The double centralizer theorem for finitary 2-representations and an application

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Double Centralizer Property

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We say that M has the *Double Centralizer Property (DCP)* if the canonical homomorphism

$$\mathrm{can}\colon A \to \mathrm{End}_{\mathrm{End}_A(M)}(M)$$

is an isomorphism of algebras.

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Example (Schur-Weyl duality)

If $n \geq r$, we have

$$\mathbb{C}[S_r] \cong \mathrm{End}_{\mathrm{End}_{\mathbb{C}[S_r]}((\mathbb{C}^n)^{\otimes r})}\left((\mathbb{C}^n)^{\otimes r}\right).$$

Double Centralizer Property

- The DCP is known to hold in lots of cases, e.g.
  - A is semisimple and M is faithful;
  - A is a self-injective algebra and M is faithful.
  - A is quasi-hereditary and M is a faithful tilting module satisfying a certain condition.

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- If M has the DCP, then the functor

$$A\text{-fmod} \to \mathrm{End}_A^{\mathrm{op}}(M)\text{-fmod},$$

$$N \mapsto \mathrm{Hom}_A(M, N)$$

estalishes a "nice" relation between A-fmod and $\mathrm{End}_A(M)$-fmod, e.g. in the semisimple case it is an equivalence.

fiat 2-categories

Let $\mathscr{C} = (\mathscr{C}, \oplus, \otimes, 0, I, {}^\star)$ be a finitary, pivotal category (a.k.a. a one-object, fiat 2-category), defined over $\mathbb{C}$.

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- There is a finite set of pairwise non-isomorphic indecomposable objects $\mathrm{B}_1, \ldots, \mathrm{B}_n \in \mathscr{C}$ such that for every object $\mathrm{F} \in \mathscr{C}$ there are unique $m_1(\mathrm{F}), \ldots, m_n(\mathrm{F}) \in \mathbb{Z}_{\geq 0}$ s.t.

$$\mathrm{F} \cong \bigoplus_{i=1}^{n} \mathrm{B}_i^{\oplus m_i(\mathrm{F})}.$$

By assumption, $I \cong \mathrm{B}_i$ for some $i = 1, \ldots, n$.

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- * defines a pivotal structure on $\mathscr{C}$.

- $\mathscr{C}$ is called *semisimple* if

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathscr{C}}(\mathrm{B}_i,\mathrm{B}_j)\right)=\delta_{ij}\quad (i,j=1,\ldots,n).$$

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- Vect_G, for a finite group G;
- $\mathbb{C}[G]$-fmod, for a finite group G;
- $U_q(\mathfrak{g})\text{-}\mathrm{fmod}_{\mathrm{ss}}$, for a complex semisimple Lie algebra $\mathfrak{g}$.

Non-semisimple examples: Dual numbers

Let $D := \mathbb{C}[x]/(x^2)$. Then D is a symmetric Frobenius algebra with non-degenerate trace form $\epsilon(x) = 1$ and $\epsilon(1) = 0$. This implies that D is also a coalgebra, with counit ϵ, and the comultiplication δ is a (D, D)-bimodule map

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i.e. $\mathscr{C}_D$ is the full subcategory of (D, D)-bimodules that are isomorphic to direct summands of direct sums of copies of D and $D \otimes D$. The monoidal structure on $\mathscr{C}_D$ is defined by $\otimes_D$ and $I = D$, e.g.

$$\begin{aligned}(D \otimes D) \otimes_D (D \otimes D) &\cong D \otimes (D \otimes_D D) \otimes D \\ &\cong D \otimes D \otimes D \\ &\cong (D \otimes D) \oplus (D \otimes D).\end{aligned}$$

Non-semisimple examples: Dual numbers

The pivotal structure on $\mathscr{C}_D$ is defined by

- $D^\star := D$, with unit and counit of adjunction defined by δ (comultiplication) and μ (multiplication), resp.
- $(D \otimes D)^\star := D \otimes D$, with unit and counit of adjunction defined by

$$D \xrightarrow{\delta} D \otimes D \xrightarrow{\mathrm{id}_D \otimes \delta(1) \otimes \mathrm{id}_D} D \otimes D \otimes_D D \otimes D$$

$$D \otimes D \otimes_D D \otimes D \xrightarrow{\mathrm{id}_D \otimes \epsilon \cdot \mu \otimes \mathrm{id}_D} D \otimes D \xrightarrow{\mu} D.$$

