# 3d Mirror Symmetry & HOMFLY-PT Homology

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virtual - Lisbon

# Review: HOMFLY-PT

st. $a = q^N$, $u = q^{1/2} - q^{-1/2}$ ⇝ WRT for $\mathfrak{sl}_N$, fund. rep

Link $K \subset S^3$ ⇝ $P_K(a^{1/2}, u) \in \mathbb{Z}[a^{\pm 1/2}, u^{\pm}]$ HOMFLY-PT poly

we'll expand in $q$: $P_K(a^{1/2}, u = q^{1/2} - q^{-1/2}) \in (a/q)^{\#}\, \mathbb{Z}(\!(q)\!)[a]$

E.g. unknot $P_U = \dfrac{a^{1/2} - a^{-1/2}}{q^{1/2} - q^{-1/2}} = (a/q)^{-1/2} \dfrac{1-a}{1-q} \rightsquigarrow (a/q)^{-1/2}(1-a)(1+q+q^2+q^3+\dots)$

Categorify the series via $\mathcal{H}_K = \bigoplus_{k,d,R \in \mathbb{Z}} \mathcal{H}_K^{\overset{a}{k},\overset{q}{d},\overset{\text{coh.}}{R}}$ HOMFLY-PT homology

(GSV predicted
KR defined)

st. $P_L = (a/q)^{\#} \sum (-1)^R q^d (-a)^k \dim \mathcal{H}_K^{k,d,R}$

## M-theory construction / prediction:

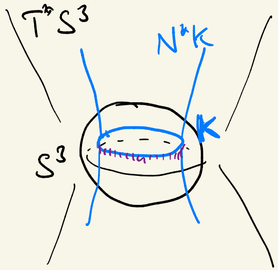
[Gukov-Schwarz-Vafa]

following Witten, Gopakumar-Vafa, Ooguri-Vafa

$\mathfrak{sl}_N$ homology $\bigoplus_{d,R\in\mathbb{Z}} \mathcal{H}^{d,R}_{(N)}$

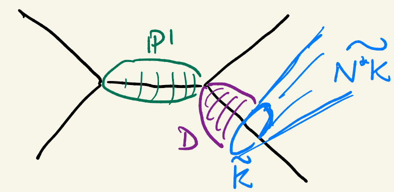
HOMFLY homology $\bigoplus_{h,d,R\in\mathbb{Z}} \mathcal{H}^{h,d,R}(K)$

$\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$




$\times\ \mathbb{C}_x\times\mathbb{C}_y\times\mathbb{R}_t$

$\longleftrightarrow$ geometric transition




$\times\ \mathbb{C}_x\times\mathbb{C}_y\times\mathbb{R}_t$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | M5 on | $N^*K$ | $\times$ | $\mathbb{C}_x$ | $\times$ | $\mathbb{R}_t$ |
| N | M5′ on | $S^3$ | $\times$ | $\mathbb{C}_x$ | $\times$ | $\mathbb{R}_t$ |
| 1 | M2 on | $S^1\times I$ | $\times$ | | | $\mathbb{R}_t$ |

degenerate holomorphic annulus connecting M5 & M5′

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | M5 on | $\widetilde{N^*K}$ | $\times$ | $\mathbb{C}_x$ | $\times$ | $\mathbb{R}_t$ |
| 1 | M2 on | $D$ | $\times$ | | | $\mathbb{R}_t$ |
| $\boxed{k}$ | M2′ on | $\mathbb{P}^1$ | $\times$ | | | $\mathbb{R}_t$ |

$\bigoplus_{d,R} \mathcal{H}^{d,R}_{(N)}$ ~ "BPS states" of M2

| | | | $\mathbb{C}_x$ | $\times\ \mathbb{C}_y$ |
|---|---|---|---|---|
| $q$-degree $d$ | ~ | $U(1)_d$ | 1 | $-1$ |
| coh. degree $R$ | ~ | $U(1)_R$ | 2 | 0 |

$\bigoplus_{d,R} \mathcal{H}^{\boxed{h},d,R}$ ~ BPS states of M2

| | | $\mathbb{C}_x$ | $\times\ \mathbb{C}_y$ |
|---|---|---|---|
| $q$-degree | $d$ | 1 | $-1$ |
| coh. degree | $R$ | 2 | 0 |

# Tight braids

$zu - wv = \mu \rightsquigarrow zu - wv = 0 \leftsquigarrow$ blow-up $\quad \begin{array}{l} z\rho = v\lambda \\ w\rho = u\lambda \end{array} \quad (\rho : \lambda) \in \mathbb{P}^1$

How to transport $N^*K$ from $T^*S^3$ to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ ?

