (1+\alpha)\left(\frac{L}{\alpha}\right)^{\frac{\alpha}{\alpha+1}} + (2\alpha+2)\,\alpha^{\frac{1}{\alpha+1}}\,(2(n-j)+1)\,L^{\frac{\alpha-1}{2\alpha+2}}$$

$$+\; O\left(|L|^{-\frac{1}{\alpha+1}}\right), \quad n \in \mathbb{N}^{[\nu]}$$

$$\mathbb{N}^{[\nu]} = \mathbb{N} \setminus \{\nu_j, \nu_{j-1}+1, \ldots, \nu_1+j-1\}.$$

$\nu = \phi, N = 0$ 1 3 5 7 9 ...

$\nu = (1), N = 1$ -1 3 5 ...

CONSTRUCTING MONSTER POTENTIAL WITH GIVEN ν-ASYMPTOTICS

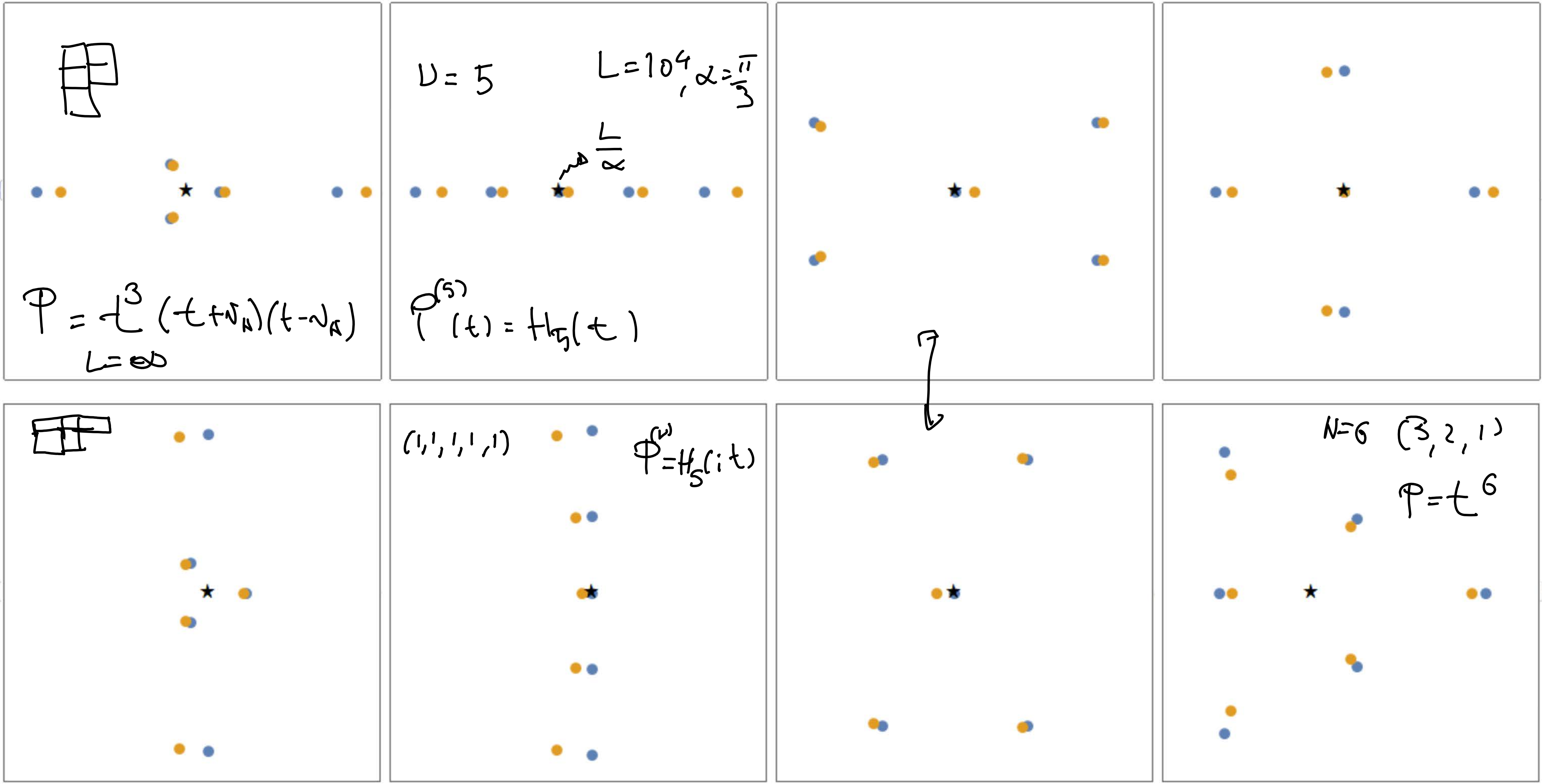
DOES -- FOR EVERY PARTITION ν OF N -- EXIST A UNIQUE ALGEBRAIC FAMILY OF MONSTER POTENTIAL $P^{[\nu]}(z;L)$ SATISFYING THE ν-ASYMPTOTICS? $\dashrightarrow$ $P^{[\nu]}(z; e^{2\pi i} L) = P^{[\nu^*]}(z, L)$

DEFINITELY YES, BUT WE ARE NOT ABLE TO PROVE IT IN GENERAL.

TECHNICAL DIFFICULTY: DEPENDING ON ν, 0 MAYBE A ROOT OF $P^{[\nu]}(t)$ WITH MULTIPLICITY GREATER THAN 1.

THIS MAKES THE PERTURBATION SERIES VERY TRICKY

WE SOLVED IT FOR MULTIPLICITY $0, 1, 3, 6$.



ν = 5
L=10^4, α=π/3
L/α
P = t^3 (t+ν_n)(t−ν_n)
L=∞
φ^(5)(t) = H_5(t)
(1,1,1,1,1)
φ^(ν)=H_5(it)
N=6 (3,2,1)
P=t^6