Non-semisimple examples: Projective bimodules

More generally, let A be a weakly symmetric f.d. Frobenius algebra (also assumed to be basic and connected). Define

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Suppose $\{e_1, \ldots, e_n\}$ is a complete set of orthogonal primitive idempotents in A, i.e. $e_i e_j = \delta_{ij} e_i$ and $e_1 + \cdots + e_n = 1$. A complete set of pairwise non-isomorphic indecomposables is given by

$$\{A\} \cup \{Ae_i \otimes e_j A \mid i, j = 1, \ldots, n\}.$$

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The monoidal structure is defined by $\otimes_A$ and $I = A$, and the pivotal structure by $A^\star = A$ and $(Ae_i \otimes e_j A)^\star = Ae_j \otimes e_i A$ with units and counits of adjunction similar to the ones above.

Let $M = (m_{ij})_{i,j=1}^n \in \mathrm{Mat}(n, \mathbb{Z}_{\geq 0})$ be a symmetric matrix such that

$$m_{ij} = \begin{cases} 1 & \text{if } i = j; \\ \geq 2 & \text{if } i \neq j. \end{cases}$$

Definition (Coxeter system)

A Coxeter system (W, S) with Coxeter matrix M is given by a set $S = \{s_1, \ldots, s_n\}$ (simple reflections) and a group (Coxeter group)

$$W := \langle s_1, \ldots, s_n \in S \mid (s_i s_j)^{m_{ij}} = e \rangle.$$

We call n the rank of (W, S).

Examples

- The Coxeter group of type $I_2(n)$ is isomorphic to the dihedral group D_{2n}:

$$D_{2n} = \langle s, t \mid s^2 = t^2 = e \ \wedge \ (st)^n = e \rangle.$$

The isomorphism with the usual presentation

$$\langle \rho, \sigma \mid \sigma^2 = \rho^n = e \ \wedge \ \rho\sigma = \sigma\rho^{-1} \rangle$$

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- The Coxeter group of type A_n is isomorphic to symmetric group S_{n+1}, generated by $s_1, \ldots, s_n$, subject to

$$\begin{aligned}
m_{ii} = 1: &\qquad (s_i s_i)^1 = e \Leftrightarrow s_i^2 = e; \\
m_{ij} = 2: &\quad (s_i s_j)^2 = e \Leftrightarrow s_i s_j = s_j s_i \quad \text{if } j \neq i \pm 1; \\
m_{i(i\pm 1)} = 3: &\quad (s_i s_{i\pm 1})^3 = e \Leftrightarrow s_i s_{i\pm 1} s_i = s_{i\pm 1} s_i s_{i\pm 1}.
\end{aligned}$$

Coxeter diagrams of finite type

Weyl type

A_n

$B_n = C_n$

D_n

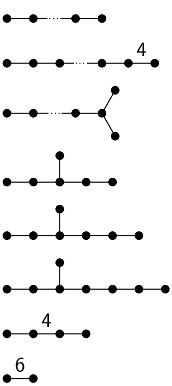
E_6

E_7

E_8

F_4

G_2

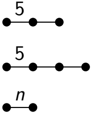


non-Weyl type

H_3

H_4

$I_2(n)$



Recall that $H = H(W, S)$ is a deformation of $\mathbb{Z}[W]$ over $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$:

$$s_i^2 = e \quad \rightsquigarrow \quad s_i^2 = (\mathrm{v}^{-2} - 1)s_i + \mathrm{v}^{-2}.$$

Let $\{b_w \mid w \in W\}$ be the Kazhdan-Lusztig basis of H and write

$$b_u b_v = \sum_{w \in W} h_{u,v,w} b_w,$$

for $h_{u,v,w} \in \mathbb{Z}_{\geq 0}[\mathrm{v}, \mathrm{v}^{-1}]$.

Let W be a finite Coxeter group and $Q = Q(W)$ the *coinvariant algebra*.

