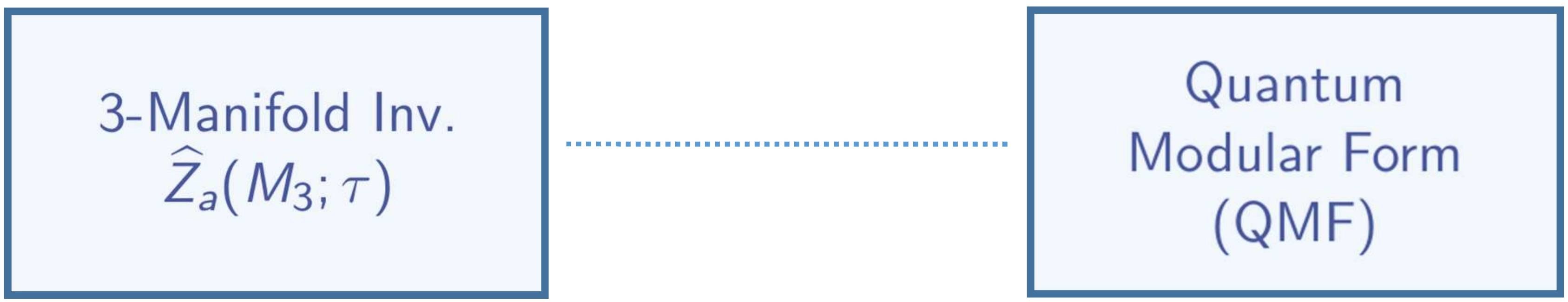
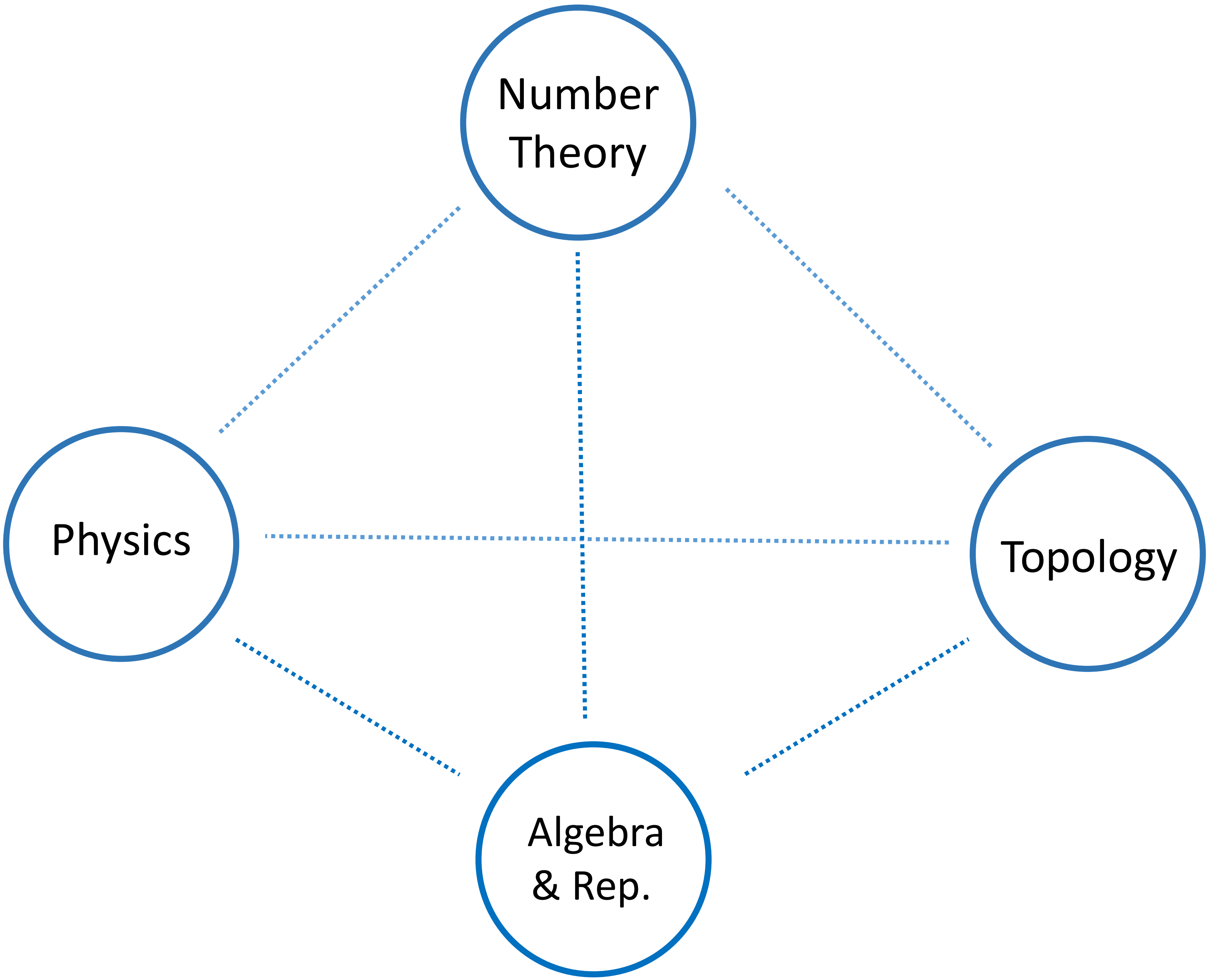
Quantum Modularity and 3-Manifolds

Miranda C. N. Cheng

University of Amsterdam


3-Manifold Inv.
$\widehat{Z}_a(M_3;\tau)$
Quantum
Modular Form
(QMF)


Number Theory
Physics
Topology
Algebra & Rep.

Main Motivations:

- QMF

natural structure beyond modular forms;

- $\widehat{Z}_a(M_3;\tau)$

q-invariants for (closed) 3-manifolds;

- $\widehat{Z}_a(M_3;\tau)$ =susy index

3d SQFT, 3d-3d, and M-theory.

- $\widehat{Z}_a(M_3;\tau) \sim \chi_R^{\mathcal{V}}(\tau)$

Novel types of vertex algebras and representations.

Based on:

- *3d Modularity*, 1809.10148
w. S. Chun, F. Ferrari, S. Gukov, S. Harrison.
- *3d Modularity and log VOA*, 20XX.XXXXX
w. S. Chun, B. Feigin, F. Ferrari, S. Gukov, S. Harrison.





- *Three-Manifold Quantum Invariants and Mock Theta Functions*, 1912.07997
w. F. Ferrari, G. Sgroi.
- *Three Manifolds and Indefinite Theta Functions*, 20XX.
w. G. Sgroi.

Outline:

I. Background

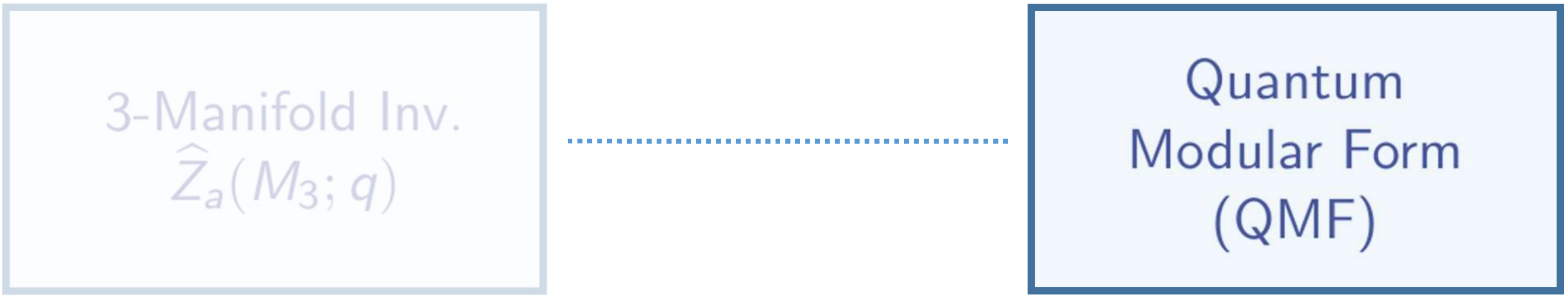
II. A (True) False Theorem

III. A Mock–False Conjecture

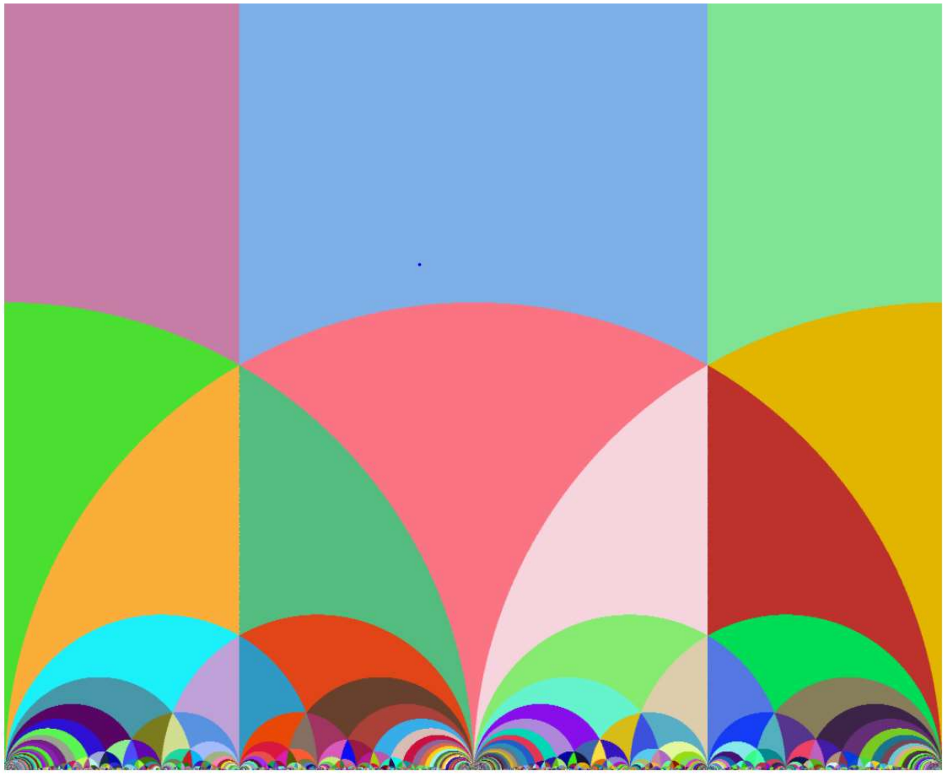
IV. Going Deeper

V. Questions for Future

I. Background



I.1 **Quantum Modular Forms (QMF)**: the Upper-Half Plane $\mathbb{H}$



Symmetry: $\tau \mapsto \gamma\tau := \dfrac{a\tau + b}{c\tau + d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \supset SL_2(\mathbb{Z})$$

$\mathbb{H}$ has natural boundary $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, the *cusps* of $SL_2(\mathbb{Z})$ which acts transitively.

