Clusters, quantum groups, higher Teichmüller theory.

(joint with Gus Schrader)

Higher Teichmüller theory.

(after Fock – Goncharov)

G – Lie gp / $\mathbb{C}$

S – 2d surface

Γ_S – mapping class gp of S

Idea: $(G, S) \rightsquigarrow Ch_{G,S} = \{\varphi : \pi_1(S) \to G\} / G$

character ~~variety~~ stack

$\mathcal{X}_{G,S}$ – "character variety",

has Atiyah–Bott Poisson structure

$\mathbb{L}^q_{G,S} := \mathrm{Fun}_q(\mathcal{X}_{G,S})$

Fock – Goncharov:

a) $\exists$ a canonical Hilb. sp. rep-n $V_{G,S}$ of $\mathbb{L}^q_{G,S}$

b) $V_{G,S}$ is a Γ_S – equiv. rep-n of $\mathbb{L}^q_{G,S}$

Conj: Modular functor conj

Assignment $(G,S) \rightsquigarrow \Gamma_S \circlearrowright (\mathbb{L}^q_{G,S}, V_{G,S})$ respects cutting / gluing of surfaces

Holds for $G = PGL_m$ (Schrader – S. in progress)

Remembering only $\Gamma_S \circlearrowright V_{G,S}$ we get a family of modular function labelled by G.

Loosely: $\mathbb{L}^q_{G,S} \hookrightarrow \mathcal{B}(V_{G,S})$

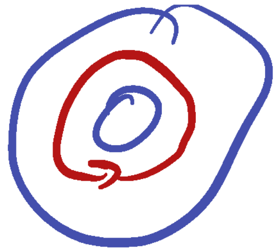
Bounded op-s

P_S lies in a completion of the image of $\mathbb{L}^q_{G,S}$

Bottom line: focus on $\mathbb{L}^q_{G,S} \circlearrowright V_{G,S}$.

Quantum group.

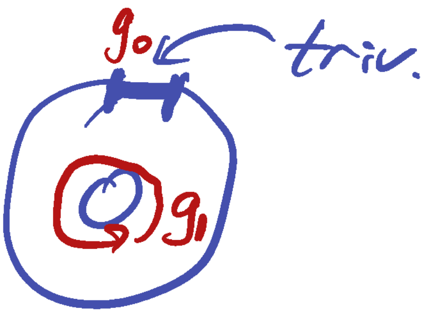
$Ch_{G,S}$ is not a variety because G does not act freely on $\{\varphi: \pi_1(S) \to G\}$

<u>Example:</u> $S =$ 

$$\Rightarrow \{\varphi : \pi_1(S) \to G\} \simeq G$$

$G \circlearrowleft G$ by conjugation

Need to trivialize G-action on a region (or several) on S

<u>Example:</u> $S =$ 

$$\mathcal{X}_{G,S} = \{(g_0, g_1) \in G \times G\} / G$$

$$h \circ (g_0, g_1) = (h g_0, h g_1 h^{-1})$$

$$\mathcal{X}_{G,S} \simeq G$$

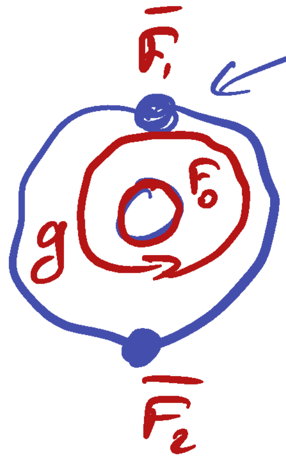
And Atiyah-Bott Poisson str. quantizes to $U_q(\mathfrak{g})$ *

Fock-Goncharov:

instead of trivialization, consider reduction to Borel

marked pts

Examples



ordering of eigenvalues of g

$$\left\{ \left(g, \underset{G/B}{\underset{\cap}{F_0}}, \underset{G/N}{\underset{\cap}{\overline{F}_1}}, \overline{F}_2 \right) \,\middle|\, gF_0 = F_0 \right\} \Big/ G$$

if $\overline{F}_1$ & $\overline{F}_2$ are in generic position, then G acts freely

$\Rightarrow$ a) S has marked pts & punctures

b) $\mathcal{X}_{G,S} = \{ \varphi : \pi_1(S) \to G \}^{\circ} / G$

$D(U_q(\mathfrak{b}))$

$$\mathbb{L}^q_{G,S} \simeq \cancel{U_q(\mathfrak{g})} \otimes U_q(\check{\mathfrak{h}})$$