Soergel bimodules

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Definition (Soergel)

Let $\mathcal{S}$ be the additive closure in $Q\text{-}\mathrm{gfbimod}\text{-}Q$ of the full monoidal subcategory generated by shifted copies of

$$\mathrm{B}_{s_i} := Q \otimes_{Q^{s_i}} Q\langle 1 \rangle \quad (i = 1, \ldots, n).$$

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Remark: $\mathcal{S}$ is not abelian, e.g. the kernel of

$$\mathrm{B}_{s_i} = Q \otimes_{Q^{s_i}} Q \xrightarrow{a \otimes b \,\mapsto\, ab} Q$$

is isomorphic to Q as a right Q-module but the left Q-action is twisted by s_i.

Let $w \in W$ and $\underline{w} = s_{i_1} \cdots s_{i_r}$ a reduced expression (rex). The *Bott-Samelson bimodule* is defined as

$$\mathrm{BS}(\underline{w}) := \mathrm{B}_{s_{i_1}} \otimes_Q \cdots \otimes_Q \mathrm{B}_{s_{i_r}}.$$

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Theorem (Soergel)

$\mathcal{S}$ is finitary. For every $w \in W$, there is an indecomposable bimodule $\mathrm{B}_w \in \mathcal{S}$, unique up to degree-preserving isomorphism, such that

- *B_w is isomorphic to a direct summand, with multiplicity one, of $\mathrm{BS}(\underline{w})$ for any rex $\underline{w}$ of w;*
- *For all $t \in \mathbb{Z}$, $\mathrm{B}_w\langle t\rangle$ is not isomorphic to a direct summand of $\mathrm{BS}(\underline{u})$ for any $u < w$ and any rex $\underline{u}$ of u.*
- *Every indecomposable Soergel bimodule is isomorphic to $\mathrm{B}_w\langle t\rangle$ for some $w \in W$ and $t \in \mathbb{Z}$.*

The categorification theorem

Theorem (Soergel, Elias-Williamson)

The $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$*-linear map given by*

$$b_w \mapsto [\mathrm{B}_w]$$

defines an algebra isomorphism between H *and* $[\mathcal{S}]_{\oplus}$ *(split Grothendieck group).*

The categorification theorem

Let $p = \sum_{i=-r}^{s} a_i \mathrm{v}^i \in \mathbb{Z}_{\geq 0}[\mathrm{v}, \mathrm{v}^{-1}]$. Define

$$\mathrm{B}^{\oplus p} := \bigoplus_{i=-r}^{s} \mathrm{B}^{\oplus a_i} \langle -i \rangle.$$

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Then the above theorem means:

Positive Integrality

For all $u, v \in W$, we have

$$\mathrm{B}_u \otimes_Q \mathrm{B}_v \cong \bigoplus_{w \in W} \mathrm{B}_w^{\oplus h_{u,v,w}},$$

whence

$$h_{u,v,w} \in \mathbb{Z}_{\geq 0}[\mathrm{v}, \mathrm{v}^{-1}].$$

2-Representations

Definition

A *finitary 2-representation* of $\mathscr{C}$ is a finitary category $\mathcal{M}$ together with a linear, monoidal functor

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Example

The monoidal structure of $\mathscr{C}$ defines the *principal 2-representation*

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- The finitary 2-representations of $\mathscr{C}$ form a 2-category, denoted $\mathscr{C}\text{-afmod}$.

Note: We will often write $\mathrm{F}\mathrm{G} := \mathrm{F} \otimes \mathrm{G}$ and $\mathrm{F}X := \mathbf{M}(\mathrm{F})(X)$.

Cell theory

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- There are natural pre-orders $\geq_L, \geq_R, \geq_{LR}$ on $\mathrm{Ind}(\mathscr{C}) := \{[\mathrm{F}] \mid \mathrm{F} \in \mathscr{C}\text{ indecomposable}\}$, e.g.

$$[\mathrm{F}] \geq_L [\mathrm{G}] \quad \text{if} \quad \exists \mathrm{H} \in \mathscr{C} : \mathrm{F} \subseteq_{\oplus} \mathrm{HG}.$$

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- These pre-orders induce equivalence relations, e.g.

$$[\mathrm{F}] \sim_L [\mathrm{G}] \quad \Leftrightarrow \quad [\mathrm{F}] \geq_L [\mathrm{G}] \ \wedge \ [\mathrm{G}] \geq_L [\mathrm{F}],$$

  and the associated equivalence classes are called *left, right and two-sided cells*, respectively.