I.1 Quantum Modular Forms (QMF): Modular Forms

Consider a holomorphic fn f on $\mathbb{H}$, G a discrete subgroup of $SL_2(\mathbb{Z})$.

Def (modular transf. of weight w): $f|_w\gamma(\tau) := f(\gamma\tau)(c\tau + d)^{-w}$

Def (modular form of weight w for G): $f|_w\gamma(\tau) = f(\tau)\ \forall\gamma \in G$

Many generalisations: *non-trivial G-characters, vector-valued, non-holomorphic etc.*

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Example: Lattice θ-functions

- $\Lambda = \mathbb{Z}$, $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}$, wt $1/2$
- $\Lambda = \sqrt{2m}\mathbb{Z}$, $\Lambda^*/\Lambda \cong \mathbb{Z}/2m$,

$$\theta^0_{m,r}(\tau) = \sum_{k\equiv r\ (2m)} q^{\frac{k^2}{4m}}, \text{ wt } 1/2$$

$$\theta^1_{m,r}(\tau) = \sum_{k\equiv r\ (2m)} kq^{\frac{k^2}{4m}}, \text{ wt } 3/2$$

$(q := e^{2\pi i\tau})$

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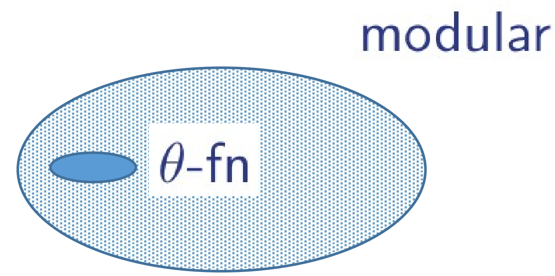


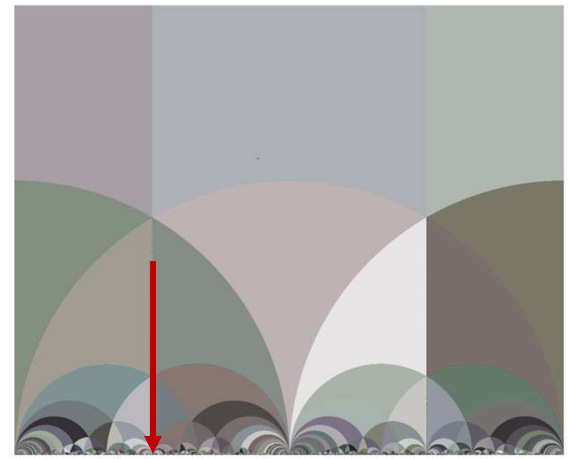
I.1 **Quantum Modular Forms (QMF)**: Radial Limit

Consider a holomorphic fn f on $\mathbb{H}$.

Taking the *radial limit*:

$$f\left(\frac{p}{q}\right) := \lim_{t \to 0^+} f\left(\frac{p}{q} + it\right)$$

defines a function on $\mathbb{Q}$.

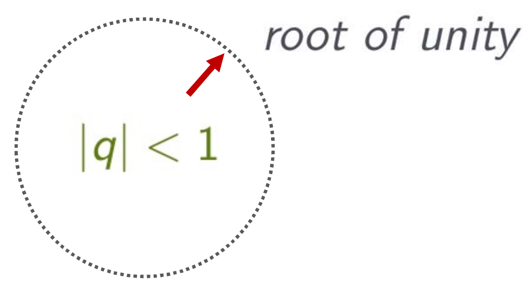


$\tau \to \alpha \in \mathbb{Q}$

Remark: Later we will see:

q-series invariant $\to\to\to$ Chern-Simons (WRT) invariant

$$q \quad \to \quad e^{2\pi i \frac{1}{k}}$$

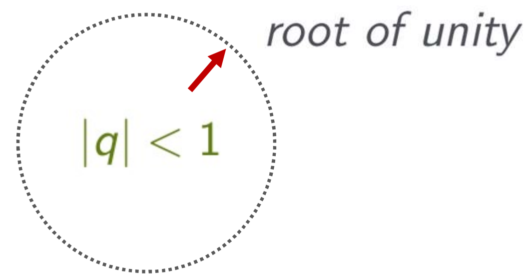


I.1 **Quantum Modular Forms (QMF)**: Modular Forms

Consider a modular form f.

Taking the *radial limit*:

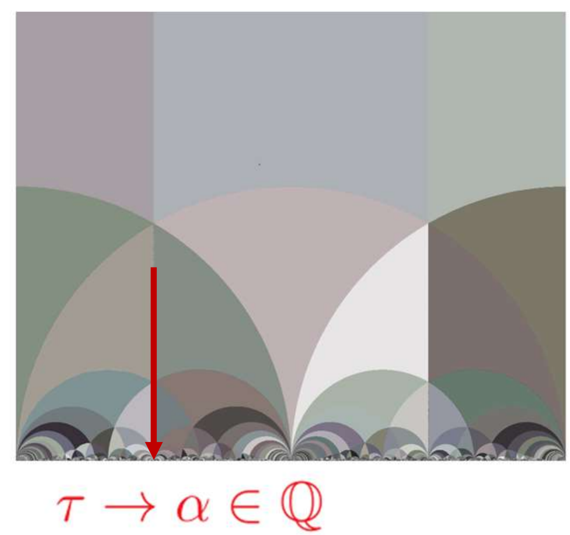
$$f\left(\frac{p}{q}\right) := \lim_{t \to 0^+} f\left(\frac{p}{q} + it\right)$$



defines a function on $\mathbb{Q}$, satisfying

$$f(x) - f|_w\gamma(x) = 0$$

for all $x \in \mathbb{Q}\backslash\{\gamma^{-1}(\infty)\}$.



I.1 Quantum Modular Forms (QMF): A First Definition

How to generalise $f(x) - f|_w\gamma(x) = 0$?

Here neither of the properties which are required of classical modular forms—analyticity and Γ-covariance—are reasonable things to require: the former because $\mathbb{P}^1(\mathbb{Q})$, viewed as the set of cusps of the action on Γ on $\mathfrak{H}$, is naturally equipped only with the discrete topology, not with its induced topology as a subset of $\mathbb{P}^1(\mathbb{R})$, so that any requirement of continuity or analyticity is vacuous; and the latter because Γ acts on $\mathbb{P}^1(\mathbb{Q})$ transitively or with only finitely many orbits, so that any requirement of Γ-covariance of a function on this set would lead to a trivial definition. So we do not demand either continuity/analyticity or modularity, but require instead that the failure of one precisely offsets the failure of the other. In other words, our quantum modular form should be a function $f : \mathbb{Q} \to \mathbb{C}$ for which the function $h_\gamma : \mathbb{Q} \smallsetminus \{\gamma^{-1}(\infty)\} \to \mathbb{C}$ defined by

$$h_\gamma(x) = f(x) - (f|_k\gamma)(x) \qquad (2)$$

has some property of continuity or analyticity (now with respect to the real topology) for every element $\gamma \in \Gamma$. This is purposely a little vague, since examples coming from different sources have somewhat different properties, and we want to consider all of them as being quantum modular forms.

[Don Zagier 2010]

I.1 Quantum Modular Forms (QMF): Strong QMF

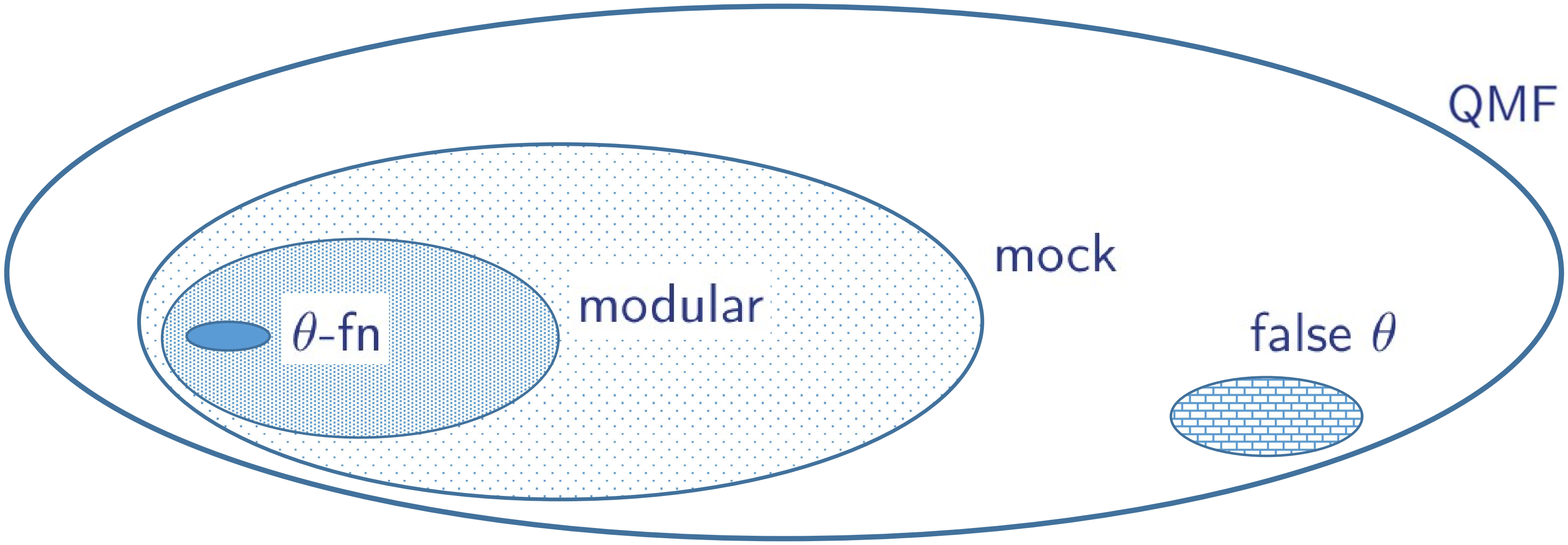
A *strong quantum modular form*—and most of our examples will belong to this category—is an object with a stronger (and more interesting) structure: it associates to each element of $\mathbb{Q}$ a formal power series over $\mathbb{C}$, rather than just a complex number, with a correspondingly stronger requirement on its behavior under the action of Γ. To describe this, we write the power series in $\mathbb{C}[[\varepsilon]]$ associated to $x \in \mathbb{Q}$ as $f(x + i\varepsilon)$ rather than, say, $f_x(\varepsilon)$, so that f is now defined in the union of (disjoint!) formal infinitesimal neighborhoods of all points $x \in \mathbb{Q} \subset \mathbb{C}$. Since the function h_γ in (2) was required to be real-analytic on the complement of a finite subset S_γ of $\mathbb{P}^1(\mathbb{R})$, it extends holomorphically to a neighborhood of $\mathbb{P}^1(\mathbb{R}) \smallsetminus S_\gamma$ in $\mathbb{P}^1(\mathbb{C})$, and in particular has a power series expansion (convergent in some disk of positive radius) around each point $x \in \mathbb{Q}$. Our stronger requirement is now that the equation

$$(3) \qquad f(z) - (f|_k\gamma)(z) = h_\gamma(z) \qquad (\gamma \in \Gamma, \quad z \to x \in \mathbb{Q})$$

holds as an identity between countable collections of formal power series.

the power series $f(0 + it) \sim$ semi-classical $\frac{1}{k}$-expansion of WRT $\sim$ Ohtsuki series of 3-manifolds

I.1 Quantum Modular Forms (QMF): Examples



Examples: False Theta Functions, Mock Modular Forms,...