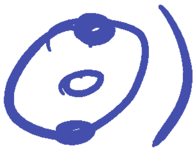
$$Z \simeq U_q(\mathfrak{h})^W$$

<u>Thm:</u> (Schrader–S., Ip) $\exists$ an embedding

$$i: D(U_q(\check{\mathfrak{b}})) \hookrightarrow \mathbb{L}^q_{G,}$$ 

Moreover, pull-back of $V_{G,S}$ of $\mathbb{L}^q_{G,S}$ onto $U_q(\mathfrak{g})$ is the so-called "positive" rep-n P_λ of $U_q(\mathfrak{g})$.

central character coming from eigenvalues of monodromy around the puncture

$(S =$  $)$

Before, (Frenkel–Ip), positive repn-s appeared as q-deform of $SL_n(\mathbb{R})$ principal series.

<u>Positive rep-s of $U_q(\mathfrak{sl}_n)$:</u>

$$i: U_q(\mathfrak{g}) \hookrightarrow \mathcal{L}^q_{G,\odot}$$

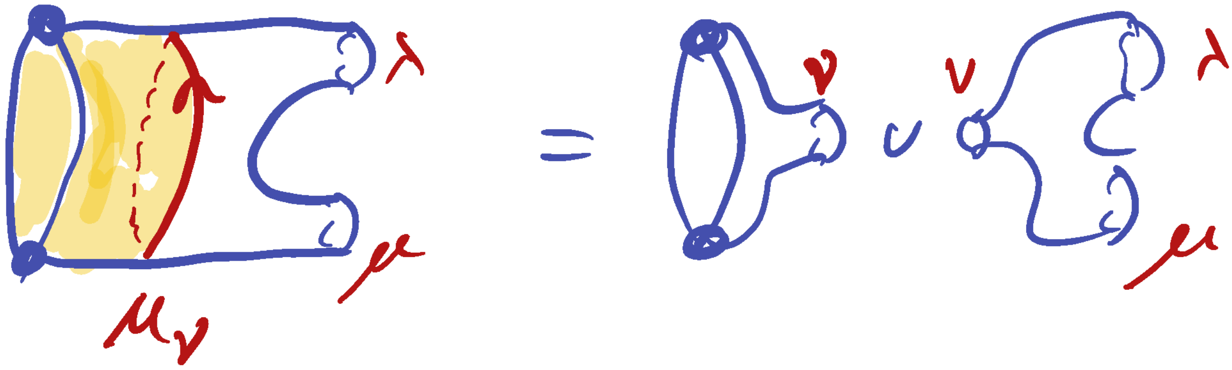
Moreover,

$$\begin{tikzcd} U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \arrow[r, hook] & \mathcal{L}^q_{G,\odot} \otimes \mathcal{L}^q_{G,\odot} \\ U_q(\mathfrak{g}) \arrow[u, hook, "\Delta"] \arrow[r, hook] & \mathcal{L}^q_{G,\odot} \arrow[u] \end{tikzcd}$$

<u>Thm</u> $$P_\lambda \otimes P_\mu \simeq \int^{\oplus} P_\nu \otimes M^\nu_{\lambda,\mu}\, m(\nu)\,d\nu$$

