Crossed modules, homotopy 2-types, knotted surfaces and welded knots

Topological Quantum Field Theory Club (IST, Lisbon)

30th October 2020

João Faria Martins (University of Leeds)

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Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

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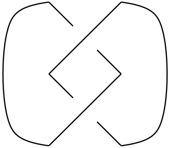
E.g. L = 

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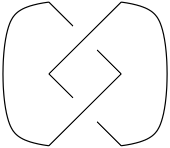

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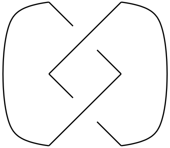

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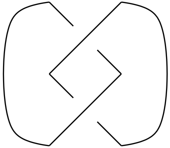
E.g. L = 

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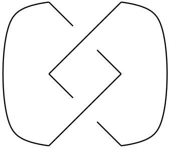

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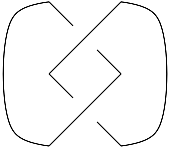

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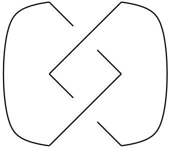
E.g. L = 

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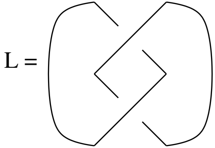

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

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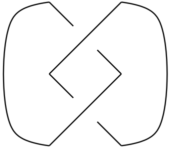
E.g. L = 

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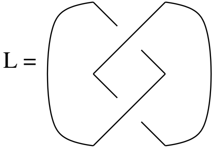

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1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

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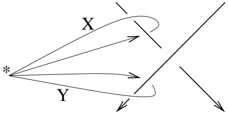

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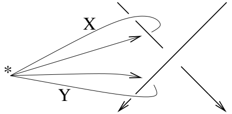

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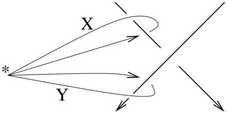

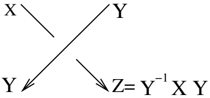

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

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Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
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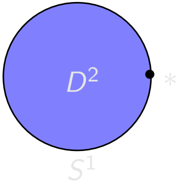

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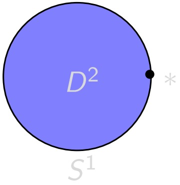

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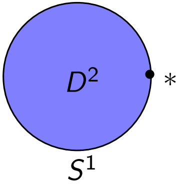

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

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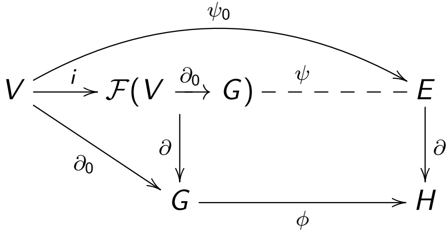

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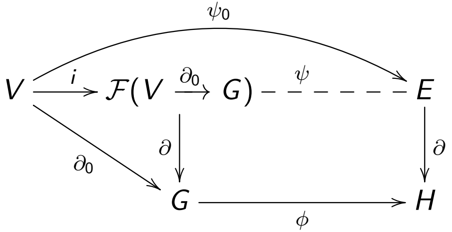

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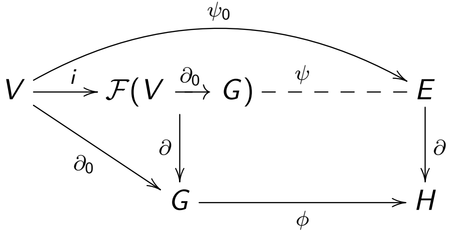

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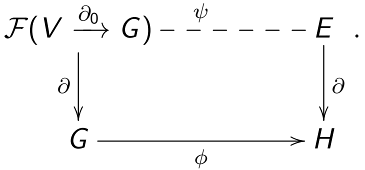

Free crossed modules and Whitehead theorem

$$Y = X \cup c_1 \cup c_2.$$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2-cells\} \xrightarrow{\partial_0} \pi_1(X)$.

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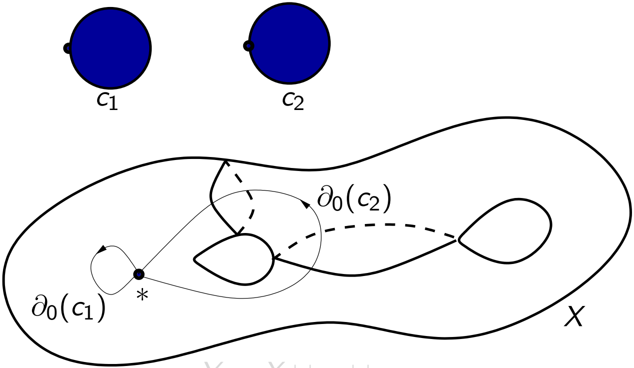

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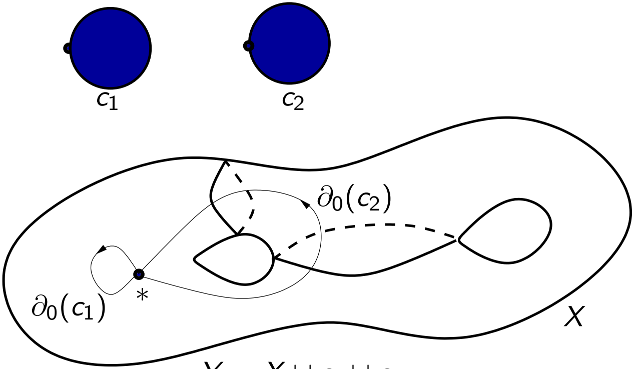

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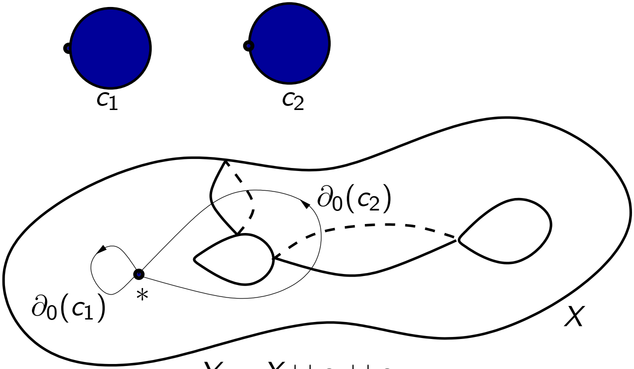

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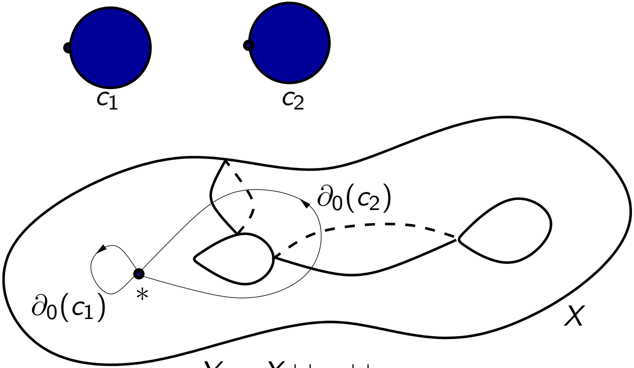