Applications: Kashaev invariants, log CFT characters, $\widehat{Z}_a(q)$, ...

I.1 Quantum Modular Forms (QMF) ⊃ False and Mock

Consider a modular form g of weight w.

Def (Eichler integrals):[*]

$$\tilde{g}(\tau) := \int_{\tau}^{i\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau' \qquad \text{(holomorphic)}$$

$$g^*(\tau) := \int_{-\bar{\tau}}^{i\infty} g(\tau')(\tau' + \tau)^{w-2} d\tau' \qquad \text{(non-holomorphic)}$$

Rk: $\tilde{g} - \tilde{g}|_{2-w}\gamma$ and $g^* - g^*|_{2-w}\gamma$ are period integrals $\to$ quantum modularity.

$$(\tilde{g}|_{2-w}\gamma)(\tau) = (c\tau + d)^{-2+w} \int_{\tau}^{\gamma^{-1}\infty} g(\gamma\tau')(\gamma\tau' - \gamma\tau)^{w-2} d(\gamma\tau')$$

$$= \int_{\tau}^{\gamma^{-1}\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau'$$

$$\Rightarrow (\tilde{g} - \tilde{g}|_{2-w}\gamma)(\tau) = \int_{\gamma^{-1}\infty}^{\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau'$$

∗ some irrelavant constant factors ignored.

I.1 Quantum Modular Forms (QMF) $\supset$ False and Mock

Consider a modular form g of weight w.

Def (Eichler integrals):*

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$$g^*(\tau) := \int_{-\bar{\tau}}^{i\infty} g(\tau')(\tau' + \tau)^{w-2} d\tau' \qquad \text{(non-holomorphic)}$$

Example: False θ-function

$$\theta^1_{m,r}(\tau) = \sum_{k \equiv r \ (2m)} k q^{\frac{k^2}{4m}} \text{, wt } 3/2$$

$$\widetilde{\theta^1_{m,r}}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k \equiv r (2m)}} \underline{\operatorname{sgn}(k)}\, q^{k^2/4m}$$

↘ false

* some irrelavant constant factors ignored.

I.1 Quantum Modular Forms (QMF) $\supset$ False and Mock

Consider a holomorphic fn f on $\mathbb{H}$.

Def (mock modular forms, mmf) [Zwegers '02]:

f is a **mmf** of weight w if there exists a modular form $g = \mathrm{shad}(f)$ (the **shadow**) of weight $2 - w$ such that $\hat{f} := f - g^*$ satisfies $\hat{f} = \hat{f}|_w\gamma \quad \forall\, \gamma \in G$.

Rk: $\hat{f} = \hat{f}|_w\gamma \Rightarrow f - f|_w\gamma = g^* - g^*|_w\gamma \rightarrow$ quantum modularity.

I.1 Quantum Modular Forms (QMF) ⊃ False and Mock

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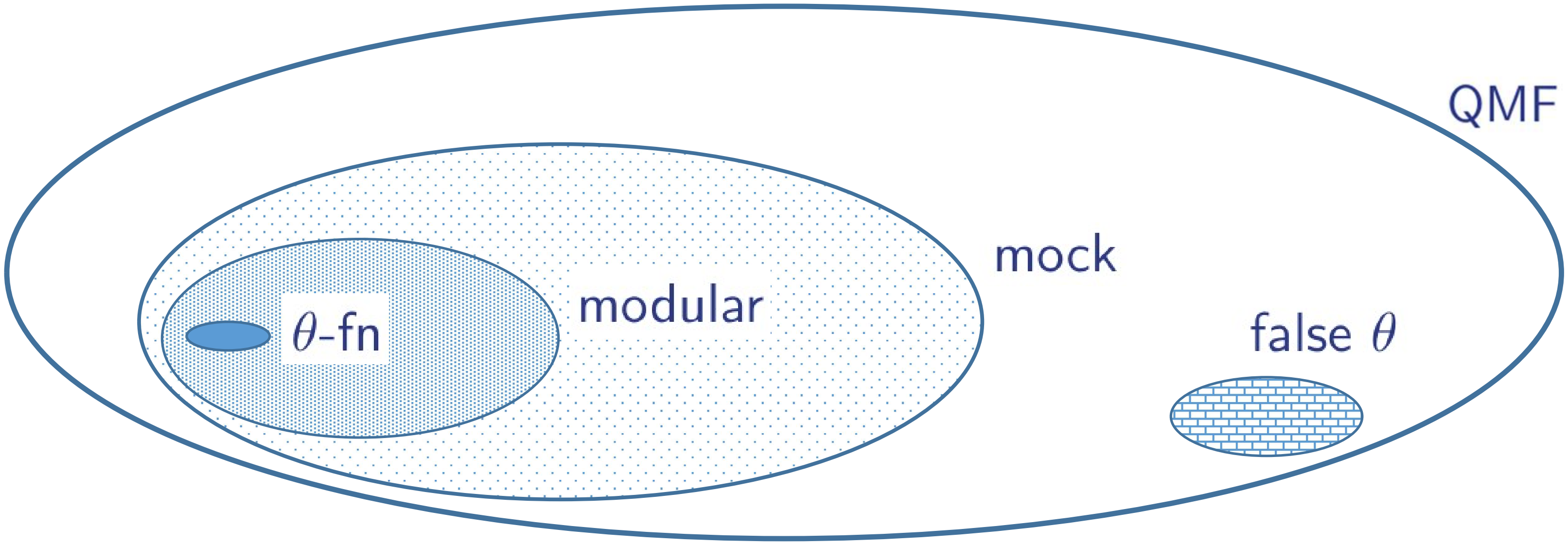
Example : modular forms

Example : Ramanujan's Mock θ Functions

$$F_0(\tau) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1 - q^{n+k})} = 1 + q + q^3 + q^4 + O(q^5)$$

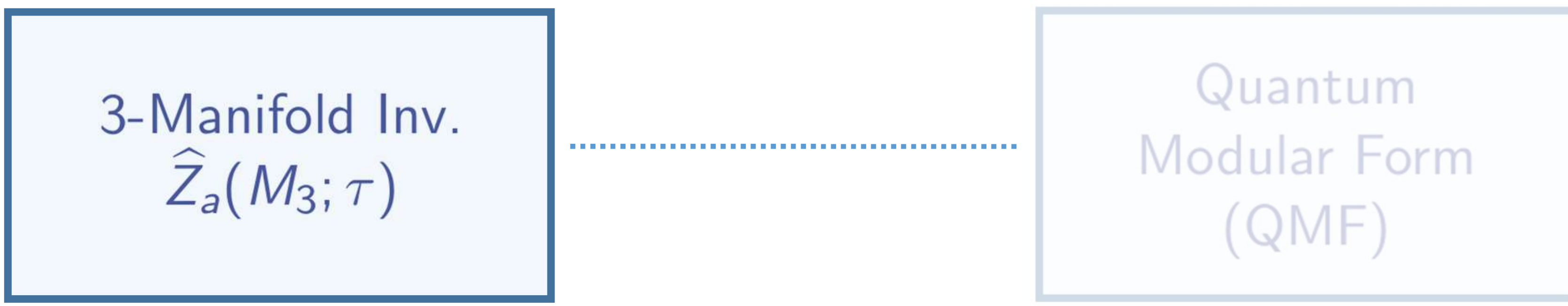
$$shad(F_0)(\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1 \ (42)}} \left(\frac{i}{21}\right) \theta^1_{42,i}(\tau)$$

I.1 Quantum Modular Forms (QMF): Examples



Questions?

I. Background



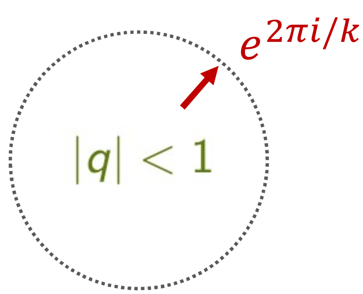
main ref. [Gukov-Pei-Putrov-Vafa '17]

$\widehat{Z}_a(M_3;\tau)$ and Z_{CS}

$Z_{\mathrm{CS}}(M_3;k)$; $k \in \mathbb{Z}$ is the (effective) level.

Question: Can we go from $\mathbb{Z}$ to $\mathbb{H}$:
a q-series inv. for 3-man. extending Z_{CS}?

$$\text{Idea: } q\text{-series} \xrightarrow[q\to e^{2\pi i/k}]{\text{radial limit}} Z_{CS}(k) \qquad (*)$$



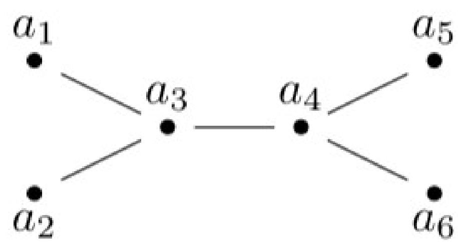
Remarks: 1. cf. previous work by Habiro. 2. $(*)$ is not sufficient to fix the q-series.

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

M_3: Plumbed 3-manifold, determined by its **plumbing graph** Γ.

weighted graph $\Gamma := (V, E, a)$, $a : V \to \mathbb{Z}$.

e.g.