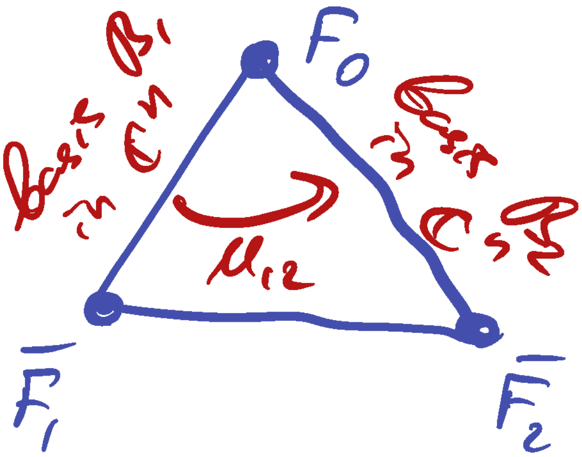
$\uparrow$ mult. space

$\uparrow$ Sklyanin measure



$(G = GL_n)$

Monodromies



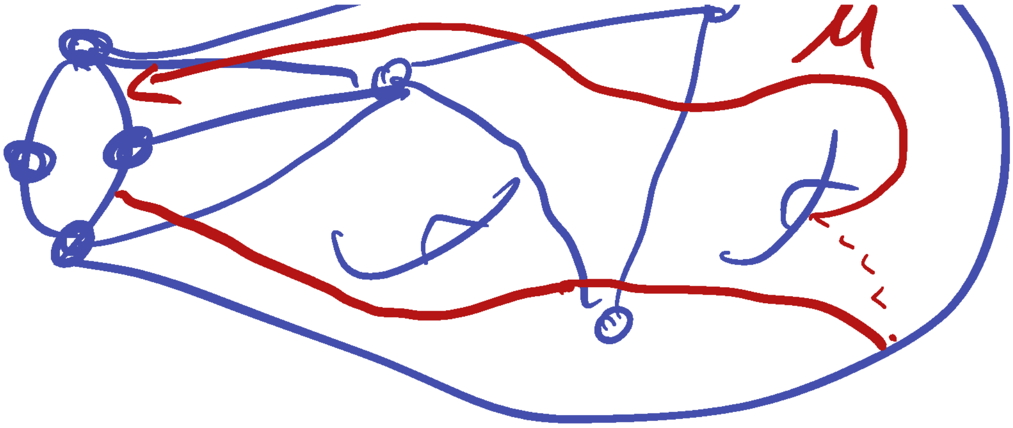
$\bar{F}_0, \bar{F}_1, \bar{F}_2 \in G/N$

$\pi: G/N \to G/B$

$F_0 := \pi(\bar{F}_0)$

$$\mu_{12} \in \underset{\substack{\cap \\ GL_n(\mathbb{C})}}{B} \otimes \mathbb{L}^q_{G,\Delta}$$

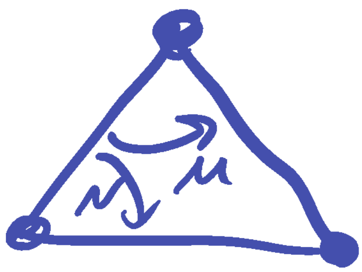
Some elt-s of $\mathcal{L}^q_{G,S}$ can be described as



entries of open monodromies & traces of closed monodromies are elt-s of $\mathcal{U}^q_{G,S}$.

1) Decomp $P_\lambda \otimes P_\mu = \int^{\oplus} P_\nu \otimes \ldots$

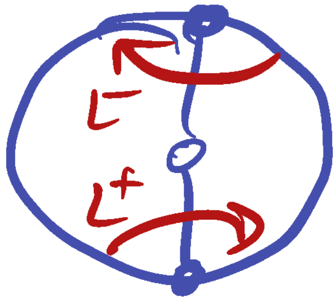
$\Longleftrightarrow$ diagonalizing traces of M_ν

2) 