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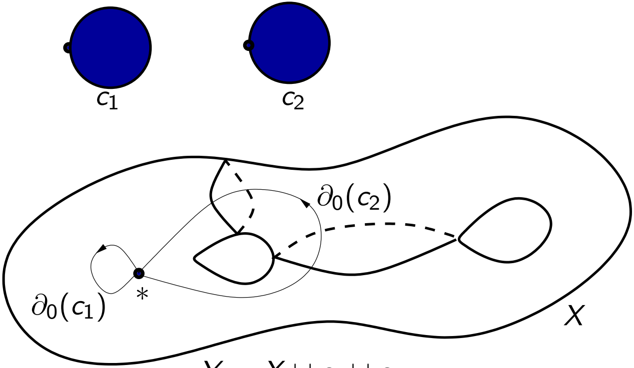

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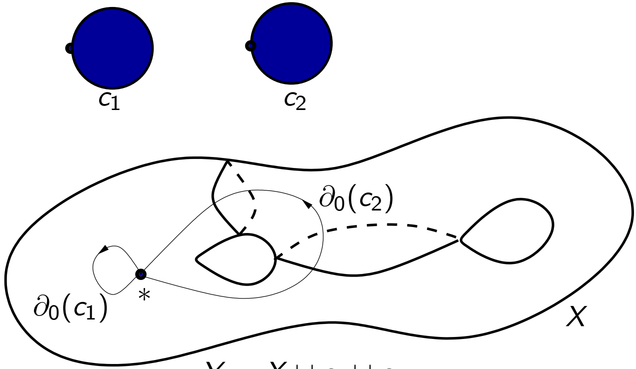

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Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\simeq$.
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Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

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Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$*.*

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

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Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

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Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

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$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
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Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

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$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

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   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle .$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \ \middle|\ \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle .$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

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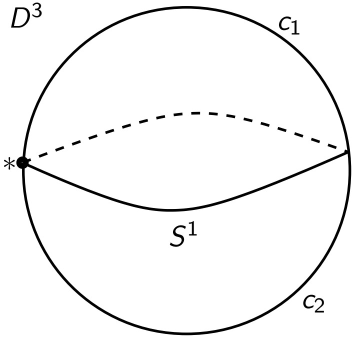
Presentation of $\Pi_2(D^3, S^1)$ by generators and relations



$$\Pi_2(S^2, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \right\rangle = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}, \triangleright_{trivial})$$

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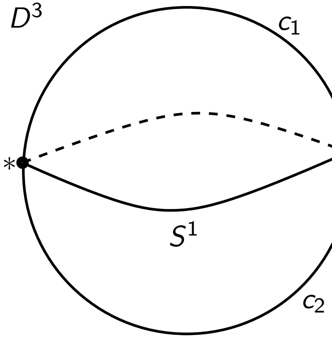

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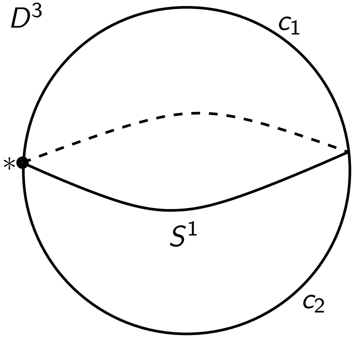

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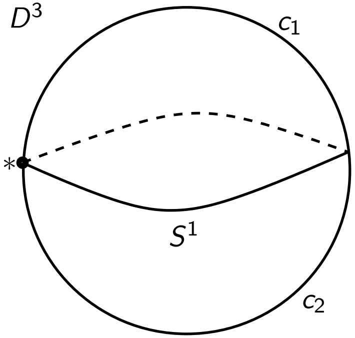

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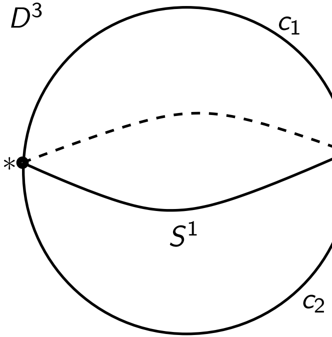

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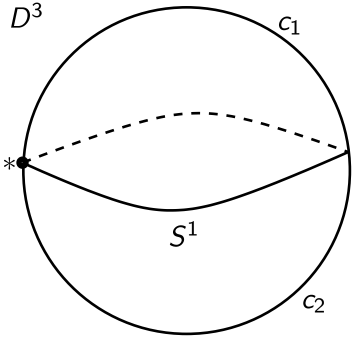

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The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

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Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

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We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

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Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
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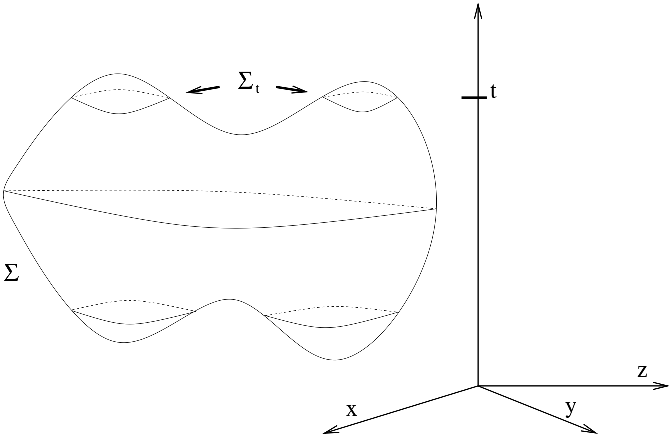

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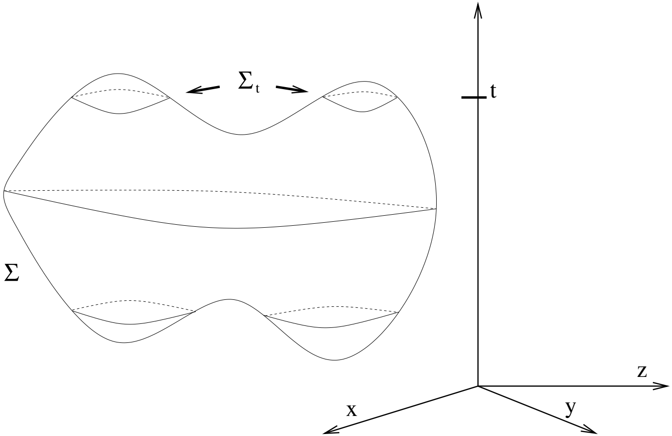

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

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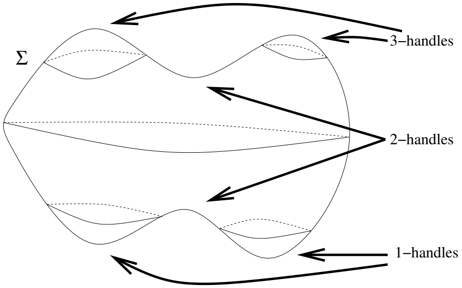

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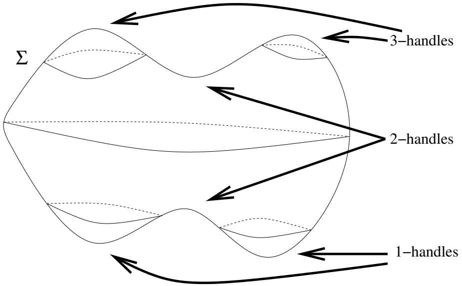