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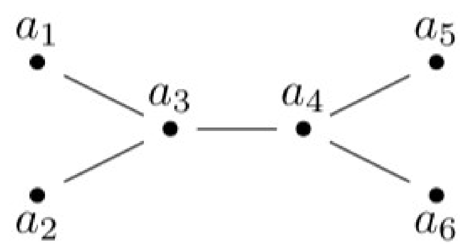


plumbing graph Γ ⟷ glue (disk bdls over S^2 w. $\chi = a_i$) = $M_4(\Gamma)$

↕

adjacency matrix M

$$M = \begin{pmatrix} a_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 1 & 1 & a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 & 1 & 1 \\ 0 & 0 & 0 & 1 & a_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & a_6 \end{pmatrix}$$

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

M_3: Plumbed 3-manifold, determined by its **plumbing graph** Γ.

glue (disk bdle over S^2 w. $\chi = a_i$)

$=$

$M_4(\Gamma)$

plumbing graph Γ $\longleftrightarrow$ $M_4(\Gamma)$

$\updownarrow$

adjacency matrix M

$\downarrow$ take bdry

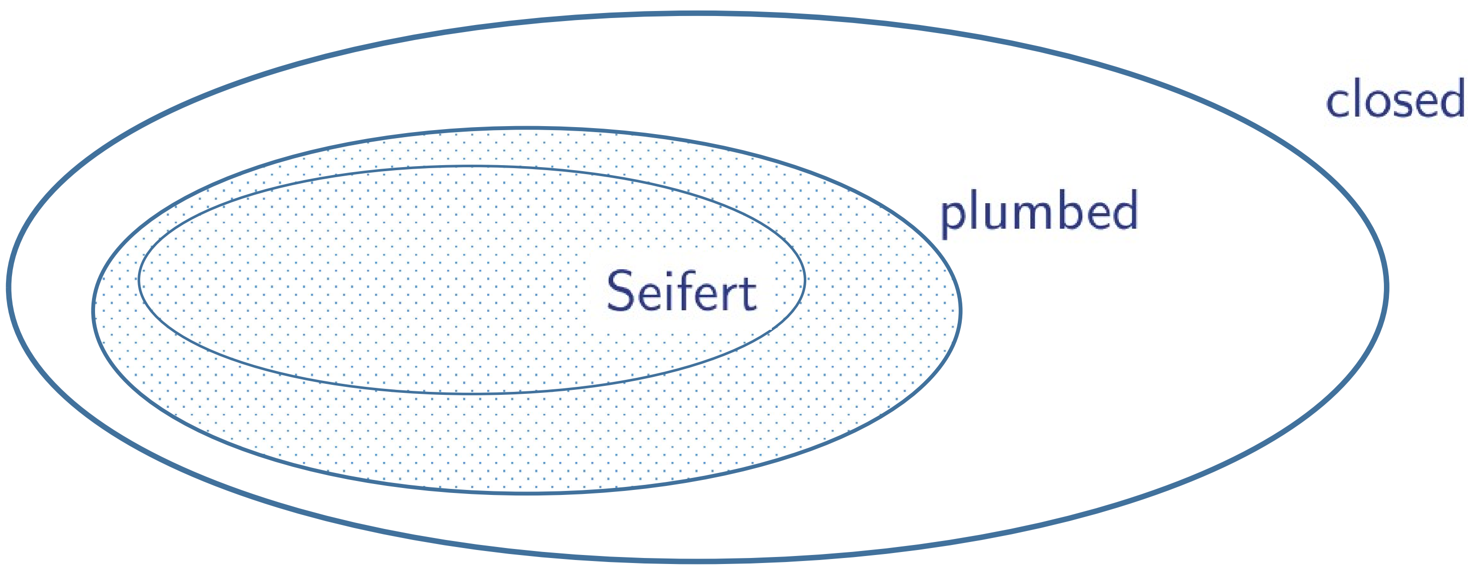
plumbed $M_{3,\Gamma}$

$$M = \begin{pmatrix} a_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 1 & 1 & a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 & 1 & 1 \\ 0 & 0 & 0 & 1 & a_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & a_6 \end{pmatrix}$$

$H_1(M_{3,\Gamma};\mathbb{Z}) \cong \mathbb{Z}^{|V|}/M\mathbb{Z}^{|V|}$ (Coker M)

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

M_3: Plumbed 3-manifold, determined by its **plumbing graph** Γ.



$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

Def: For a weighted graph Γ with a neg.-def. M, and for a given $a \in \mathrm{Cork}(M)$, define the theta function

$$\Theta_a^M(\tau;\mathbf{z}) := \sum_{\ell \in 2M\mathbb{Z}^{|V|} \pm a} q^{-\ell^T M^{-1} \ell}\, \mathbf{z}^{\ell}.$$

$$\boxed{\widehat{Z}_a(M_{3,\Gamma};\tau) := (\pm)\, q^{\Delta} \oint \prod_{v \in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^M(\tau;\mathbf{z})}$$

$$\sim [\mathbf{z}^0] \left(\prod_{v \in V} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^M(\tau;\mathbf{z}) \right)$$

Def: For a weighted graph Γ with a neg.-def. M, and for a given $a \in \mathrm{Cork}(M)$, define the theta function

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$$\sim [\mathbf{z}^0]\left(\prod_{v\in V}\left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta^M_a(\tau;\mathbf{z})\right)$$

Remarks:

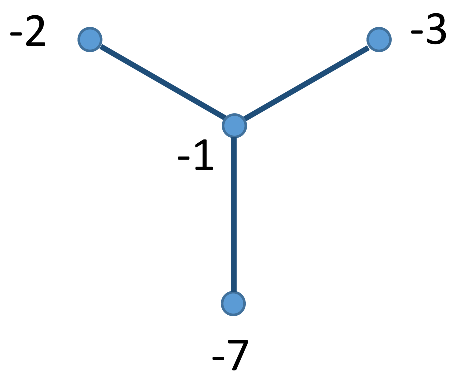
1. a *set* of q-invariants;
2. $a \in \mathrm{Cork}(M) \cong H_1(M_3,\mathbb{Z}) \cong \{\text{inequiv. } SU(2) \text{ Ab. flat connections}\}^*$;
3. neg.-def. $M^{**} \Leftrightarrow$ pos.-def. lattice $\Leftrightarrow$ Θ and hence $\widehat{Z}_a$ converges when $\tau \in \mathbb{H}$;
4. $q^c \widehat{Z}_a(\tau) \in \mathbb{Z}[[q]]$ for a $c \in \mathbb{Q}$ dependening only on M_3.

* up to Weyl group $\mathbb{Z}_2$ action

** this condition can be relaxed : M^{-1} only needs to be neg.-def. in the subspace spanned by the vertices with at least 3 edges

$\widehat{Z}_a(M_3;\tau)$: Mathematical Definition

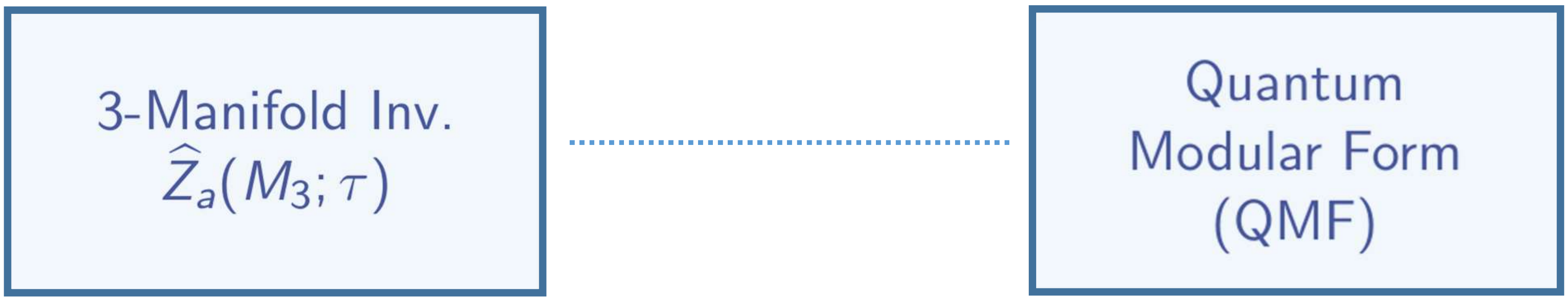
eg.



$$M_{3,\Gamma} = \Sigma(2,3,7) = \{x^2 + y^3 + z^7 = 0\} \cap S^5$$

$$q^{-\frac{83}{168}}\,\hat{Z}_0(\Sigma(2,3,7),\tau) = \sum_{\substack{i\in\mathbb{Z}/42 \\ i^2\equiv 1\ (42)}} \left(\frac{i}{21}\right) \widetilde{\theta^1_{42,i}}(\tau) = \widetilde{shad(F_0)}(\tau)$$

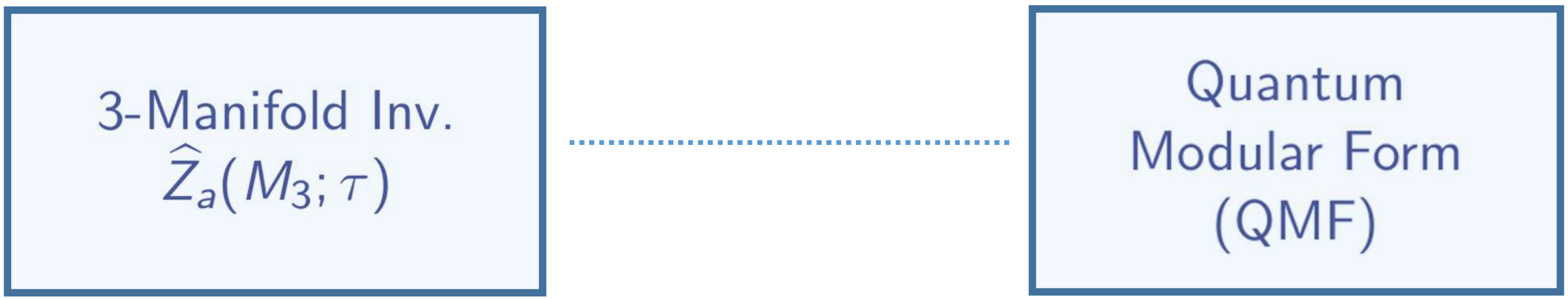
Questions?



Applications:

Quantum modularity

- helps to determine the q-invariants;
- leads to new ways of retrieving topological information;
- gives hints about the physical theories.



See also important previous and ongoing work on a related topic (Kashaev invariants of knots):
Zagier '10, Garoufalidis-Zagier '13 and new, Dimofte-Garoufalidis '15, Hikami-Lovejoy '14, ….

