Homotopy Quantum Field Theories

Alexis Virelizier
(University of Lille)

Topological Quantum Field Theory Seminar
Técnico Lisboa - September 11, 2020

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Joint work with ***Vladimir Turaev***

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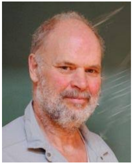

Dedicated to the memory of Vaughan Jones

Homotopy quantum field theories (HQFTs)

Idea: TQFTs for manifolds endowed with maps to a fixed target topological space X (with base point $*$)

The category $X\text{-}\mathrm{Cob}_n$ is a symmetric monoidal category

- an object is a pair (Σ, f)
  - Σ closed oriented pointed $(n-1)$-manifold
  - $f\colon (\Sigma, \Sigma_\bullet) \to (X, *)$ pointed map
- a morphism $f\colon (\Sigma_1, f_1) \to (\Sigma_2, f_2)$ is equiv. class of (M, f)
  - M an oriented n-cobordism $\Sigma_1 \to \Sigma_2$
  - h an homotopy class $M \to X$ with $h_{|\Sigma_i} = f_i$

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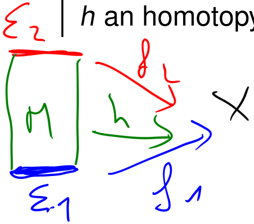

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A *n-dim HQFT with target X* is a symmetric monoidal functor

$$\tau\colon X\text{-}\mathrm{Cob}_n \to \mathrm{Vect}_{\Bbbk}$$

Data:

- $\Bbbk$-vector spaces $\tau\Big($

$\xrightarrow{f} X\Big)$
- $\Bbbk$-linear maps $\tau\left(\right.$

$\left.\xrightarrow{h} X\right) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
- isomorphisms $\tau\big((\Sigma, f) \amalg (\Sigma', f')\big) \simeq \tau(\Sigma, f) \otimes_{\Bbbk} \tau(\Sigma', f')$
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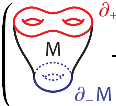
- $\Bbbk$-linear maps $\tau\Bigg($  $\overset{h}{\longrightarrow} X\Bigg) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
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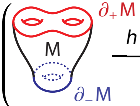
- $\mathbb{k}$-linear maps $\tau\Big($  $\xrightarrow{h} X\Big) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
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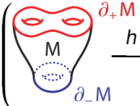
- $\mathbb{k}$-linear maps $\tau\Bigg($ 

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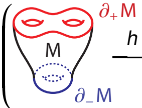
- $\mathbb{k}$-linear maps $\tau\left(\right.$  $\left.\xrightarrow{h} X\right) : \tau(\partial_- M, h_-) \to \tau(\partial_+ M, h_+)$
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- $X = \{\mathrm{pt}\} \rightsquigarrow$ TQFT
- M closed oriented n-manifold, $h \in [M, X]$
  $\tau(M, h) \in \mathrm{End}_{\Bbbk}(\tau(\emptyset)) \simeq \Bbbk$ is a numerical invariant of h
- $\tau(\Sigma, f)$ is finite-dimensional and $\tau(\Sigma, f)^* \simeq \tau(-\Sigma, f)$
- τ induces finite-dimensional representation of
  $$\mathrm{MCG}(\Sigma, f) = \left\{\phi\colon \Sigma \to \Sigma \text{ o.p. diffeo } \middle| f\phi = f\right\}_{/\text{isotopy}}$$
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HQFTs of dimension 1

There are bijective correspondences between:

1. 1-dimensional HQFTs with target X
2. finite-dimensional representations of $\pi_1(X)$
3. finite-dimensional flat vector bundles over X

Rk: HQFTs may be seen as higher-dimensional generalizations of finite-dimensional flat vector bundles

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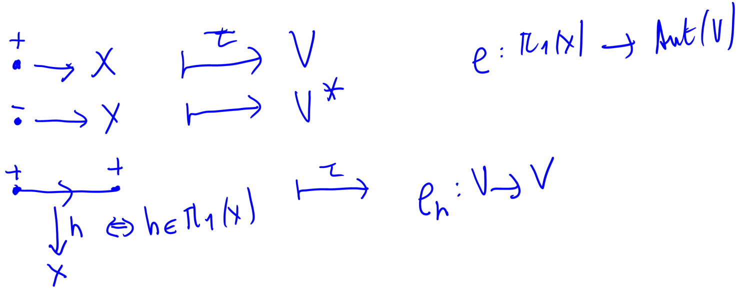