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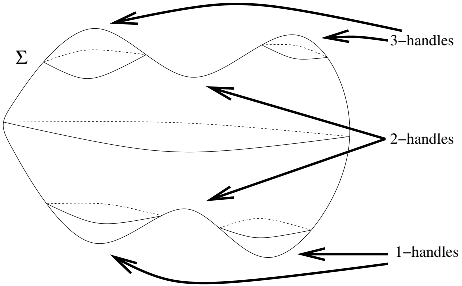

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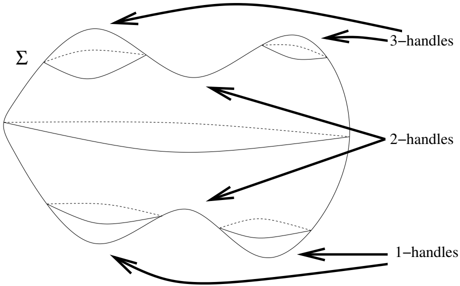

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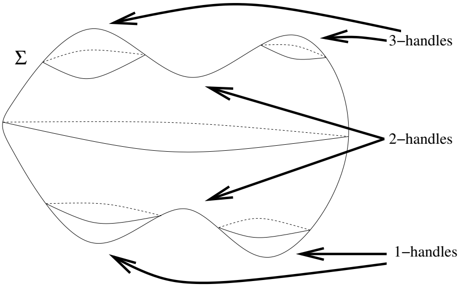

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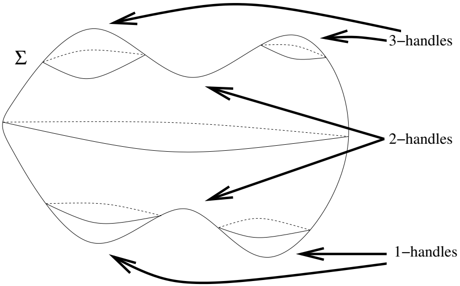

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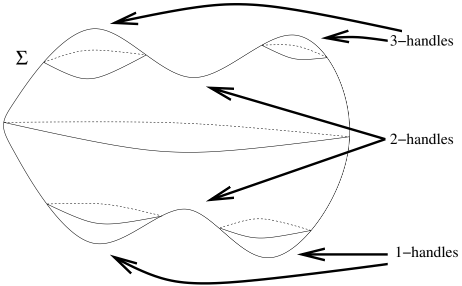

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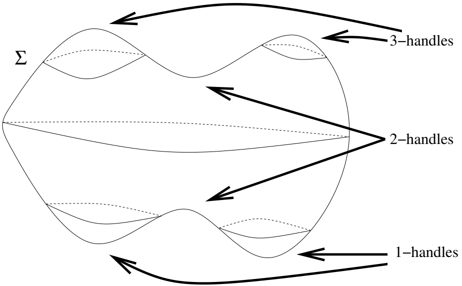

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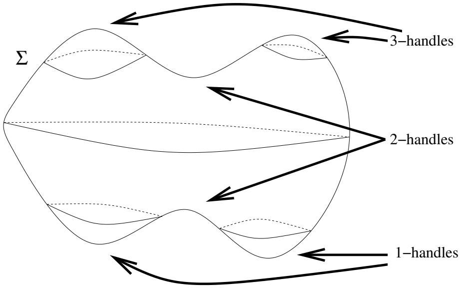

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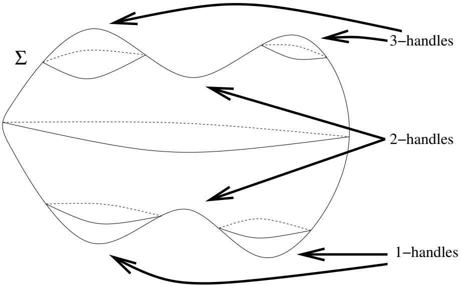

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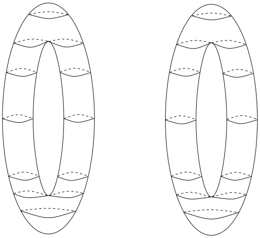

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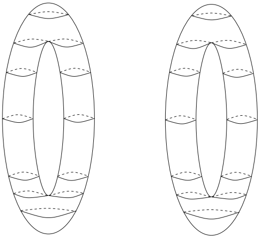

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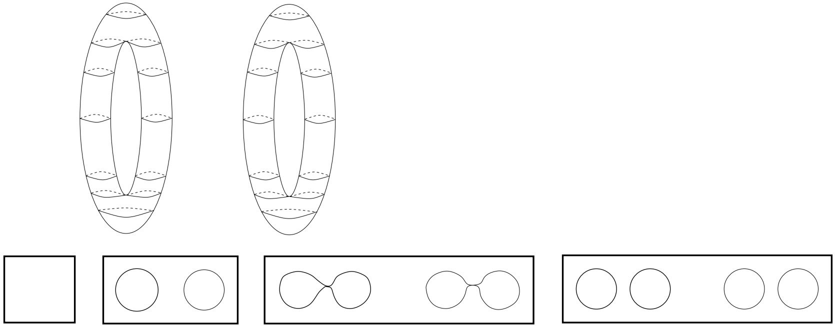

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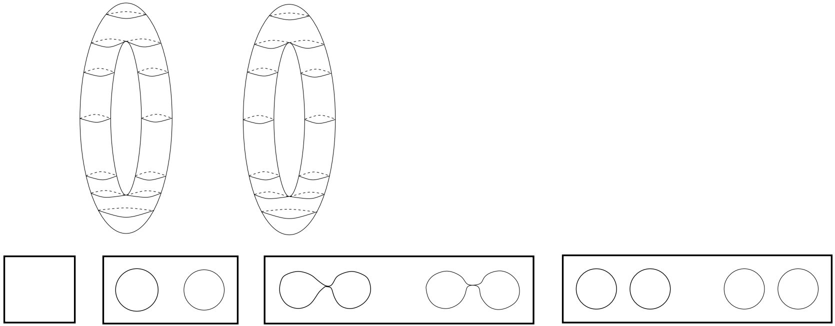

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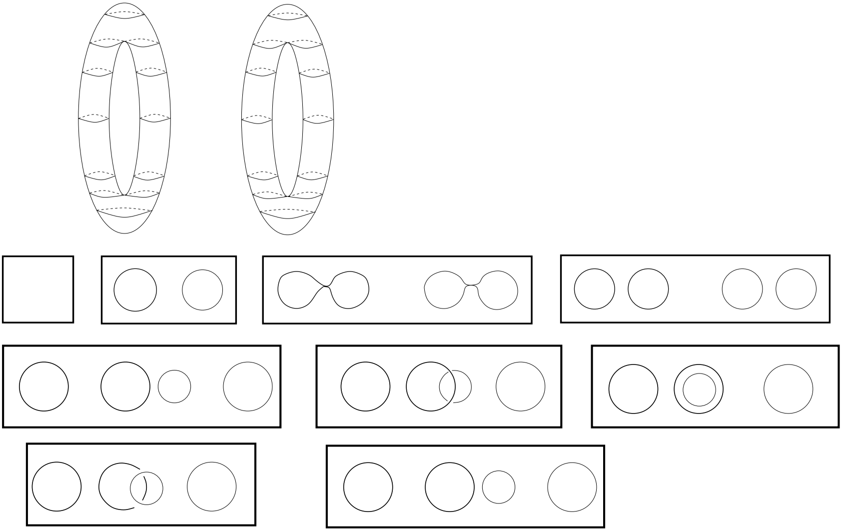