I. Background

II. A (True) False Theorem

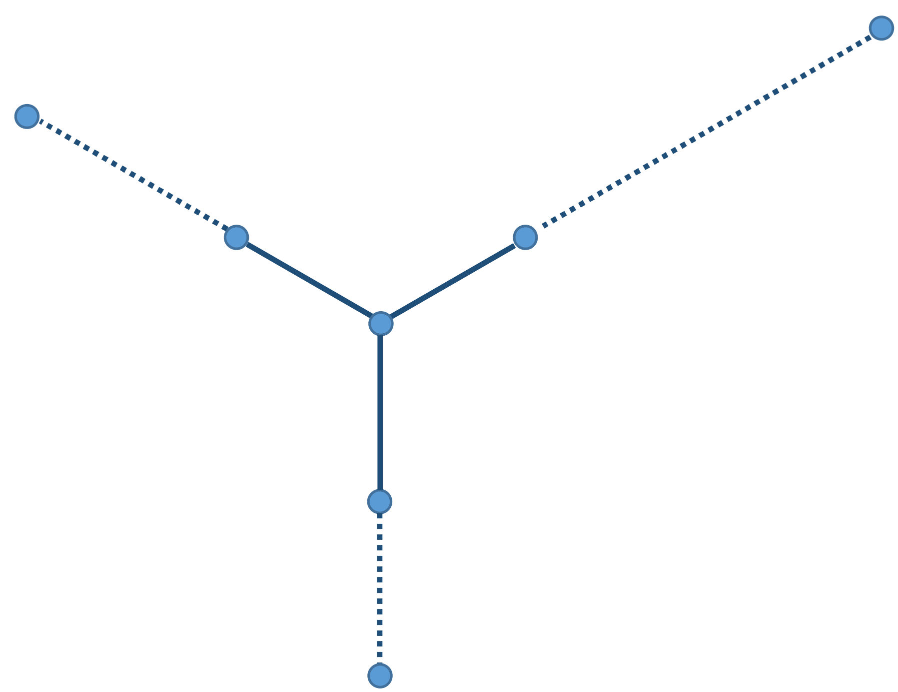
III. A Mock–False Conjecture

IV. Going Deeper

V. Questions for Future

First we focus on the most tractable family of examples:

Γ = 3-pronged star



A False Theorem

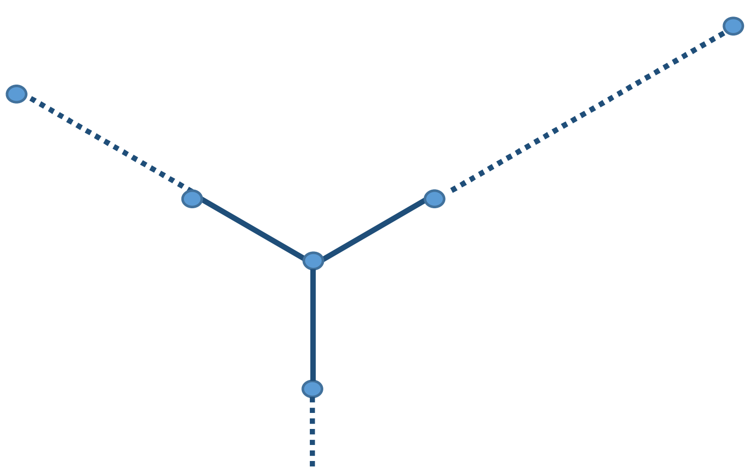
Theorem : Negative three-stars are false.

[MC-Chun-Ferrari–Gukov-Harrison, Bringmann-Mahlburg-Milas '18]

For any three-pronged star weighted graph Γ of negative type, the functions $\widehat{Z}_a(M_{3,\Gamma};\tau)$ *are false theta functions*. In particular, there exists an $m = m(\Gamma) \in \mathbb{Z}_{>0}$ such that (up to a finite polynomial)

$$q^{c}\,\widehat{Z}_a(\tau) \in \mathrm{span}_{\mathbb{Z}} \left\{ \widetilde{\theta^1_{m,r}},\, r \in \mathbb{Z}/2m \right\} \quad \forall\ a.$$

Rk: See also earlier work by [Lawrence–Zagier '99] and Hikami in the context of CS inv.



Recall: (false) theta functions

$$\theta^1_{m,r} = \sum_{k \equiv r\ (2m)} k\, q^{\frac{k^2}{4m}}$$

$$\widetilde{\theta^1_{m,r}} = \sum_{k \equiv r\ (2m)} \mathrm{sgn}(k)\, q^{\frac{k^2}{4m}}$$

$\widehat{Z}_a = \textbf{QMF}$

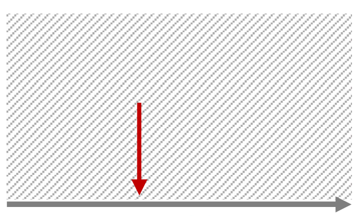
$$\widehat{Z}_a(\tau) = \left(\widetilde{\theta^1_{m,r}} + \widetilde{\theta^1_{m,r'}} + \widetilde{\theta^1_{m,r''}} + \dots\right), \quad r, r', \dots \in \mathbb{Z}/2m$$

Recall that the false theta functions like $\widetilde{\theta^1_{m,r}}$ are quantum modular forms, which means

$$\left(\widehat{Z}_a - \widehat{Z}_a|_{1/2}\gamma\right)(\tau) \qquad (*)$$

when the radially limit is properly taken, has analytic properties.

$$\boxed{\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\text{radial limit}} Z_{\mathrm{CS}}}$$



$\tau \to \frac{1}{k}$

$(*) \Rightarrow$

$$Z_{CS}(k) \sim \widehat{Z}(\frac{1}{k}) = \widehat{Z}(-k) + \text{pert. series in } \frac{1}{k}$$

sadd. pnt contr. from $SL(2,\mathbb{C})$ flat connections

gives the Ohtsuki series

$\widehat{Z}_a$ = Log Characters

Theorem : Negative three-stars are false.

[MC-Chun-Ferrari–Gukov-Harrison, Bringmann-Mahlburg-Milas '18]

For any three-pronged star weighted graph Γ of negative type, the functions $\widehat{Z}_a(M_{3,\Gamma};\tau)$ *are false theta functions*. In particular, there exists an $m = m(\Gamma) \in \mathbb{Z}_{>0}$ such that (up to a finite polynomial)

$$q^{c}\, \widehat{Z}_a(\tau) \in \mathrm{span}_{\mathbb{Z}} \left\{ \widetilde{\theta^1_{m,r}},\, r \in \mathbb{Z}/2m \right\} \quad \forall\, a.$$

$\sim$ log VOA character

Log VOAs:

- contain modules not decomposable into irreducibles;

- a nice playground to study the mathematical properties of non-rational vertex algebras.

A Simple Log VOA: the $(1, m)$ Algebra

Given a positive integer m, let $\alpha_\pm = \pm\sqrt{2m^{\pm 1}}$, $\alpha_0 = \alpha_+ + \alpha_-$

free boson : $\varphi(z)\varphi(w) \sim \log(z - w)$

stress energy tensor : $T = \frac{1}{2}(\partial\varphi)^2 + \frac{\alpha_0}{2}\partial^2\varphi$, $c = 1 - 3\alpha_0^2$

screening charges : $Q_- = (e^{\alpha_- \varphi})_0$

> triplet $(1, m)$ algebra: $\mathcal{W}(m) := \ker_{\mathcal{V}_L} Q_-$
>
> singlet $(1, m)$ algebra: $\mathcal{M}(m) := \ker_{H} Q_-$

where $\mathcal{V}_L$ = lattice VOA for $L = \sqrt{2m}\mathbb{Z}$, H = Heisenberg algebra.

$$\begin{array}{ccc} H & \subset & \mathcal{V} \\ \cup & & \cup \\ \mathcal{M}(m) & \subset & \mathcal{W}(m) \end{array}$$

A Simple Log VOA: the $(1, m)$ Algebra

The triplet $(1, m)$ algebra $\mathcal{W}(m)$ has $2m$ irreducible modules.

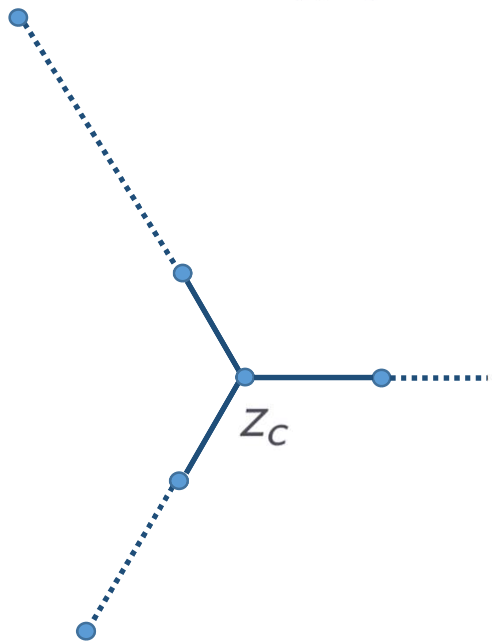
We are especially interested in m of them, with graded character

$$\chi_s^{\mathcal{W}(m)} = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{(2mn+m-s)^2}{4m}} \frac{z^{2n+1} - z^{-2n-1}}{z - z^{-1}}, \; s = 1, \ldots, m.$$

$$\frac{\widehat{Z}_a(M_{3,\Gamma}; \tau)}{\eta(\tau)} \sim \frac{1}{\eta(\tau)} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^M(\tau; \mathbf{z})$$

$\widehat{Z}_a$ and Log VOA Characters

$$\frac{\widehat{Z}_a(M_{3,\Gamma};\tau)}{\eta(\tau)} \sim \frac{1}{\eta(\tau)} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^M(\tau;\mathbf{z})$$