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Cohomological HQFTs

$\theta \in H^n(X, \Bbbk^*) \quad \rightsquigarrow \quad n$-dim HQFT τ^θ with target X

τ^θ is characterized by :

- M closed oriented n-manifold, $h \in [M, X]$

  $$\tau^\theta(M, h) = \langle h^*(\theta), [M] \rangle \in \Bbbk$$

  where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M

- Σ closed oriented $(n-1)$-manifold, $f \colon \Sigma \to X$

  $\tau^\theta(\Sigma, f)$ is one-dimensional

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The case of aspherical targets

From now, assume that X is aspherical (i.e., $\pi_i(X) = 0$ for $i \geq 2$)

$\rightsquigarrow$ X is a $K(G, 1)$-space with $G = \pi_1(X)$

(Turaev, 2000)

2-dim HQFTs with target X $\Leftrightarrow$ G-graded Frobenius algebras

(Sozer, 2019)

Classification of 2-dim extended HQFTs with target X

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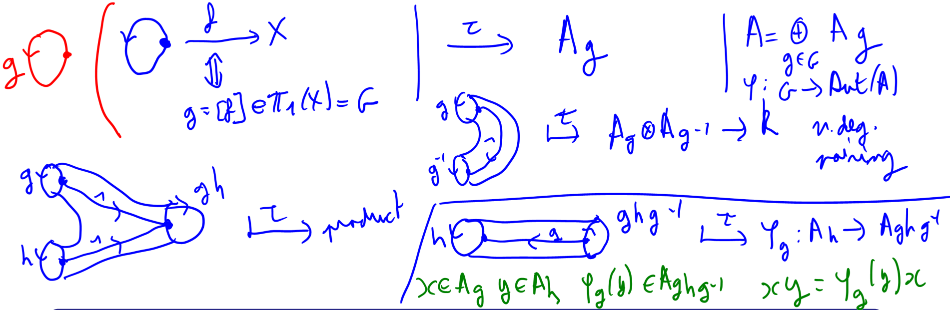

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Classification of 2-dim extended HQFTs with target X

3-dimensional TQFTs

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim TQFT

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triangulation  + $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_{\mathcal{C}}$

- **Reshetikhin-Turaev (91)**

surgery + $\mathcal{B}$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{B}}$

- **Müger (03):** $\mathcal{Z}(\mathcal{C})$ modular fusion category $\rightsquigarrow$ $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$

Theorem (Turaev-V. & Balsam-Kirillov, 2010)

$\mathrm{TV}_{\mathcal{C}}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ are isomorphic TQFTs

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$\mathcal{C}$= spherical fusion G-graded category:

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$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\Bigg($  $\Bigg)$

Example: $G\text{-}\mathrm{vect}_{\Bbbk}^{\theta}$ with $\theta \in H^3(G, \Bbbk^*)$

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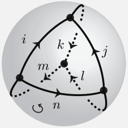
$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

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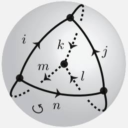
$\rightsquigarrow$ 6j-symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F_{\mathcal{C}}\left(\right.$  $\left. \right)$

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State sum HQFT with target $X = K(G,1)$

$C = \bigoplus_{g \in G} C_g$ spherical fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Pick a triangulation of M and $g \in h$ with $g(\text{vertices}) = * \in X$

e oriented edge $\rightsquigarrow$ $g(e)$ loop in X $\rightsquigarrow$ $[g(e)] \in \pi_1(X, *) = G$

c= **G-coloring** of the edges : $c_e \in C_{[g(e)]}$ simple object

$$\mathrm{HTV}_C(M,h) = \sum_c \quad \mathrm{coef}(c) \quad \mathrm{ctr}_f(\otimes_\Delta \; |\Delta|) \in \Bbbk$$