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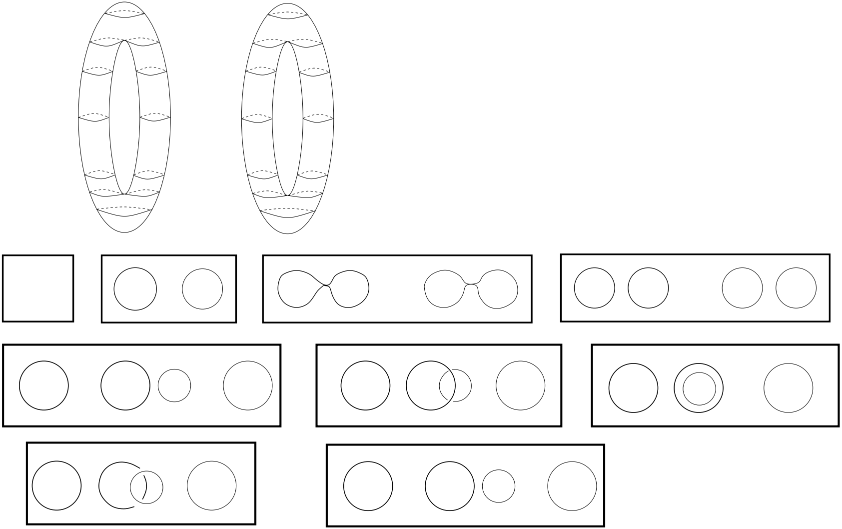

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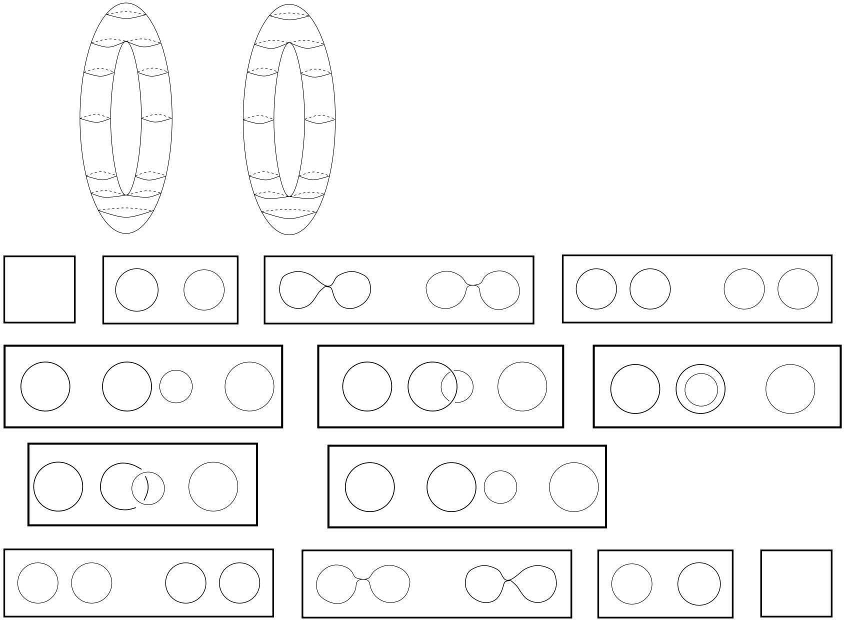

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
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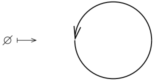
Locally, an oriented minimal point looks like: 

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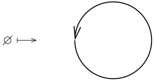

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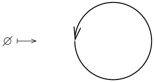

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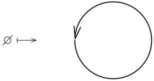

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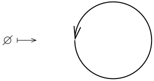

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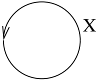

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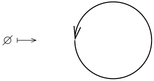

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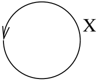

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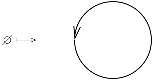

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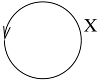

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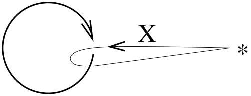

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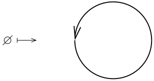

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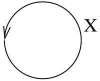

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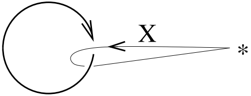

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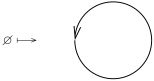

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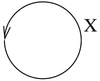

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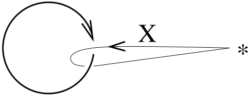

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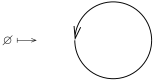

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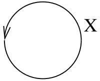

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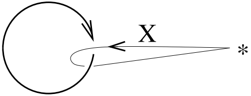

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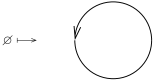

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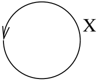

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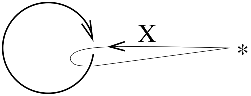

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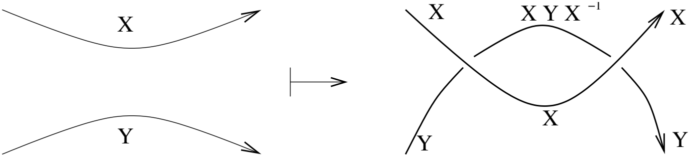

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Locally, an (oriented) saddle point looks like:

When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.

Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

Bands are to be kept and evolve throughout the rest of the movie.
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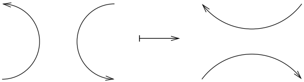

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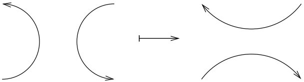

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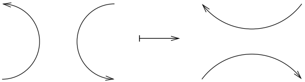

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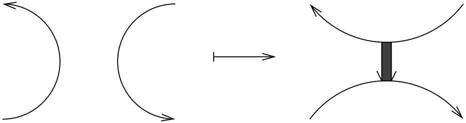

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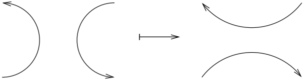

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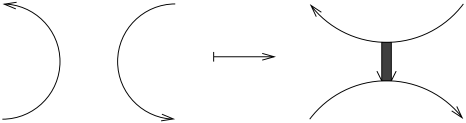

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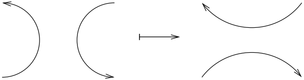

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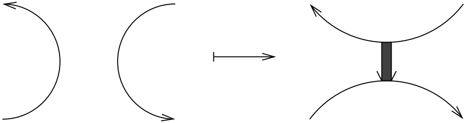

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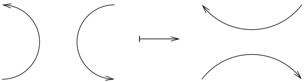

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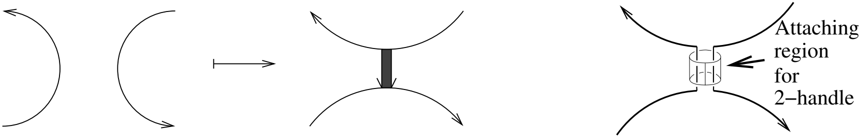

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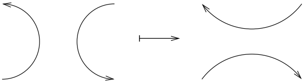

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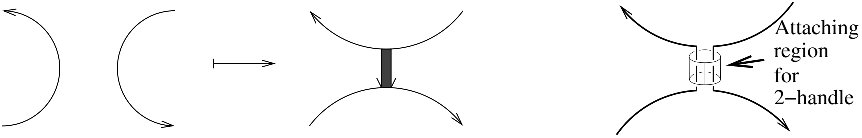

Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

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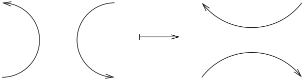

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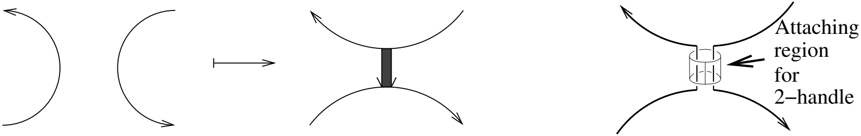

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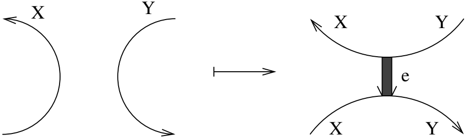

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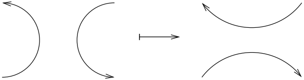

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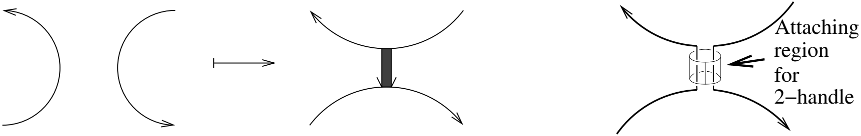

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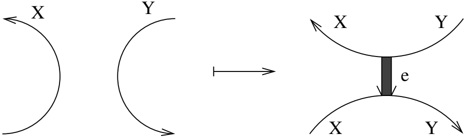

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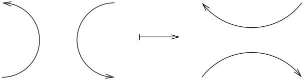

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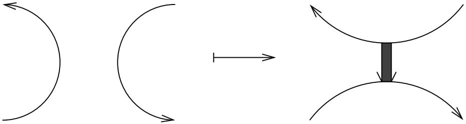

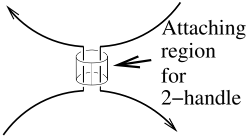

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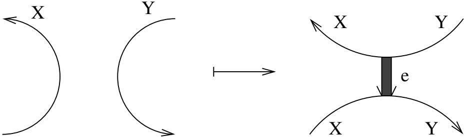

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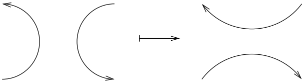

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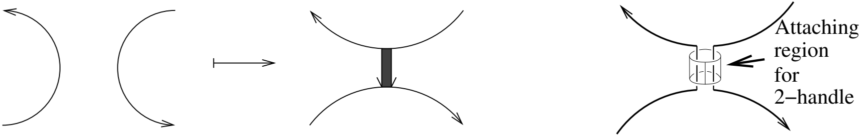

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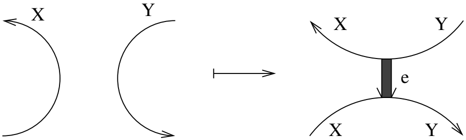

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Maximal points

Locally, an oriented maximal point looks like:

Some bands will possibly be present.
Before maximal point, configuration looks like:

In this case the 2-relations are as below:

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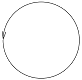
Locally, an oriented maximal point looks like:  $\mapsto \emptyset$

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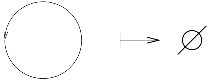
Locally, an oriented maximal point looks like: 

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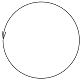
Locally, an oriented maximal point looks like:  $\longmapsto \emptyset$

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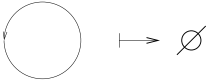
Locally, an oriented maximal point looks like: 

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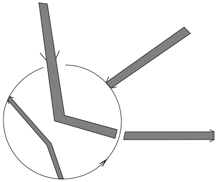
Before maximal point, configuration looks like: 

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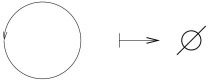
Locally, an oriented maximal point looks like: 

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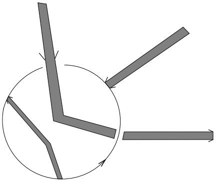
Before maximal point, configuration looks like: 

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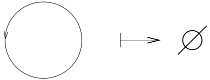

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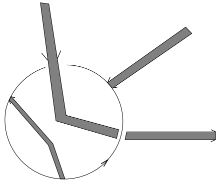

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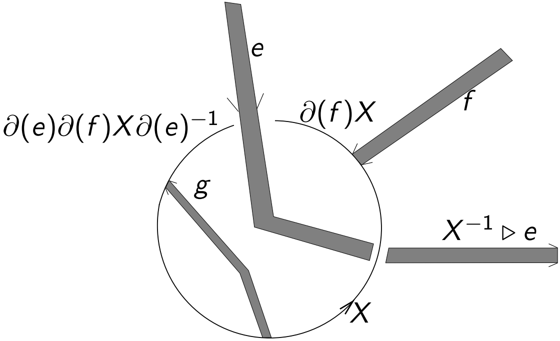

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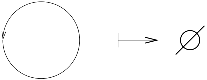
Locally, an oriented maximal point looks like: 

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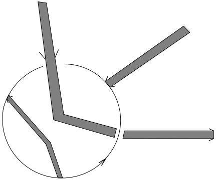
Before maximal point, configuration looks like: 

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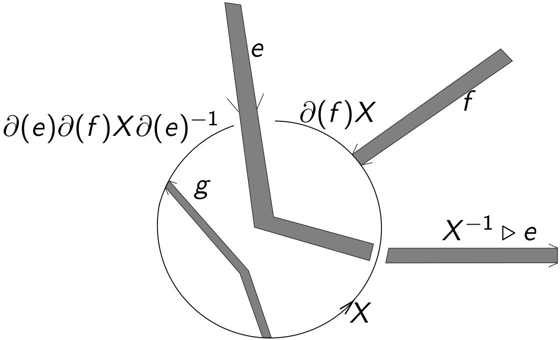

2-relation:

$$e\ f\ (X^{-1} \triangleright e^{-1}) = 1$$

A movie for a knotted union Σ of two tori

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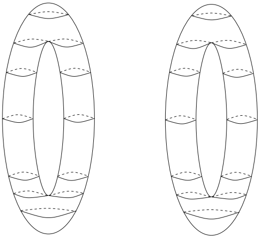

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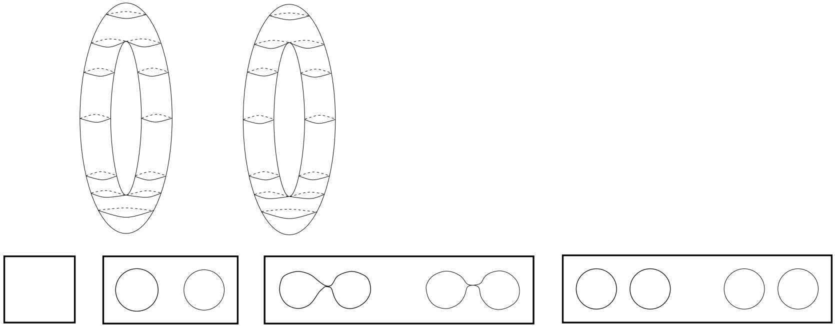