Integrate over all but the central node z_c

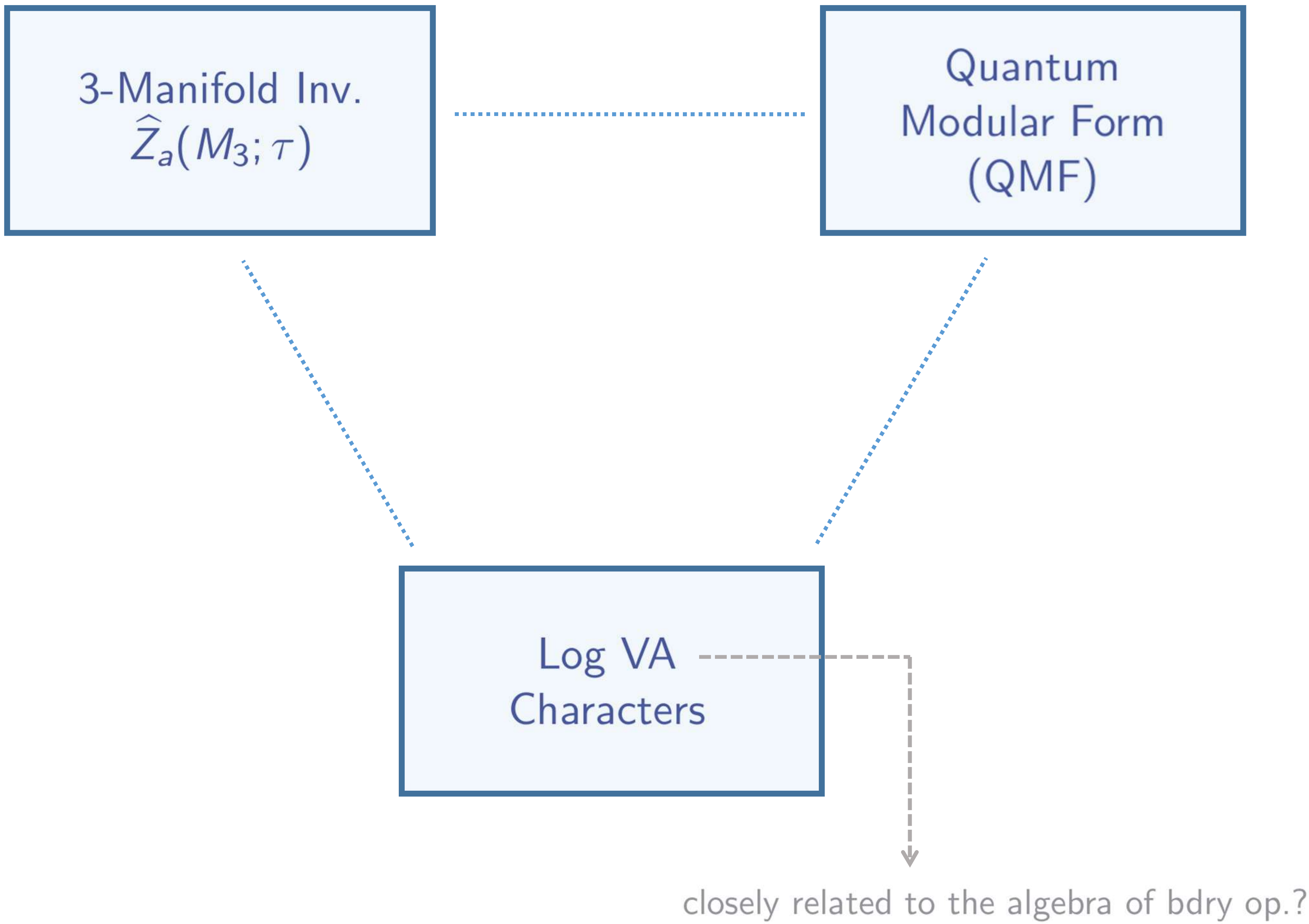
$$= [z_c^0]\left(\chi_s^{\mathcal{W}(m)} + \chi_{s'}^{\mathcal{W}(m)} + \chi_{s''}^{\mathcal{W}(m)} + \ldots\right)(\tau, z_c)$$

triple $(1, m)$ alg. characters

$$= \left(\chi_s^{\mathcal{M}(m)} + \chi_{s'}^{\mathcal{M}(m)} + \chi_{s''}^{\mathcal{M}(m)} + \ldots\right)(\tau)$$

single $(1, m)$ alg. characters

$$= \frac{1}{\eta(\tau)}\left(\widetilde{\theta^1_{m,m-s}} + \widetilde{\theta^1_{m,m-s'}} + \widetilde{\theta^1_{m,m-s''}} + \ldots\right)(\tau)$$


3-Manifold Inv.
$\widehat{Z}_a(M_3;\tau)$
Quantum
Modular Form
(QMF)
Log VA
Characters
closely related to the algebra of bdry op.?

Questions?

I. Background

II. A (True) False Theorem

III. A Mock–False Conjecture

IV. Going Deeper

V. Questions for Future

A Puzzle

Recall

$$\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\substack{\tau \to \frac{1}{k} \\ \text{radial limit}}} Z_{\mathrm{CS}}$$

Upon flipping orientation, we have

$$Z_{\mathrm{CS}}(-M_3; k) = Z_{\mathrm{CS}}(M_3; -k)$$

A Puzzle

Recall

$$\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\text{radial limit}} Z_{\rm CS} \qquad \tau \to \frac{1}{k}$$

Upon flipping orientation, we have

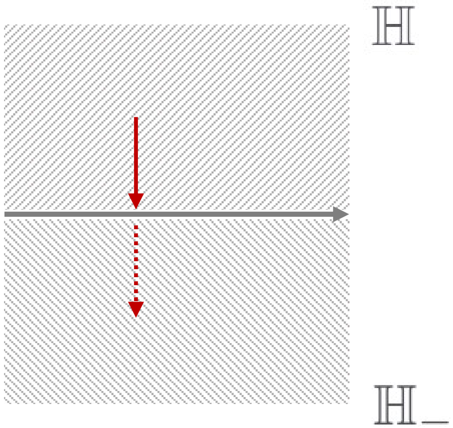
$$Z_{\rm CS}(-M_3; k) = Z_{\rm CS}(M_3; -k)$$

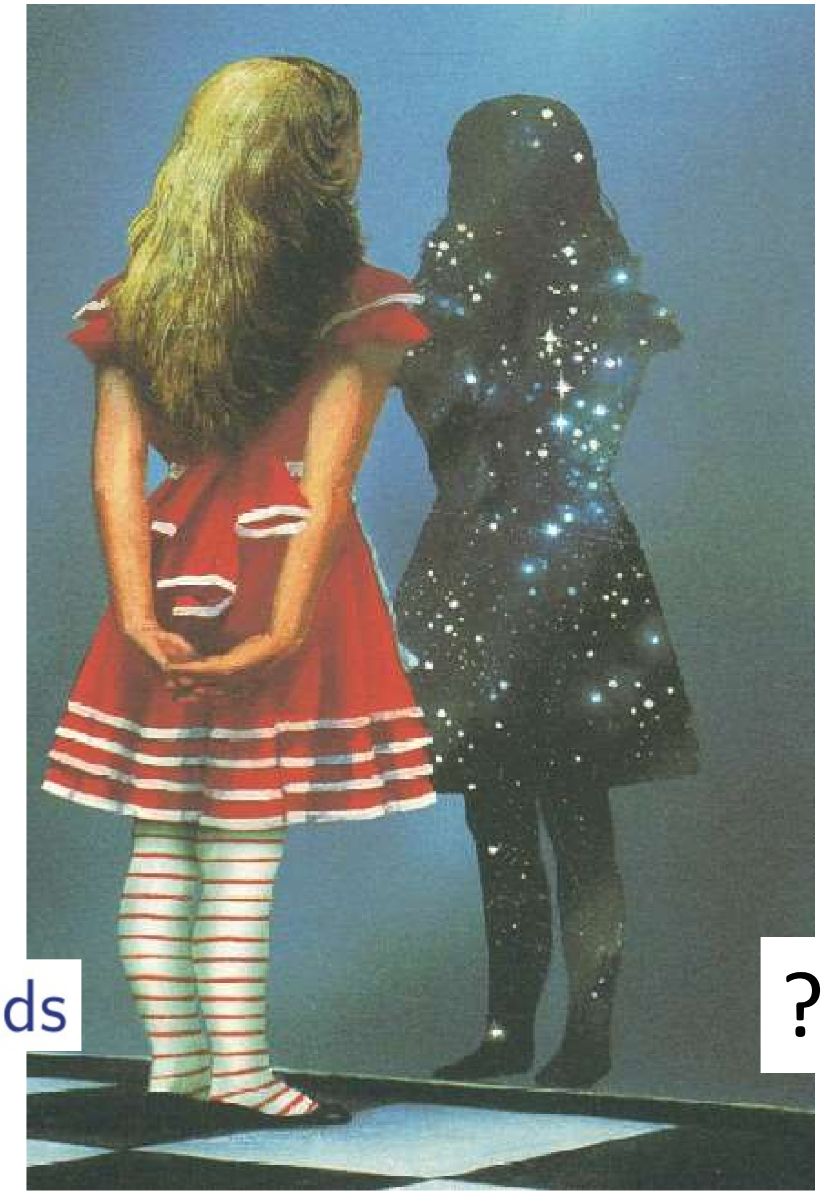
From $(k \leftrightarrow -k) \Leftrightarrow (\tau \leftrightarrow -\tau) \Leftrightarrow (q \leftrightarrow q^{-1})$, we expect

$$\boxed{\widehat{Z}_a(-M_3; \tau) = \widehat{Z}_a(M_3; -\tau)}$$

But what's this? Can we define $\widehat{Z}_a(M_3; \tau)$ *for both* $(|q| < 1 \Leftrightarrow \tau \in \mathbb{H})$ *and* $(|q| > 1 \Leftrightarrow \tau \in \mathbb{H}_-)$ *?*

Going to the Other Side





False 3-Manifolds

????????

Troubles with Thetas

$$\widehat{Z}_a(M_{3,\Gamma};\tau) := (\pm)\, q^{\Delta} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^M(\tau;\mathbf{z})$$

$$\Theta_a^M(\tau;\mathbf{z}) := \sum_{\ell\in 2M\mathbb{Z}^{|V|}\pm a} q^{-\ell^T M^{-1}\ell}\, \mathbf{z}^{\ell}.$$

↓

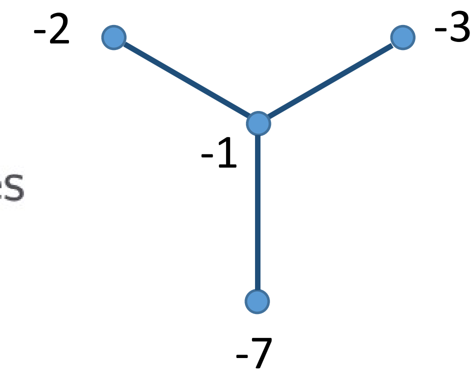
$M_3 \leftrightarrow -M_3 \Leftrightarrow q \leftrightarrow q^{-1} \Leftrightarrow$ flipping the lattice signature $M \leftrightarrow -M$

no longer convergent for $|q| < 1$!