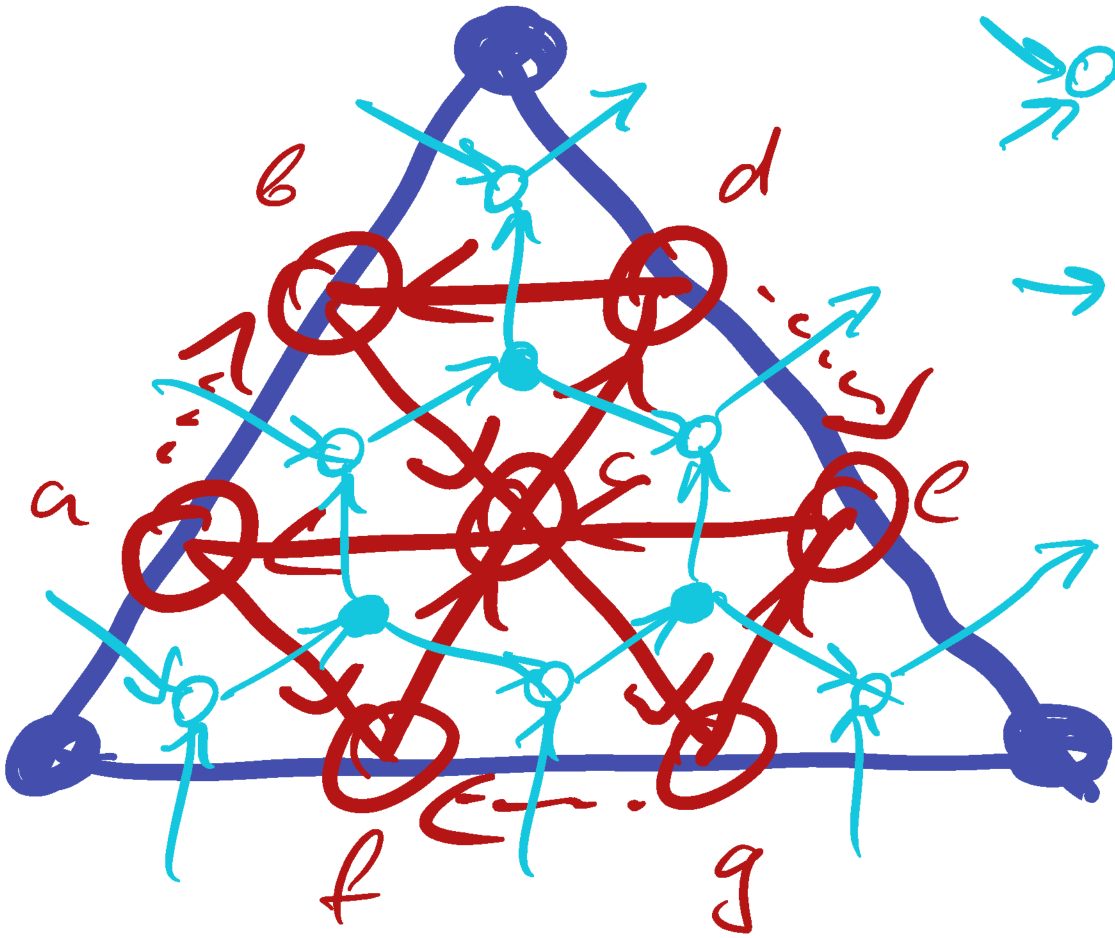
$$R M_1 M_2 = M_2 M_1 R$$
$$R N_1 N_2 = N_2 N_1 R$$
$$M_1 N_2 = N_2 M_1 R$$

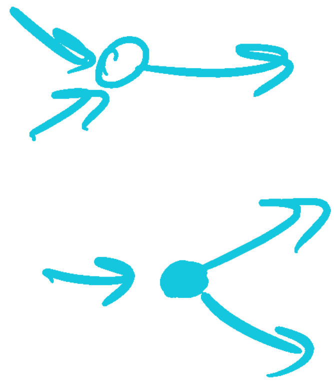
R — universal R-matrix eval. at fundam. rep-n

$\mu_1 = \mu \otimes id_{\mathbb{C}^n} \qquad \mu_2 = id_{\mathbb{C}^n} \otimes \mu$



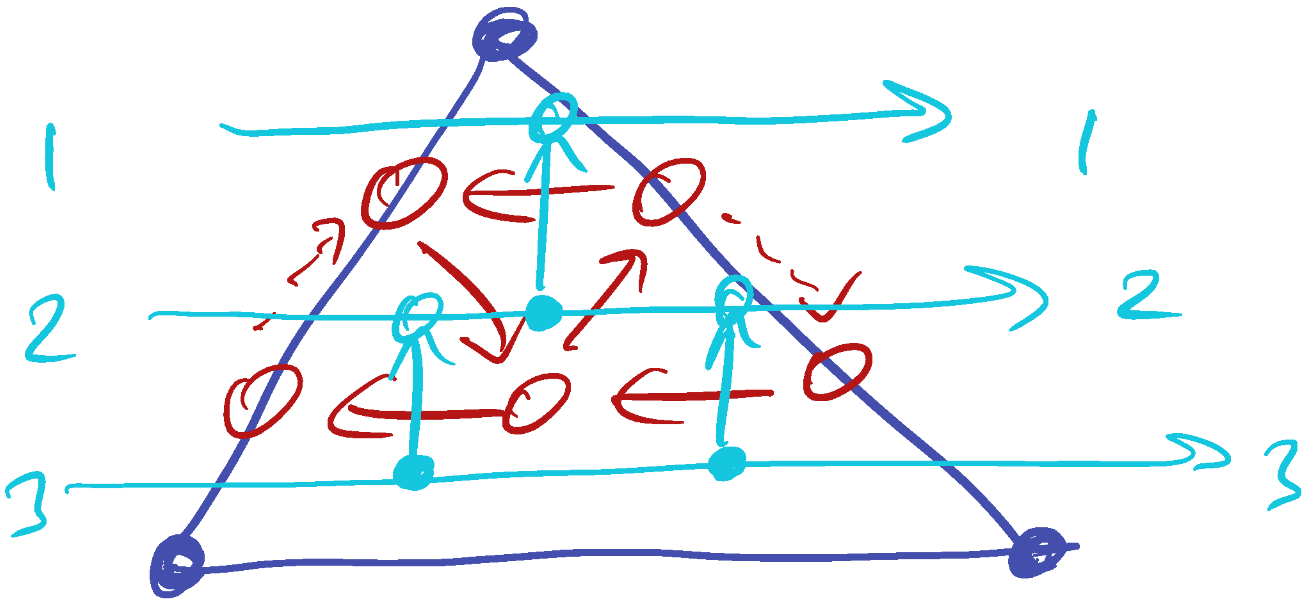
Cluster structure (SL_3)