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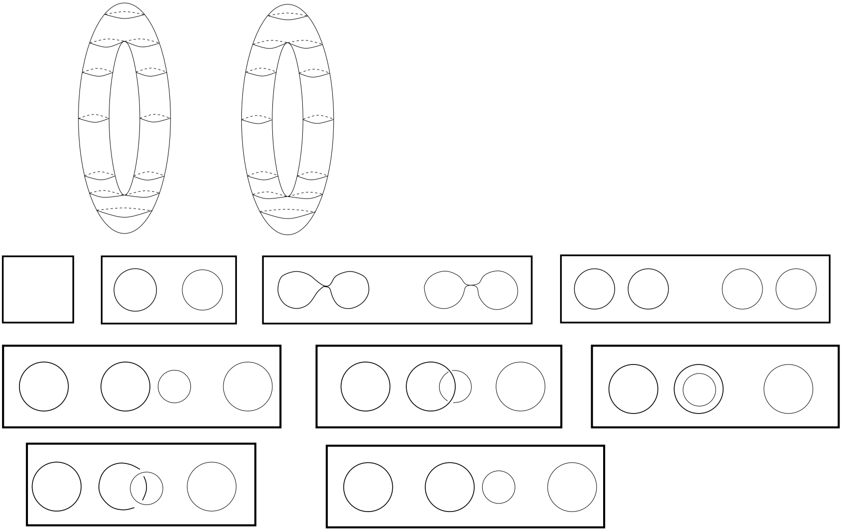

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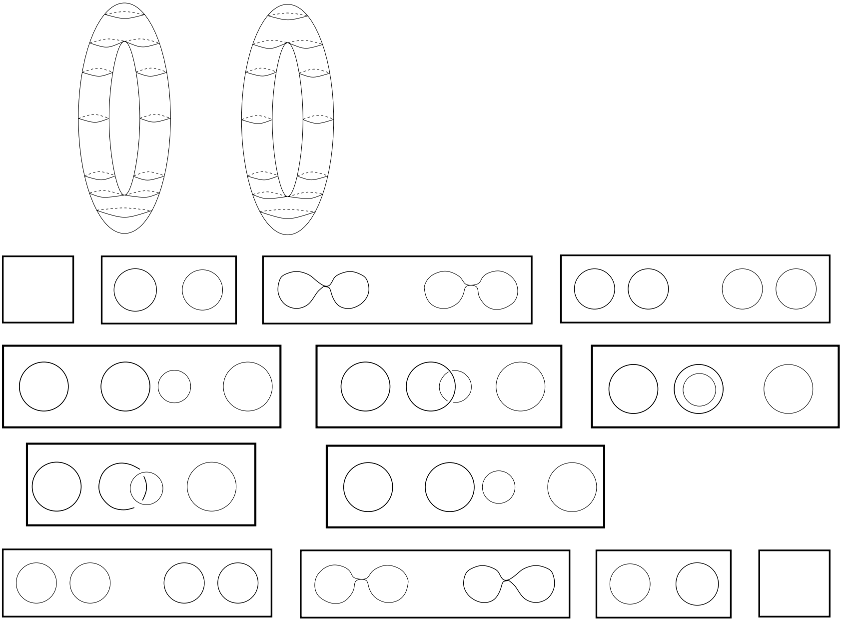

Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise

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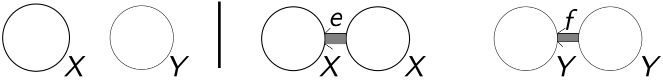

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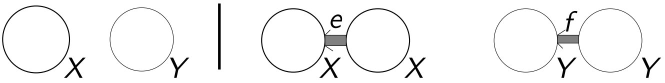

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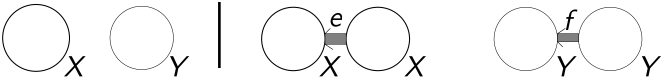

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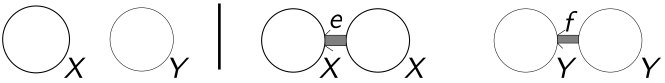

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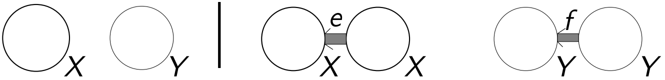

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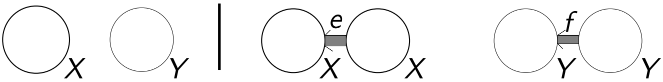

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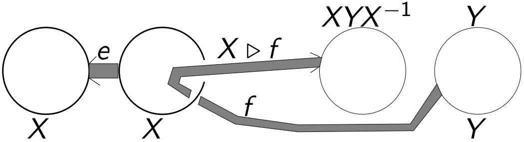

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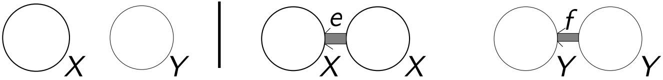

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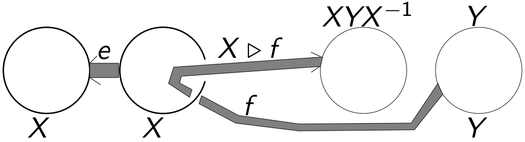

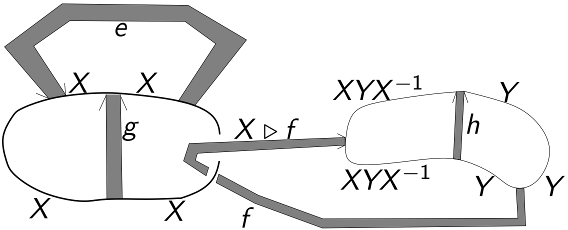

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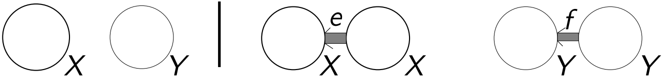

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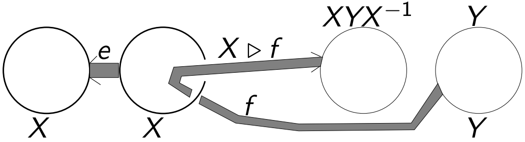

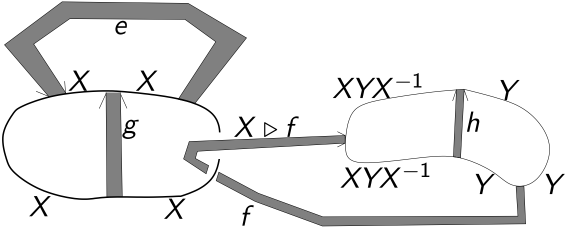

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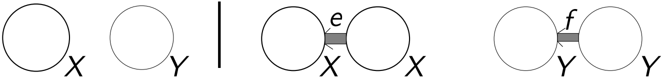

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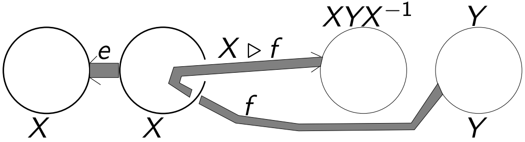

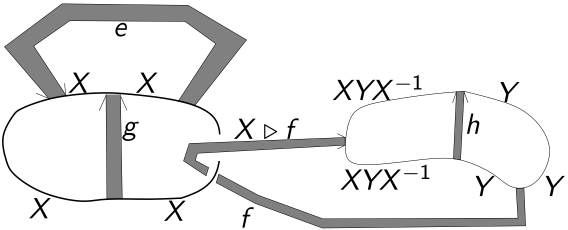

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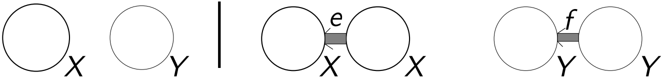