The definition for $\widehat{Z}_a(\tau)$ no longer applies after $M_3 \to -M_3$.

A Small Miracle

$$\widetilde{shad(F_0)}(\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1 \ (42)}} \left(\frac{i}{21}\right) \widetilde{\theta^1_{42,i}}(\tau) = q^{-\frac{83}{168}} \hat{Z}_0(\Sigma(2,3,7),\tau)$$



It admits an expression as q-hypergeometric series

$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^{n+k})}$$

A Small Miracle

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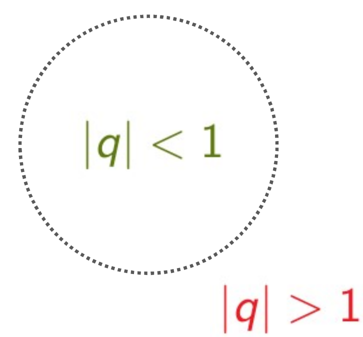
$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^{n+k})}$$

which moreover converges both inside and outside (but not on) the unit circle:

$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{q^{-n^2}}{\prod_{k=1}^{n}(1-q^{-(n+k)})}$$



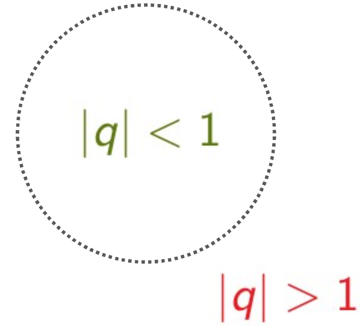
A Small Miracle

Recall : Ramanujan's Mock θ Functions

$$F_0(\tau) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^{n+k})} = 1 + q + q^3 + q^4 + O(q^5)$$

$$shad(F_0)(\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1\ (42)}} \left(\frac{i}{21}\right) \theta^1_{42,i}(\tau)$$

A Small Miracle



$$q^{-\frac{83}{168}} \hat{Z}_0(\Sigma(2,3,7),\tau) = \sum_{\substack{i \in \mathbb{Z}/42 \\ i^2 \equiv 1 \ (42)}} \left(\frac{i}{21}\right) \widetilde{\theta^1_{42,i}}(\tau)$$

$$= q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^{n+k})} = q^{\frac{1}{168}} \sum_{n=0}^{\infty} \frac{q^{-n^2}}{\prod_{k=1}^{n}(1-q^{-(n+k)})}$$

$$= F_0(-\tau)$$

cf. Ramanujan's mock theta function

$$F_0(\tau) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^{n+k})} = 1 + q + q^3 + q^4 + O(q^5)$$

A Small Miracle

The q-hypergeometric series defines a function $F : \mathbb{H} \cup \mathbb{H}^- \to \mathbb{C}$, satisfying

$$F(\tau) = \begin{cases} \widetilde{shad(F_0)}(\tau) & \text{when } \tau \in \mathbb{H} \\ F_0(-\tau) & \text{when } \tau \in \mathbb{H}^-. \end{cases}$$

Moreover, it gives the same asymptotic expansion as $\tau \to \pm it$
$\Rightarrow$ they lead to the same *quantum modular form*.

Conjecture:

$$\hat{Z}_0(-\Sigma(2,3,7),\tau) = \hat{Z}_0(\Sigma(2,3,7),-\tau)$$

$$= q^{-\frac{1}{2}} F_0(\tau) = q^{-\frac{1}{2}}(1 + q + q^3 + q^4 + O(q^5))$$

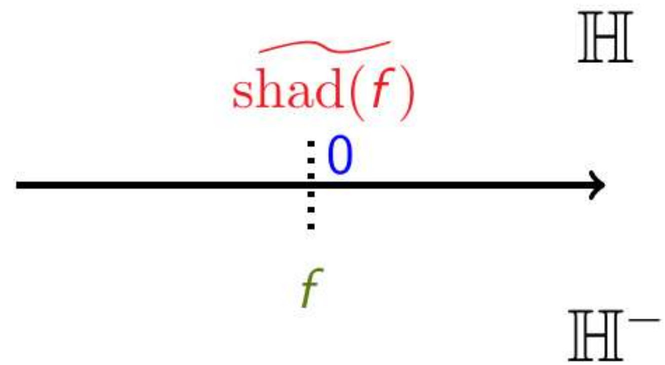
A Mock-False Conjecture

Theorem :* [MC–Duncan '13, Rhoads '18] A Rademacher sum (a regularised sum over $SL_2(\mathbb{Z})$ images) defines a function F in $\mathbb{H}$ and $\mathbb{H}^-$, satisfying

false

$$F(\tau) = \begin{cases} \widetilde{\mathrm{shad}(f)}(\tau) & \text{when } \tau \in \mathbb{H} \\ f(-\tau) & \text{when } \tau \in \mathbb{H}^-. \end{cases}$$

mock



∗ at weight 1/2.

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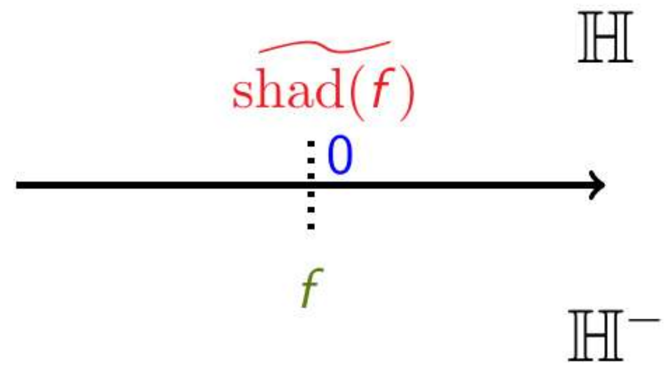
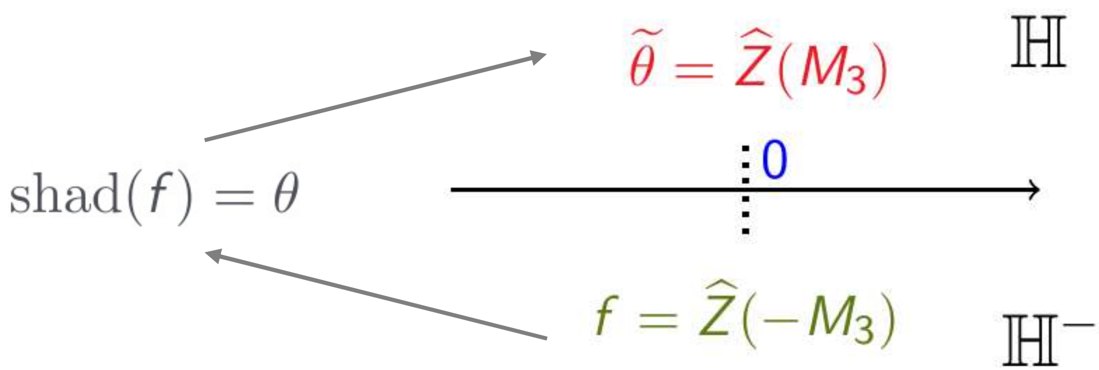
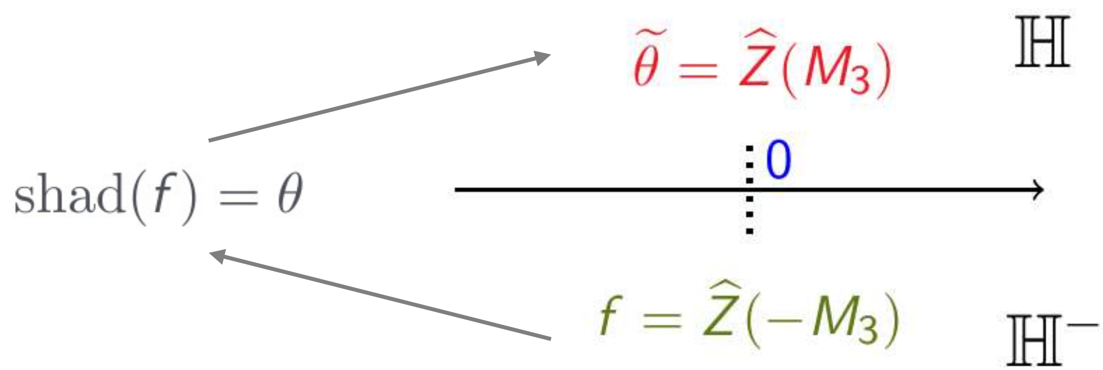
mock





$*$ at weight 1/2.

A Mock-False Conjecture



The False–Mock Conjecture: [CCFGH'18]
If $q^{-c}\widehat{Z}_a(M_3;\tau) = \widetilde{\theta}(\tau)$ for some $c \in \mathbb{Q}$ is a false theta function, then

$$\boxed{q^c\widehat{Z}_a(-M_3;\tau) = f(\tau)}$$

is a mock theta function with $\mathrm{shad}(f) = \theta$.

$*$ at weight 1/2.

False–Mock Conjecture: A Test Case

Conjecture:

$$\hat{Z}_0(-\Sigma(2,3,7),\tau) = \hat{Z}_0(\Sigma(2,3,7),-\tau)$$

$$= q^{-\frac{1}{2}} F_0(\tau) = q^{-\frac{1}{2}}(1+q+q^3+q^4+O(q^5))$$

Independent verification: [Gukov-Manolescu '19]

Using $-\Sigma(2,3,7) = S^3_{-1}(\text{figure } 8)$ and the surgery formula, one obtains

$$\widehat{Z}_0(-\Sigma(2,3,7),\tau) = q^{-\frac{1}{2}}(1+q+q^3+q^4+q^5+2q^7+\dots)$$

Nice! But is there a way to obtain the mock answer from a more direct definition?

Defining $\widehat{Z}_a(-M_3)$

$$\widehat{Z}_a(M_{3,\Gamma};\tau) := (\pm)\, q^{\Delta} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta^M_a(\tau;\mathbf{z})$$

$$\Theta^M_a(\tau;\mathbf{z}) := \sum_{\ell\in 2M\mathbb{Z}^{|V|}\pm a} q^{-\ell^T M^{-1}\ell}\, \mathbf{z}^{\ell}.$$

↓

$M_3 \leftrightarrow -M_3 \Leftrightarrow q \leftrightarrow q^{-1} \Leftrightarrow$ flipping the lattice signature $M \leftrightarrow -M$

no longer convergent for $|q| < 1$!