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- For each two-sided cell $\mathcal{J}$, we have

$$\mathcal{J} = \coprod_{\mathcal{L} \subseteq \mathcal{J}} \mathcal{L} \quad , \quad \mathcal{J} = \coprod_{\mathcal{R} \subseteq \mathcal{J}} \mathcal{R}.$$

Cell theory: examples

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- If $\mathscr{C}$ is semisimple, then there is only one cell, because I is simple and $I \subseteq_{\oplus} LL^{\star}$ for any simple L.
- $\mathscr{C}_A$ has the following cell structure.
  - Two-sided cells:

$$\mathcal{J} := \{A\}\ ,\ \mathcal{J}' := \{Ae_i \otimes e_j A \mid i, j = 1, \ldots, n\}.$$

  - Left cells:

$$\{A\}\ ,\ \mathcal{L}_j := \{Ae_i \otimes e_j A \mid i = 1, \ldots, n\} \text{ for fixed } j.$$

  - Right cells:

$$\{A\}\ ,\ \mathcal{R}_i := \{Ae_i \otimes e_j A \mid j = 1, \ldots, n\} \text{ for fixed } i.$$

Kazhdan-Lusztig cells

The cells of $\mathscr{S}$ correspond exactly to the Kazhdan-Lusztig cells of H, e.g. in type $I_2(n)$:

e	
$s, sts, \ldots$	$st, stst, \ldots$
$ts, tsts, \ldots$	$t, tst, \ldots$
w_0	

Remark: d is the so called **Duflo involution**.

- Let $\mathcal{L}$ be a left cell. The cell 2-representation $\mathbf{C}_{\mathcal{L}}$ is given by the natural 2-action of $\mathscr{C}$ on

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- The cell 2-representations are important examples of so-called *simple transitive 2-representations*. However, the former do not exhaust the latter in general.

Simple transitive 2-representations

- A finitary 2-representation **M** of $\mathscr{C}$ is *transitive* if every $X \in \mathcal{M}$ is a *generator*, i.e.

$$\mathcal{M} \simeq \mathrm{add}\left(\{\mathrm{F}X \mid \mathrm{F} \in \mathscr{C}\}\right).$$

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- Mazorchuk and Miemietz proved a categorical Jordan-Hölder theorem for finitary 2-representations, where the role of the simples is played by the simple transitive 2-representations.

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- A finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ is *transitive* if every $X \in \mathcal{M}$ is a *generator*, i.e.

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Projective endofunctors

Let $\mathcal{J}$ be a two-sided cell and $\mathbf{M} \in \mathscr{C}\text{-stmod}_{\mathcal{J}}$. Then:

- $\mathcal{M} \simeq A\text{-fproj}$ for some f.d. self-injective algebra A.

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$$(Ae_i \otimes e_jA) \otimes_A Ae_k \cong Ae_i \otimes e_jAe_k \cong (Ae_i)^{\oplus \dim_{\mathbb{C}}(e_jAe_k)}.$$

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- An endomorphism of $\mathbf{M}$ is called *projective* if it corresponds to tensoring over A with a projective (A, A)-bimodule, i.e. one of the form $\oplus_{i,j=1}^{n}(Ae_i \otimes e_jA)^{\oplus m_{ij}}$ for some $m_{ij} \in \mathbb{Z}_{\geq}$.

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Theorem (Kildetoft-M-Mazorchuk-Zimmermann)

The 2-representation $\mathbf{M}$ *restricts to a monoidal semifunctor*

$$\mathbf{M}\colon \mathrm{add}(\mathcal{J}) \to \mathscr{E}\mathrm{nd}_{\mathscr{C}}^{\mathrm{proj}}(\mathbf{M}).$$

Strong $\mathcal{H}$-reduction

Fix a two-sided cell $\mathcal{J}$.