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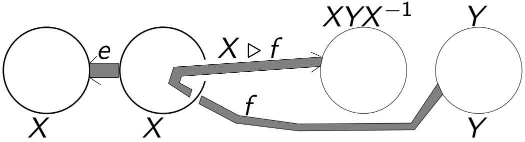

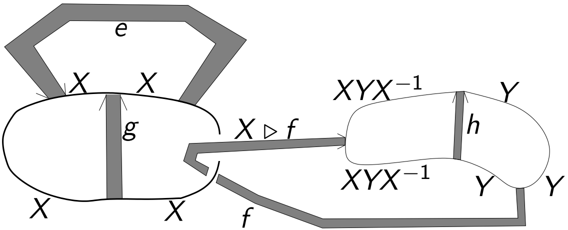

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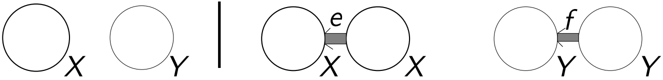
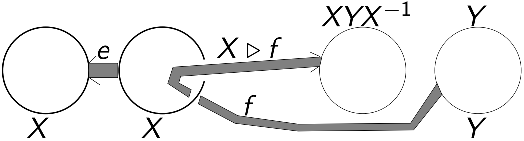
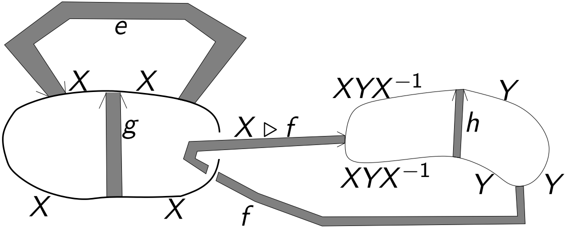


$$\partial(e) = 1$$
$$\partial(f) = 1$$

$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.





$$\partial(g) = 1$$
$$\partial(h) = XYX^{-1}Y^{-1}$$
$$e\, e^{-1}\,(X \triangleright f^{-1})\; f = 1$$
$$(X \triangleright f)f^{-1} = 1.$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto 1 \\ h \mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\} / < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

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Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:

$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

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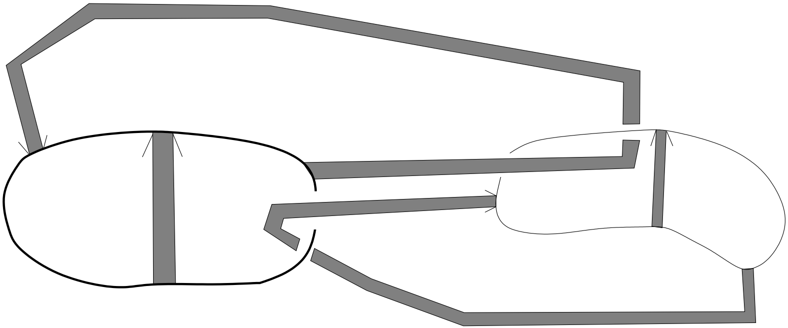

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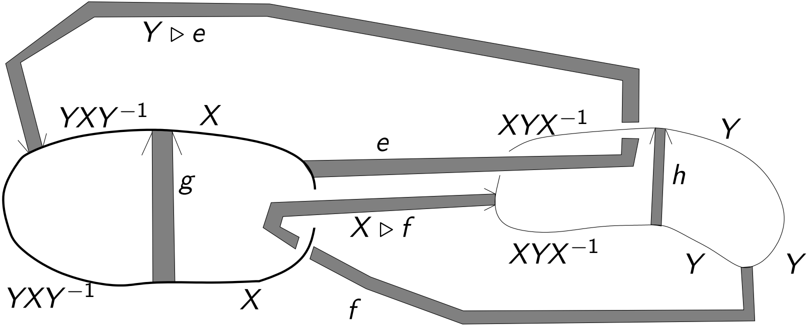

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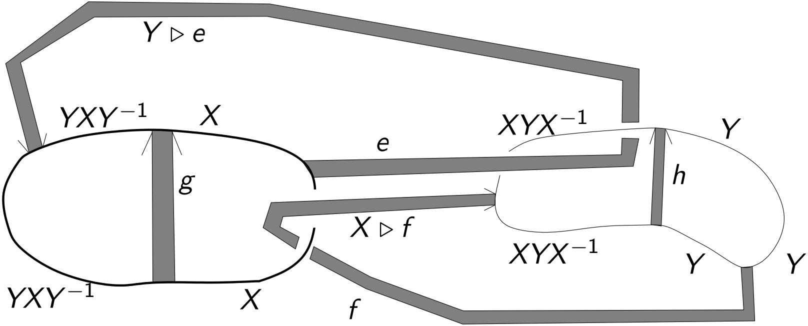

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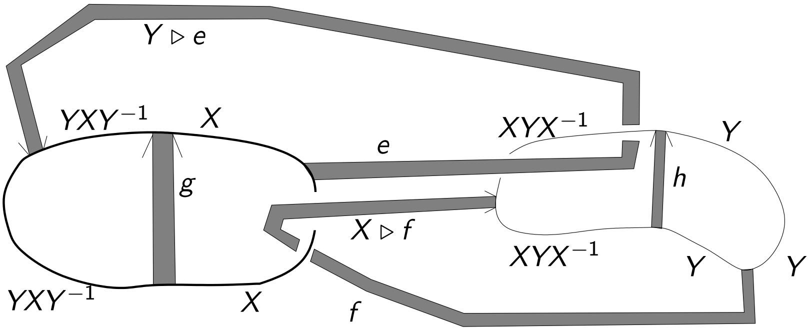

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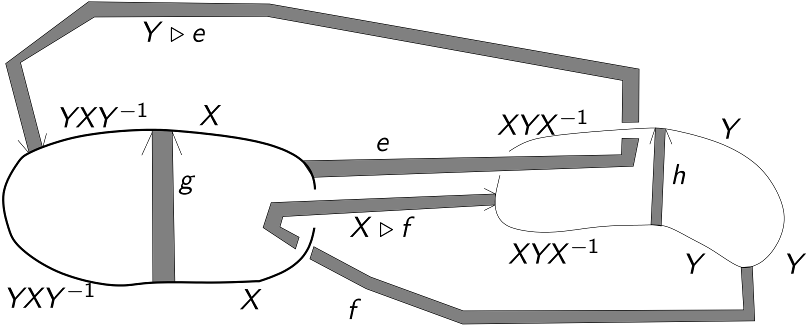

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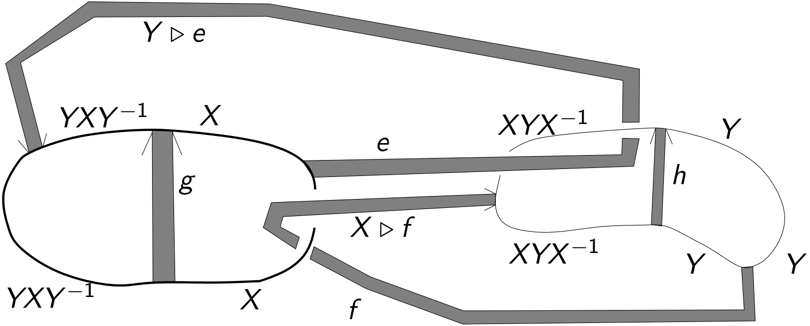