Regularised θ-function: [Zwegers '02]

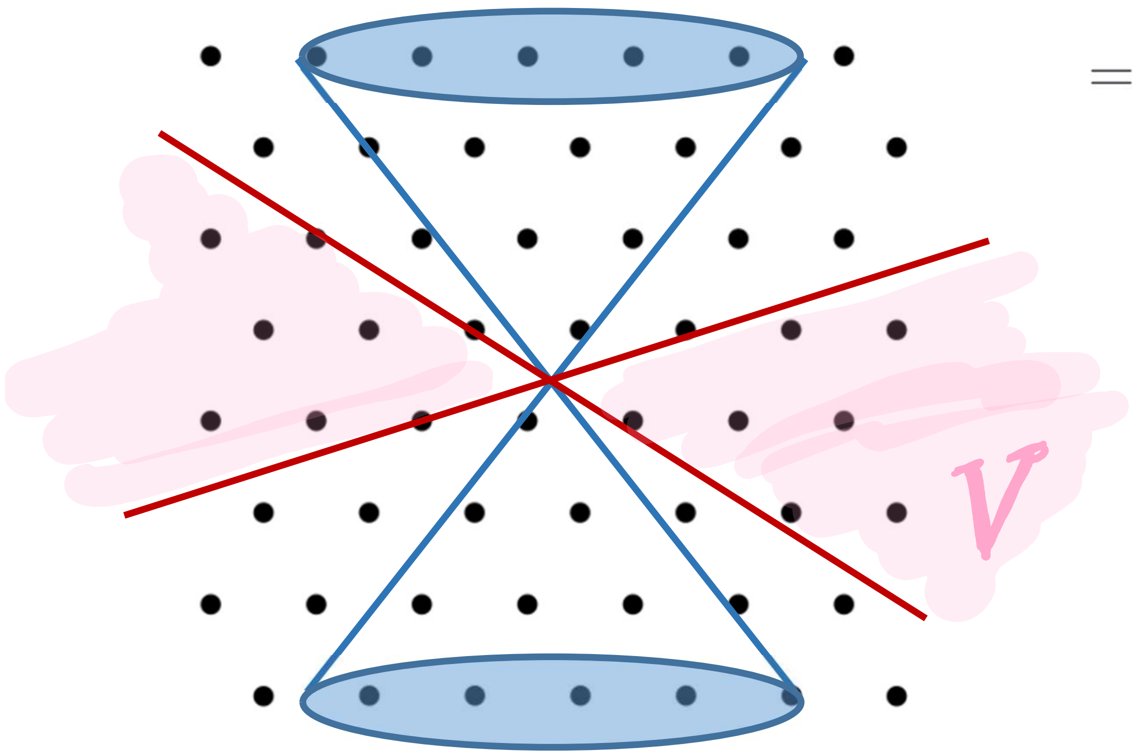
$$\Theta_a^{-M,\mathrm{reg}}(\tau;\mathbf{z}) := \sum_{\ell\in a+2M\mathbb{Z}^{|V|}} \rho(\ell)\, q^{+(\ell,M^{-1}\ell)}\mathbf{z}^{\ell}$$

Indefinite Theta Functions

Regularised θ-function: [Zwegers '02]

$$\Theta_a^{-M,\mathrm{reg}}(\tau;\mathbf{z}) := \sum_{\ell \in 2M\mathbb{Z}^{|V|} \pm a} \rho(\ell)\, q^{(\ell, M^{-1}\ell)} \mathbf{z}^{\ell}$$

$$= \sum_{\substack{\ell \in 2M\mathbb{Z}^{|V|} \pm a \\ \ell \in V}} q^{(\ell, M^{-1}\ell)} \mathbf{z}^{\ell}$$



Defining $\widehat{Z}_a(-M_3)$

Regularised θ-function:

$$\Theta_a^{-M,\mathrm{reg}}(\tau;\mathbf{z}) := \sum_{\ell\in a+2M\mathbb{Z}^{|V|}} \rho(\ell)\, q^{+(\ell,M^{-1}\ell)}\mathbf{z}^{\ell}$$

$$\widehat{Z}_a(-M_{3,\Gamma};q) := (\pm)\, q^{\Delta} \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^{-M,\mathrm{reg}}(\tau;\mathbf{z})$$

[MC-Sgroi, to appear]

[MC-Ferrari-Sgroi '19]

Using the above definition:

$$\widehat{Z}_0(-\Sigma(2,3,7),\tau) = q^{-\frac{1}{2}} F_0(\tau) = q^{-\frac{1}{2}}(1+q+q^3+q^4+O(q^5))$$

What we have seen:

- Explicit examples of QMF play the role of 3-manifold inv.;
- Modularity considerations lead to new examples of q-series inv. ;
- What is the physical meaning of the regularisation?

Questions?

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The $(1, m)$ Algebra for Lie Algebra $\mathfrak{g}$

Given a positive integer m, let $\alpha_\pm = \pm\sqrt{2m^{\pm 1}}$, $\alpha_0 = \alpha_+ + \alpha_-$

free boson : $\varphi(z)\varphi(w) \sim \log(z - w)$

stress energy tensor : $T = \frac{1}{2}(\partial\varphi)^2 + \frac{\alpha_0}{2}\partial^2\varphi$, $c = 1 - 3\alpha_0^2$

screening charges : $Q_- = (e^{\alpha_-\varphi})_0$

triplet $(1, m)$ algebra: $\mathcal{W}(m) := \ker_{\mathcal{V}_L} Q_-$

singlet $(1, m)$ algebra: $\mathcal{M}(m) := \ker_H Q_-$

where $\mathcal{V}_L$ = lattice VOA for $L = \sqrt{2m}\mathbb{Z}$, H = Heisenberg algebra.

→ corresponding to $\mathfrak{g} = A_1$

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where $\mathcal{V}_L =$ lattice VOA for $L = \sqrt{2m}\mathbb{Z}$, $H =$ Heisenberg algebra.

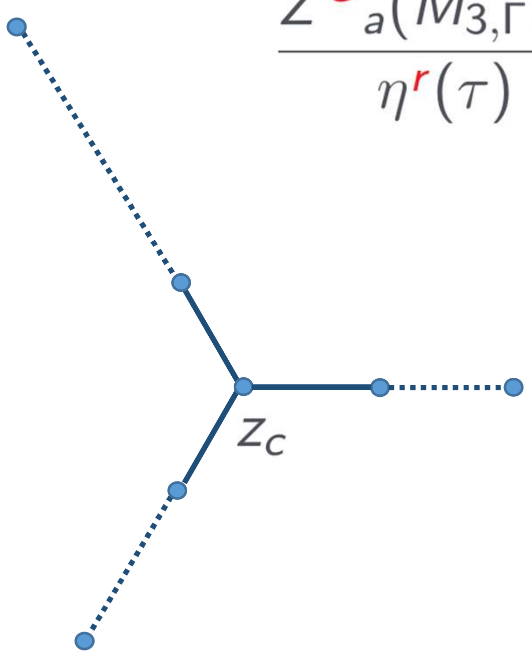
→ corresponding to $\mathfrak{g} = A_1$

More generally, we have

$r =$rank$(\mathfrak{g})$ bosons, and $L = \sqrt{m}\,\Lambda_{\text{root}}$.

$\widehat{Z}_a^G(\tau)$ and $\mathfrak{g}$-Log VOA Characters

From the M-theory origin of $\widehat{Z}_a$, it is clear that there is a higher rank generalisation $\widehat{Z}_a^G(\tau)$.



Integrate over all but the central node $\vec{z}_c$

$$\frac{\widehat{Z}^{G}{}_a(M_{3,\Gamma};\tau)}{\eta^r(\tau)} = [(\vec{z}_c)^0]\Big(\text{triplet } \mathfrak{g}\text{-Log VOA characters}\Big)$$

$$= \text{singlet } \mathfrak{g}\text{-Log VOA characters}$$

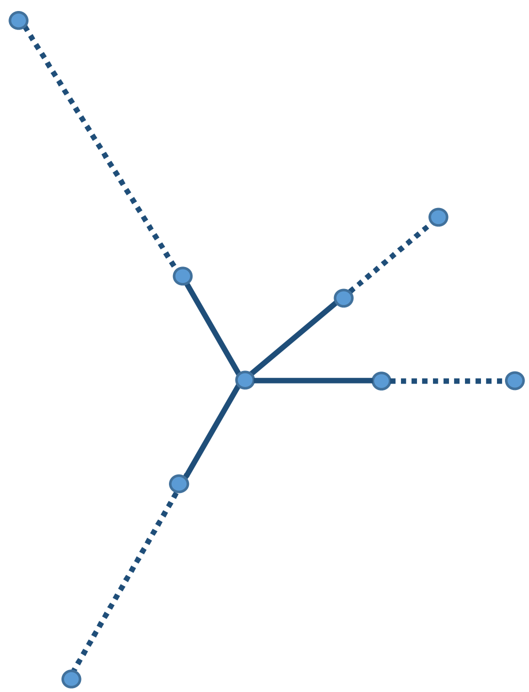
[MC-Chun-Feigin-Ferrari-Gukov-Harrison, t.a.]

Another generalisation: (p, p') Log VOA

When $p \neq 1$, the corresponding minimal model is non-trivial.

(p, p') min. model $\sim$ the cohomology of screening op.
(p, p') log model $\sim$ the kernel of screening op.

They correspond to 4-pronged stars in the $\widehat{Z}_a$–VOA correspondence.



[MC-Chun-Feigin-Ferrari-Gukov-Harrison, t.a.]

More General Quantum Modularity

Def (Depth 1 QMF): $f : \mathbb{Q} \to \mathbb{C}$ s.t. $h_\gamma := f - f|_w\gamma$ have some properties of analyticity $\forall \gamma \in G$.

Def (Depth N QMF): a function $f \in \mathbb{Q}$ such that $h_\gamma := f - f|_w\gamma$ is a sum of QMFs of depth less than N (multiplied by some real-analytic functions) $\forall \gamma \in G$.

- $\widehat{Z}_a^{A_2}(\tau)$ is a QMF of *depth 2* when M_3 is given by a 3-pronged star.
- $\widehat{Z}_a(\tau)$ is a sum of QMFs of *different weights* when M_3 is given by a 4-pronged star.

[MC-Chun-Feigin-Ferrari-Gukov-Harrison, t.a.] and see earlier work by Bringmann, Milas, Kaszian ('17-'18).

Questions?

Future Questions

just the beginning ...

• a mathematical definition for more families of 3-manifolds;

• boudary algebra of $\mathcal{T}[M_3]$;

• mock and false are exceptionally simple, more involved quantum modularity for general M_3;

• what does quantum modularity say about physics/topology?

$$\vdots$$