Definition

For every left cell $\mathcal{L} \subseteq \mathcal{J}$, define the *diagonal $\mathcal{H}$-cell* $\mathcal{H} = \mathcal{H}(\mathcal{L})$ as

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- $\mathcal{S}(D_{2n})$: $\{\mathrm{B}_e\}$, $\{\mathrm{B}_s, \mathrm{B}_{sts}, \ldots\}$, $\{\mathrm{B}_t, \mathrm{B}_{tst}, \ldots\}$, $\{\mathrm{B}_{w_0}\}$.

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Fix a two-sided cell $\mathcal{J}$ of $\mathscr{C}$.

- For any diagonal $\mathcal{H}$-cell $\mathcal{H} \subseteq \mathcal{J}$, there is an $\mathcal{H}$-*simple* monoidal subquotient category of $\mathscr{C}$, denoted $\mathscr{C}_{\mathcal{H}}$, whose only cells are $\{I\}$ and $\mathcal{H}$.

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Theorem (M-Mazorchuk-Miemietz-Tubbenhauer-Zhang)

For any diagonal $\mathcal{H}$-cell in $\mathcal{H} \subseteq \mathcal{J}$, there is a biequivalence

$$\mathscr{C}\text{-afmod}_{\mathcal{J}}^{\mathrm{ex}} \simeq \mathscr{C}_{\mathcal{H}}\text{-afmod}_{\mathcal{H}}^{\mathrm{ex}},$$

which preserves Jordan-Hölder constituents and restricts to a biequivalence

$$\mathscr{C}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{C}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}.$$

Note: We can pick *any* $\mathcal{H} \subseteq \mathcal{J}$!

Double centralizer theorem

Fix a diagonal $\mathcal{H}$-cell $\mathcal{H} \subseteq \mathcal{J}$ and let $\mathbf{M} \in \mathscr{C}_{\mathcal{H}}\text{-}\mathrm{stmod}_{\mathcal{H}}$.

- Denote by $\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})$ the endomorphism category of $\mathbf{M}$ in $\mathscr{C}_{\mathcal{H}}\text{-}\mathrm{stmod}_{\mathcal{H}}$.

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- The 2-actions of $\mathscr{C}_{\mathcal{H}}$ and $\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})$ on $\mathbf{M}$ commute.

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The canonical monoidal functor

$$\mathrm{can}\colon \mathscr{C}_{\mathcal{H}} \to \mathscr{E}\mathrm{nd}^{\mathrm{ex}}_{\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}).$$

is fully faithful on 2-morphism and essentially surjective on 1-morphisms when restricted to $\mathrm{add}(\mathcal{H})$ *and corestricted to* $\mathscr{E}\mathrm{nd}^{\mathrm{proj}}_{\mathscr{E}\mathrm{nd}_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M})$.

Observations

- The above double centralizer theorem was inspired by the analog for faithful exact module categories of tensor categories, due to Ostrik and Etingof-Ostrik.

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- The following conjecture is also inspired by a result of Ostrik and Etingof-Ostrik:

Conjecture

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Classification Problem

Classify all graded simple transitive 2-representations of $\mathscr{S}$ up to equivalence.

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  - Coxeter type $I_2(n)$ and arbitrary apex, for all $n \geq 2$ [Kildetoft-M-Mazorchuk-Zimmermann, M-Tubbenhauer].

Application of the double centralizer theorem

Fix a two-sided cell $\mathcal{J}$ and choose a diagonal $\mathcal{H}$-cell $\mathcal{H} \subseteq \mathcal{J}$. In this case, the double centralizer theorem implies:

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- In almost all cases, $\mathcal{H}$ can be chosen such that $\mathscr{A}_{\mathcal{H}}$ is well-known and its simple transitive 2-representations have been classified.

Lusztig's **a**-function

Fact: $h_{x,y,z}$ is symmetric in v and v^{-1}.