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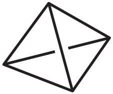

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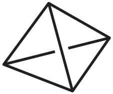 $\xrightarrow{g} X$

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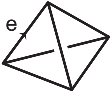

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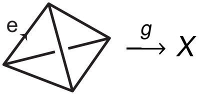

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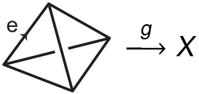

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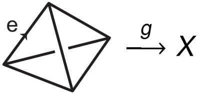

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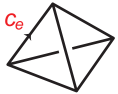

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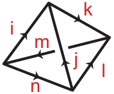

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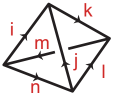

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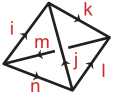

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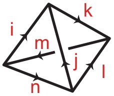

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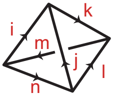

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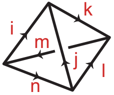

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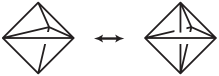
Pachner moves



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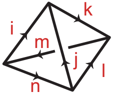

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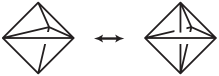
Pachner moves 

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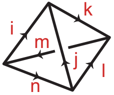

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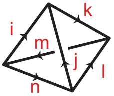

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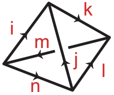

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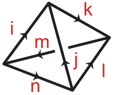

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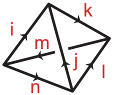

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HQFT decomposition of Turaev-Viro TQFT

Let $\mathcal{C}$ spherical fusion category $\rightsquigarrow$ $\mathrm{TV}_\mathcal{C}$

Γ=**graduator** of $\mathcal{C}$ (= largest group making $\mathcal{C}$ faithfully graded)

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$$\mathrm{TV}_\mathcal{C}(\Sigma) = \bigoplus_{f\in[\Sigma,B\Gamma]} \mathrm{HTV}_\mathcal{C}(\Sigma,f) \quad \text{and} \quad \mathrm{TV}_\mathcal{C}(M) = \sum_{h\in[M,B\Gamma]} \mathrm{HTV}_\mathcal{C}(M,h)$$

Example: $\theta \in H^3(G,\Bbbk^*)$ $\rightsquigarrow$ $G\text{-}\mathrm{vect}_\Bbbk^\theta$ spherical fusion category whose graduator is G

$$\mathrm{TV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M) = \sum_{h\in[M,BG]} \mathrm{HTV}_{G\text{-}\mathrm{vect}_\Bbbk^\theta}(M,h) = \sum_{h\in[M,BG]} \tau^\theta(M,h)$$

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3-dimensional HQFTs with target $X = K(G, 1)$

presentation of M^3 + algebraic data $\rightsquigarrow$ 3-dim HQFT

- **Turaev-V. (2012)**

triangulation  + $\mathcal{C}$ spherical fusion G-graded category $\rightsquigarrow$ $\mathrm{HTV}_{\mathcal{C}}$

- **Turaev-V. (2014)**

surgery  + $\mathcal{B}$ modular fusion G-graded category $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{B}}$

- **Gelaki-Naidu-Nikshych (2009):**

G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

Theorem (Turaev-V., 2019)

$\mathrm{HTV}_{\mathcal{C}}$ and $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$ are isomorphic HQFTs

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$\mathcal{B}$= modular fusion G-graded category:

- $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is spherical fusion G-graded
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$$\tau_{X,Y} \colon X \otimes Y \to \varphi_g(Y) \otimes X$$

- the S-matrix of fusion category $\mathcal{B}_1$ is invertible

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$$(L, f \colon \pi_1(L) \to G)$$

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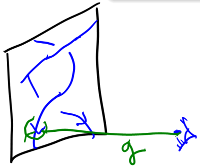

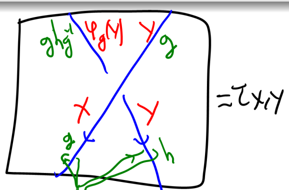

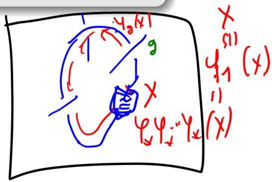

Surgical HQFT with target $X = K(G,1)$

$\mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$ modular fusion G-graded category