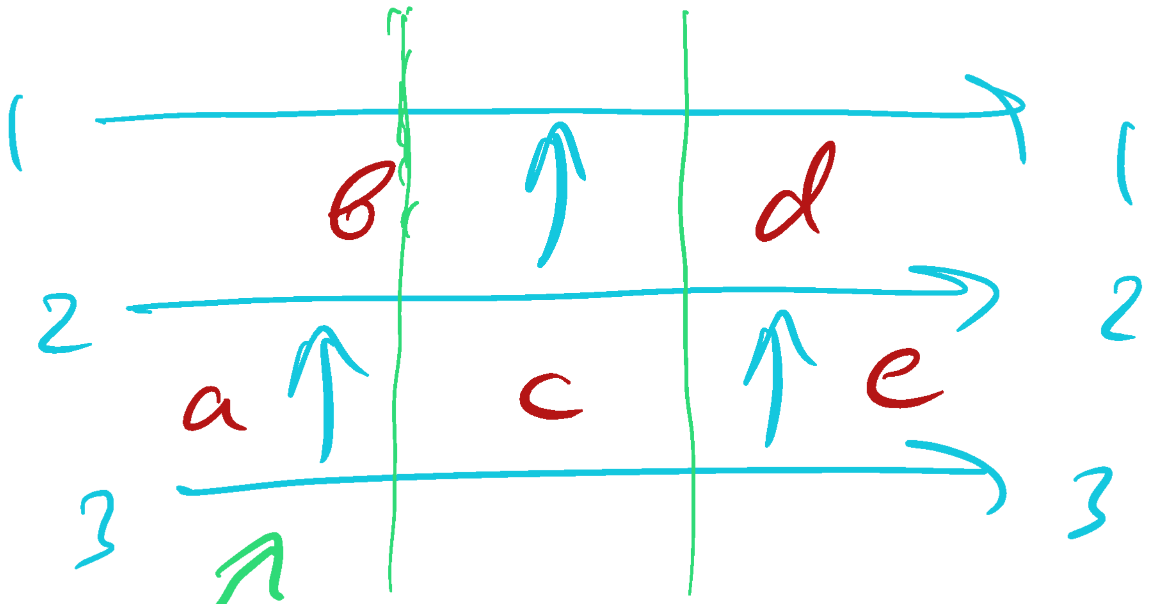




$a \dashrightarrow b \qquad \{a, b\} = \frac{1}{2} ab \qquad X_a X_b = q X_b X_a$