[Taubes, Diaconescu-Shende-Vafa, ... ]

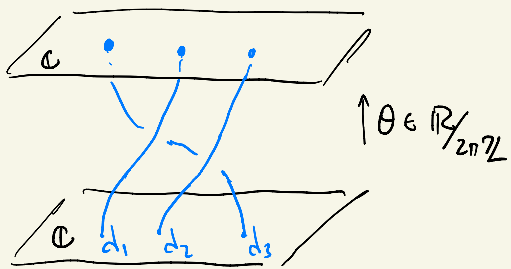
- lift $N^*K$ off $S^3$ (Lag. isotopy)
- find families of lagrangians on the two sides that match (smoothly) at singular conifold

Implemented decades ago for the unknot $\leftsquigarrow$ toric [Aganagic-Vafa] Lags on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We generalized the unknot construction to tight braids

braid (closure):

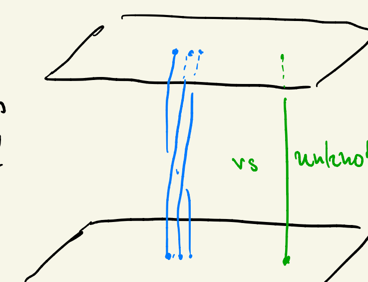
$n$ strands




$\{d_i(\theta)\}_{i=1}^n$ positions of strands

tight braid

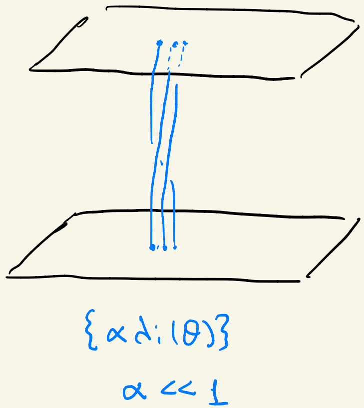
$\sim$ deformation of $n$ copies of unknot




$\{\alpha d_i(\theta)\}$

$\alpha \ll 1$

tight braid
~ deformation
of $n$ copies of
unknot

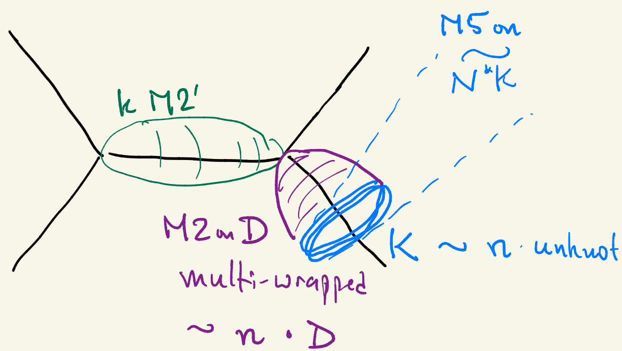



$\rightsquigarrow$ $N^*K$ deformation $\subset T^*S^3$
of $n$ copies of $N^*$unknot

$\rightsquigarrow$ $\widetilde{N^*K}$ def. of $\subset \mathcal{O}(-1)\oplus\mathcal{O}(-1)$
$n$ copies of toric AV brane

For tight braids, end up with

$\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$

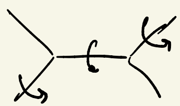


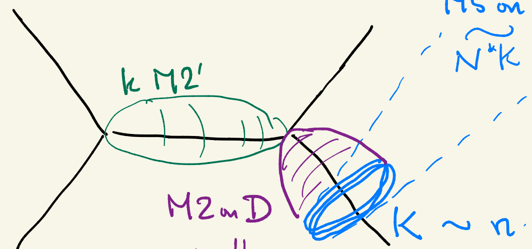
# Conifold → field theory

Goal: extract a 3d QFT on $D \times \mathbb{R}_t$ capturing the physics (BPS states) of M2 branes