M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f\colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L

Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

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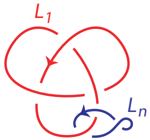

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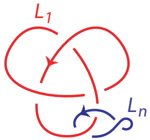

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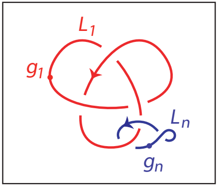

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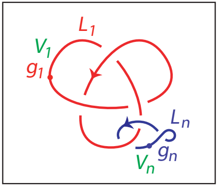

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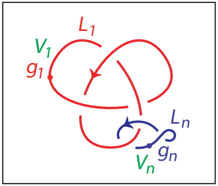

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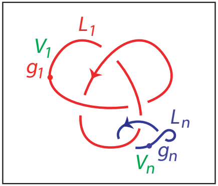


Any $\underline{V} = (V_1, \cdots, V_n) \in \mathcal{B}_{g_1} \times \cdots \times \mathcal{B}_{g_n}$ makes (L, f) $\mathcal{B}$-colored

$$\mathrm{HRT}_{\mathcal{B}}(M, h) = \sum_{\underline{V},\, V_i \text{ simple}} \left(\prod_{i=1}^{n} \dim_q(V_i) \right) \mathcal{I}_{\mathcal{B}}(L, f, \underline{V})$$

Surgical HQFT with target $X = K(G, 1)$

$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ modular fusion G-graded category

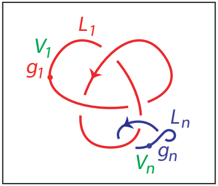
M closed oriented 3-manifold, $h \in [M, X]$

Present M by surgery along a framed link $L = L_1 \cup \cdots \cup L_n$

Let $f \colon \pi_1(L) \to G$ induced by $S^3 \setminus L \hookrightarrow M$ and h

Let $g_i \in G$ be the color of a point in each component L_i of a diagram of L

blackboard framing →



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G-center $\mathcal{Z}_G(\mathcal{C})$ modular fusion G-graded $\rightsquigarrow$ $\mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

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The graded center

$\mathcal{C}$ monoidal category, $\mathcal{D}$ monoidal subcategory of $\mathcal{C}$

The **center of $\mathcal{C}$ relative to $\mathcal{D}$** is the monoidal category $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

- objects of $\mathcal{Z}(\mathcal{C},\mathcal{D})$:

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Steps of the proof of $\mathrm{HTV}_{\mathcal{C}} \simeq \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}$

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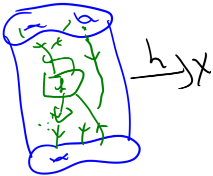
   - ▹ provides basis of $\mathrm{TV}_{\mathcal{C}}(S^1 \times S^1, f_\alpha)$



$\mathcal{Z}_G(\mathcal{C})$-colored

2. $\mathrm{HTV}_{\mathcal{C}}(\Sigma, f) \cong \mathrm{HRT}_{\mathcal{Z}_G(\mathcal{C})}(\Sigma, f)$ for G-surfaces (Σ, f)
   - ▹ via a description of $\mathcal{Z}_G(\mathcal{C})$ by *graded Hopf monad*

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2-dim HQFTs

with target X are classified by Frobenius alg graded by the 2-group of X

$\theta \in H^3(X, k^*) \mapsto \tau^\theta$

- X aspherical, $\left[\begin{array}{l} H^3(X, k^*) \cong H^3(G, k^*) \\ G = \pi_1(X) \end{array}\right.$ $\tau^\theta = HTV_{G\text{-Vect}^\theta}$

- X not aspherical? $\pi_2(X) = 0$ $\Rightarrow$ $H^3(X, k^*)$ described in terms of $G = \pi_1(X)$, $A = \pi_3(X)$ (Eilenberg-MacLane ~ 47)

$\alpha: G^3 \to k^*$ $\quad \rho: A \to k^*$

$k: G^4 \to A$ 4-cocycle

$\partial \alpha = \rho k$