$c \longrightarrow a \qquad \{c, a\} = ca \qquad X_c X_a = q^2 X_a X_c$





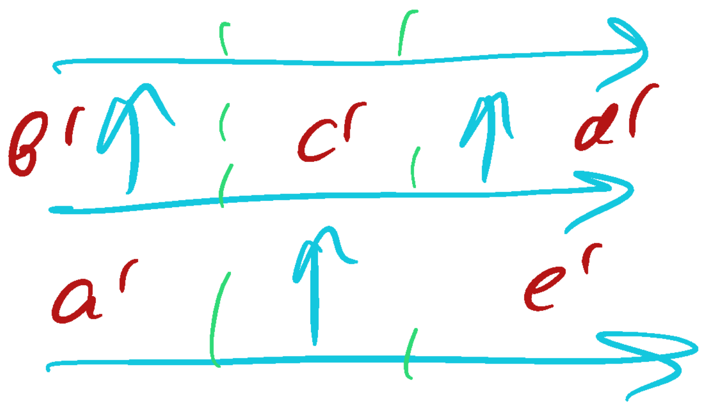
$$w_0 = s_1 s_2 s_1$$

$$M = (M_{ij})_{i,j=1}^{3} \qquad M_{ij} = wt(i \to j)$$

$$M = \begin{pmatrix} 1 & e + ce & cde \\ 0 & ace & acde \\ 0 & 0 & abcde \end{pmatrix}$$

$$\begin{pmatrix} a & & \\ & a & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} b & & \\ & 1 & \\ & & 1 \end{pmatrix} \in PGL_n$$

$w_0 = s_2 s_1 s_2$



$\mu = \mu'$

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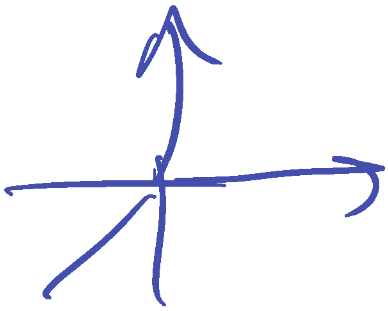
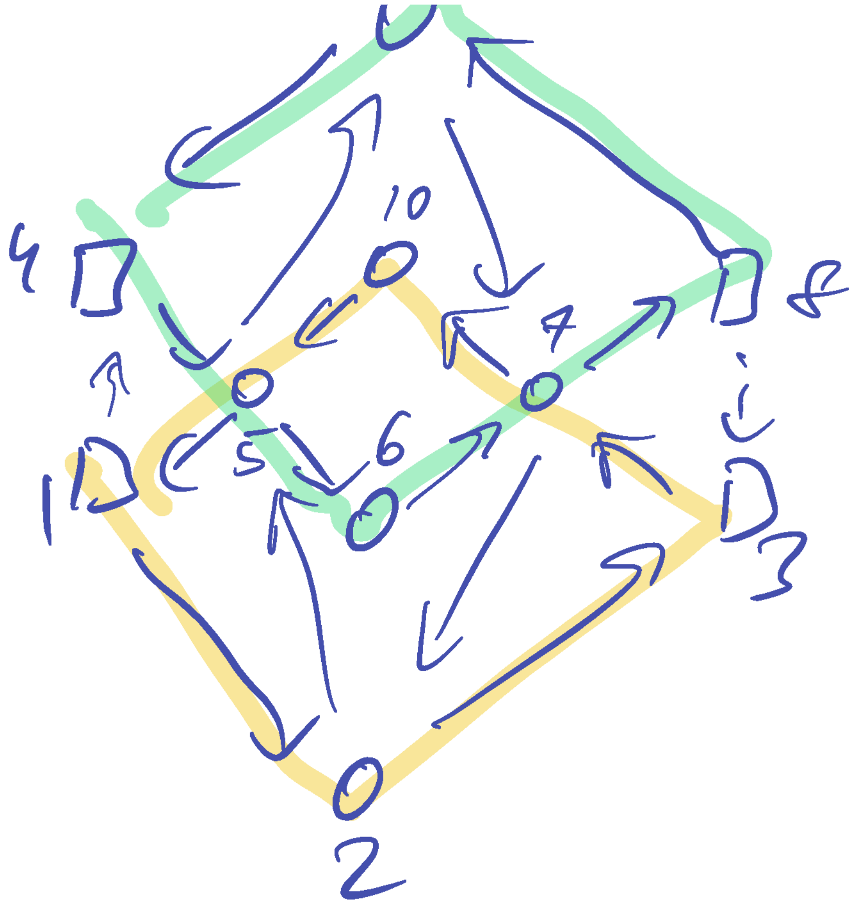
1) Wall crossing



2) Coulomb branches of 4d $\mathcal{N}=2$ gauge theories

$\underline{sl_{3}:}$

1)



$$E_1 \mapsto x_1(1+qx_2) \qquad k_1 = q^2 x_1 x_2 x_3$$

$$E_2 \mapsto x_4(1+qx_5)\cdots(1+qx_7) \quad k_2 = q^4 x_4 \cdots x_8$$

$$F_i \quad k' \qquad \underline{kk'=1}$$

$\forall$ Chevalley gen. $\exists$ a cluster chart in which this gen. is a cluster monomial