Proposition (Lusztig)

Let $\mathcal{H}$ be a diagonal $\mathcal{H}$-cell. There exists $\mathbf{a} \in \mathbb{Z}_{\geq 0}$ such that for all $x, y, z \in \mathcal{H}$:

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} \mathrm{v}^{\mathbf{a}} + \cdots + \gamma_{x,y,z^{-1}} \mathrm{v}^{-\mathbf{a}}.$$

Moreover, there exists a unique $d \in \mathcal{H}$ (Duflo involution) such that $d^2 = e$ in W and

$$\gamma_{d,x,y^{-1}} = \gamma_{x,d,y^{-1}} = \gamma_{x,y^{-1},d} = \delta_{x,y}$$

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Asymptotic limit:

$$\gamma_{x,y,z^{-1}} = \lim_{\mathrm{v} \to +\infty} \mathrm{v}^{-\mathbf{a}} h_{x,y,z} \in \mathbb{Z}_{\geq 0}.$$

Soergel's hom-formula

Theorem (Soergel, Elias-Williamson)

$$\dim\left(\hom(\mathrm{B}_x, \mathrm{B}_y\langle t\rangle)\right) = \begin{cases} \delta_{x,y}, & \textit{if } t = 0; \\ 0 & \textit{if } t < 0. \end{cases}$$

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This implies that $\mathscr{S}_{\mathcal{H}}$ is a filtered category. By the properties of $h_{x,y,z}$, the part

$$\mathcal{X}_{\leq -\mathbf{a}} := \mathrm{add}\big(\{\mathrm{B}_w\langle k\rangle \mid w \in \mathcal{H}, k \leq -\mathbf{a}\}\big)$$

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is lax monoidal: It is strictly associative with lax identity object $\mathrm{B}_d\langle -\mathbf{a}\rangle$. Define the *asymptotic Soergel category*:

$$\mathcal{A}_{\mathcal{H}} := \mathcal{X}_{\leq -\mathbf{a}} / \big(\mathcal{X}_{< -\mathbf{a}}\big).$$

The asymptotic Soergel category

Theorem (Lusztig, Elias-Williamson)

$\mathscr{A}_{\mathcal{H}} = (\mathscr{A}_{\mathcal{H}}, \star, I, \vee)$ *is a fusion category:*

- *for every* $x \in \mathcal{H}$*, the object* $\mathrm{A}_x := [\mathrm{B}_x\langle -a \rangle]$ *is simple;*
- $\{\mathrm{A}_x \mid x \in \mathcal{H}\}$ *is a complete set of pairwise non-isomorphic simple objects;*
- *for any* $x, y \in \mathcal{H}$*, we have*

$$\mathrm{A}_x \star \mathrm{A}_y \cong \bigoplus_{z \in \mathcal{H}} \mathrm{A}_z^{\oplus \gamma_{x,y,z^{-1}}};$$

- $I := \mathrm{A}_d$*;*
- *for every* $x \in \mathcal{H}$*, we have* $\mathrm{A}_x^{\vee} \cong \mathrm{A}_{x^{-1}}$.

Theorem (Bezrukavnikov-Finkelberg-Ostrik, Ostrik, Elias)

In almost all cases, $\mathcal{H}$ can be chosen such that $\mathscr{A}_{\mathcal{H}}$ is biequivalent to one of the following fusion categories:

- Vect_G *or* $\mathbb{C}[G]\text{-}\mathrm{fmod}$, *with* $G = (\mathbb{Z}/2\mathbb{Z})^k, S_3, S_4, S_5$;
- $U_q(\mathfrak{so}_3)\text{-}\mathrm{fmod}_{\mathrm{ss}}$ *for* $q = e^{\frac{\pi i}{n}}$ *for some* $n \in \mathbb{Z}_{\geq 2}$.

- The simple transitive 2-representations of these fusion categories have been completely classified by Kirillov-Ostrik and Ostrik.

Theorem (M-Mazorchuk-Miemietz-Tubbenhauer-Zhang)

There is a biequivalence

$$\mathscr{E}\mathrm{nd}^{(0)}_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}) \simeq \bigoplus_{t\in\mathbb{Z}} \mathscr{A}_{\mathcal{H}}\langle t\rangle.$$

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Corollary

There is a biequivalence

$$\mathscr{S}^{(0)}_{\mathcal{H}}\text{-stmod}/\mathbb{Z} \simeq \mathscr{A}_{\mathcal{H}}\text{-stmod}.$$

In particular, there is a bijection

$$\{\text{g.s.t. 2-reps of } \mathscr{S}_{\mathcal{H}}\}/\simeq \quad \xleftrightarrow{1:1} \quad \{\text{s.t. 2-reps of } \mathscr{A}_{\mathcal{H}}\}/\simeq .$$

THANKS!!!