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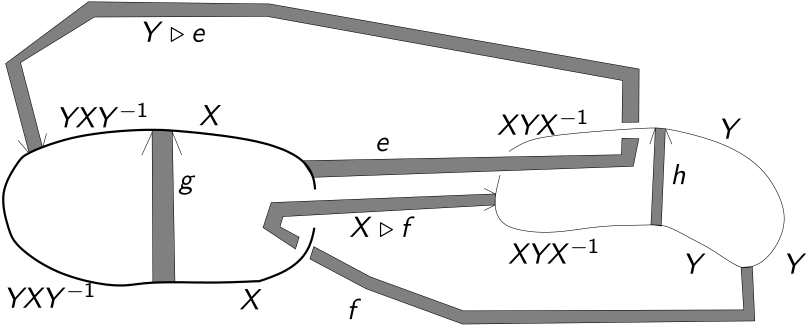

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$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\, f\\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

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If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{(X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX,\\ (Y\triangleright e)-e-(X\triangleright f)+f=0}\right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

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$\Sigma' = $ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X,Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\; f\\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X,Y\} | [X,Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f)\; + f = 0 >}.$$

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More results on $I_{\mathcal{G}}(S^4 \setminus \Sigma)$

Let $\mathcal{G} = (\partial\colon E \to G)$ be a finite crossed module.

Recall $I_{\mathcal{G}}(M) = \frac{1}{\# E^{b_1(M^1)}} \# \hom(\Pi_2(M, M^1), \mathcal{G})$

- The invariant of knotted surfaces:

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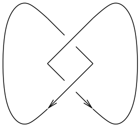

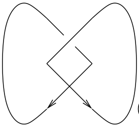

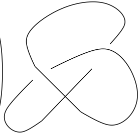

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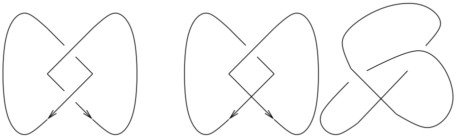

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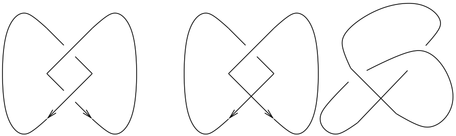

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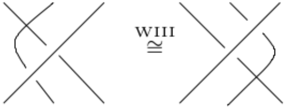

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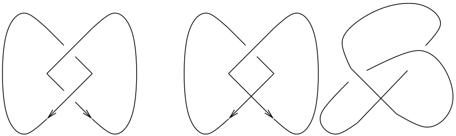

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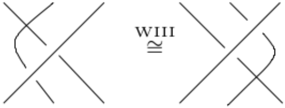

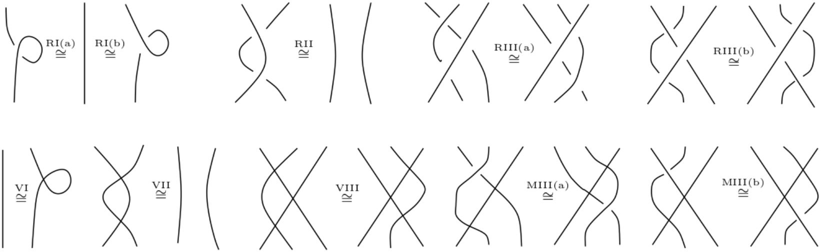

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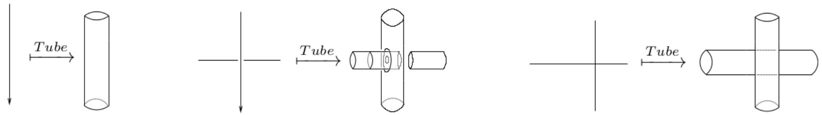

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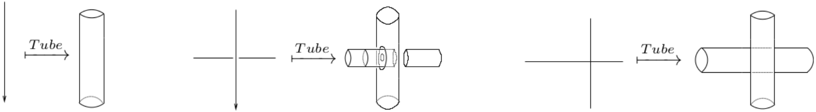

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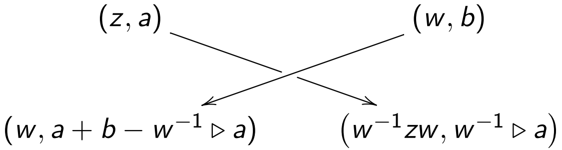

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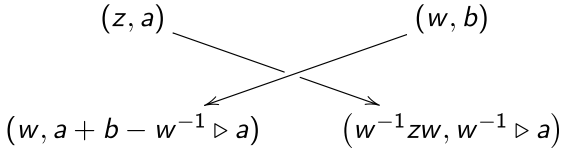

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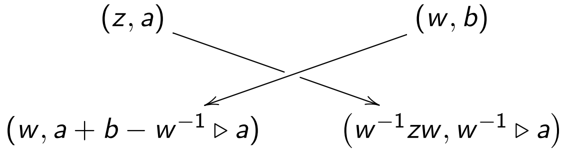

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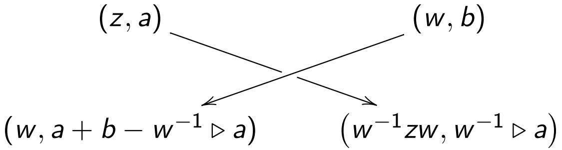

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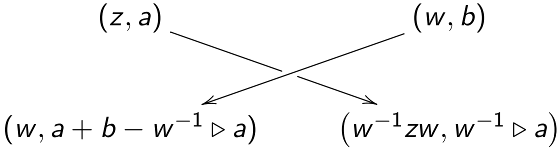

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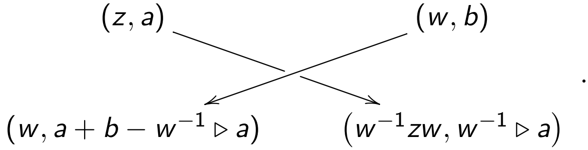

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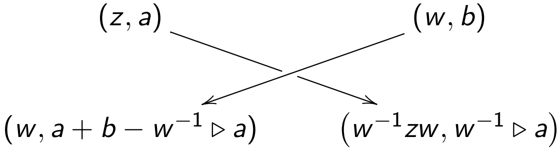

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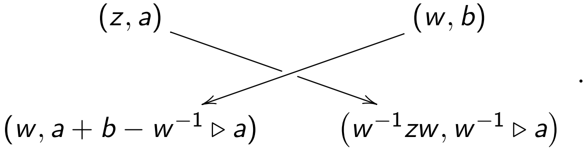

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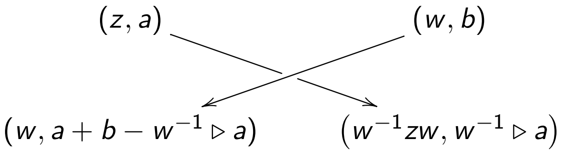
- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):



.

has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

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THANKS!