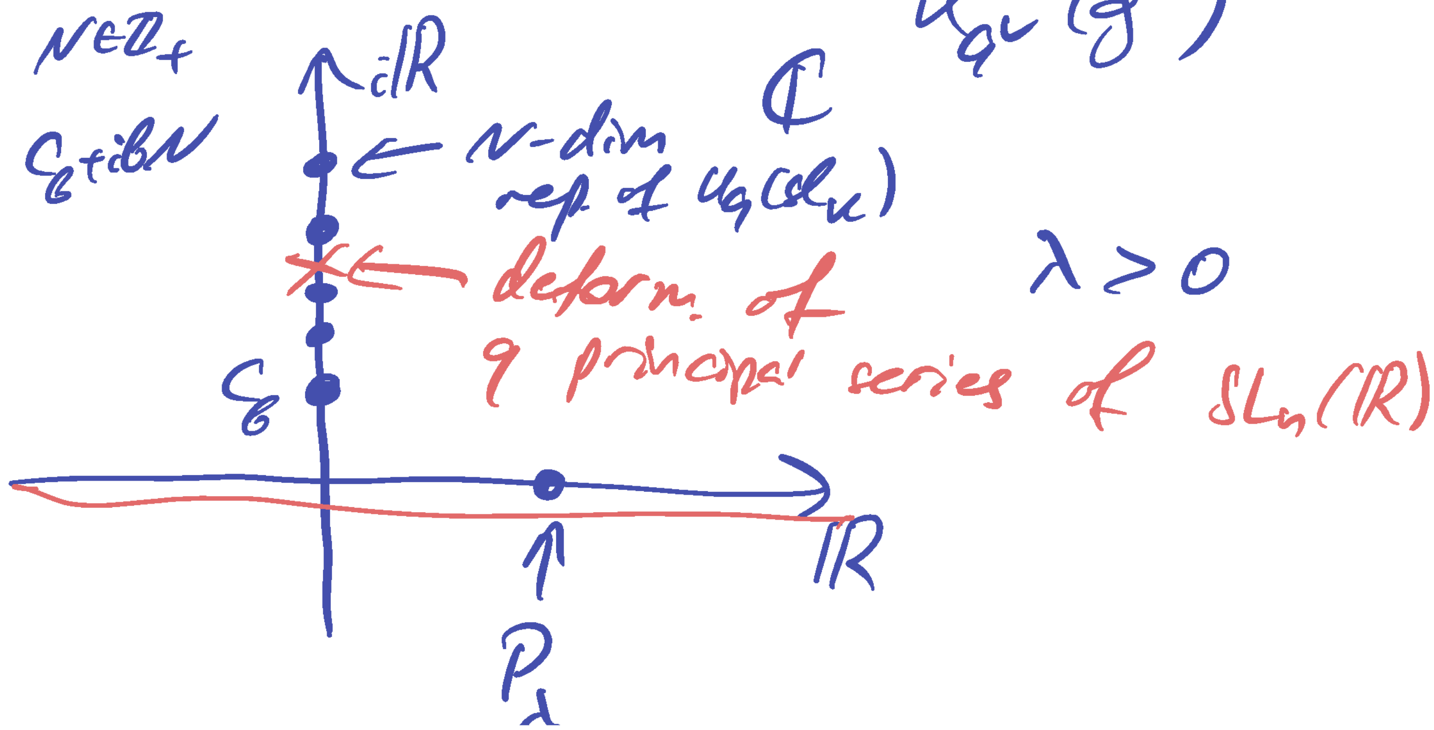
$U_q(sl_n)$ — q. Coulomb branch

of $\overset{1}{\circ} \to \overset{2}{\circ} \to \dots \to \overset{n-1}{\circ} \to \boxed{n}$

P_λ they are $b^{\pm 1}$ deforms

$$q = e^{\pi i b^2} \quad q^\vee = e^{\pi i b^{-2}}$$

P_λ — bimod for $U_q(\mathfrak{g})$ & $U_{q^\vee}(\mathfrak{g}^\vee)$



$$C_6 = i\frac{6+6^{-1}}{2}$$