$\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$

M-theory






$\times\, \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{R}_t$

$S^1$ in conifold

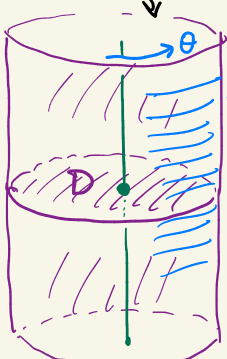
$S^1 \to \mathbb{C}_x \times \mathbb{C}_y$

$\downarrow$

$\mathbb{R}^3_{|x|^2-|y|^2,\ xy}$

IIA on $\mathbb{R}^5 \times \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{R}_t$

IIA on $(\mathcal{O}(-1)\times\mathcal{O}(-1)\to\mathbb{P}^1) \times \mathbb{R}^3 \times \mathbb{R}_t$

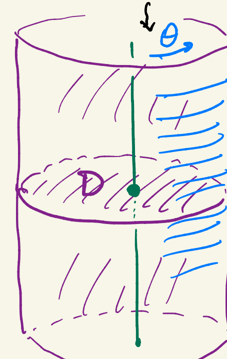



$\partial D \times \mathbb{R}_t$
boundary condition depends on $\lambda_i(\theta)$

3d B-model

$\lambda_k$ line operator $\{0\}\times\mathbb{R}_t$

3d mirror sym




$\partial D \times \mathbb{R}_t$
boundary condition
$\lambda_i(\theta)$

3d A-model

$\lambda_k$ line operator $\{0\}\times\mathbb{R}_t$

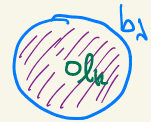
# 3d TQFT

Ultimately, the 3d ($\mathcal{N}=4$) gauge theories, boundary conditions, & line operators can all be identified explicitly — in physical terms.

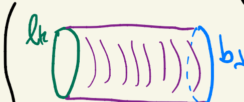
How to extract a (putative) mathematical definition of HOMFLY-PT homology?

Work in the framework of 3d TQFT.

Want: Hilbert space on a disc $D$, w/ bdy cond$^{n}$ $b_\lambda$ at $\partial D$, "punctured" by $\ell_k$ at $0$

$\simeq Z\Bigg($ 

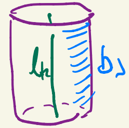
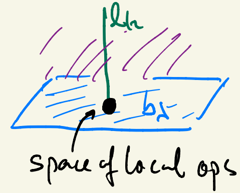
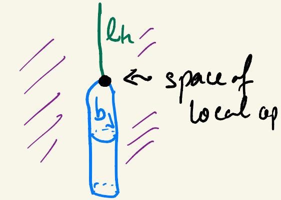
$\Bigg)$

Attempt 1 : $\simeq Z\Bigg($ 

$\Bigg) = \mathrm{Hom}_{Z(S^1)}(\ell_k, b_\lambda)$

$\uparrow$ category of line operators

Note $Z(S^1)$ is a braided $\otimes$ category

Dual physics picture: Hilb(  ) $\simeq$  $\simeq$ 

= Hom($\ell_k$, $b_\beta$)
lines



Let's try to apply this.

$\boxed{B}$ 3d gauge theory has $G = GL_n$ (really $G_\mathbb{C}$), "matter" $T^*V$ $V = \mathfrak{gl}_n \times \mathbb{C}^n$

"Higgs branch" is $T^*V/\!\!/G \simeq \mathrm{Hilb}_n\,\mathbb{C}^2$ (resolved)

$\mathbb{C}_x \times \mathbb{C}_y$ in M-thy!

$Z(S^1) \simeq D^b\mathrm{Coh}(\mathrm{Hilb}_n\,\mathbb{C}^2)$

[Rozansky-Witten, Kapustin-Rozansky-Saulina]

$\ell_k$ = Wilson line in rep $\Lambda^k\mathbb{C}^n$ of $GL_n$

$\rightsquigarrow$ object $\Lambda^k\mathrm{Taut} \in \mathrm{Coh}(\mathrm{Hilb}_n\,\mathbb{C}^2)$

$b_\beta$ = a (complex of) sheaves on $\mathrm{Hilb}_n\,\mathbb{C}^2$ determined by the braid

HARD to construct directly

[Oblomkov-Rozansky, Gorsky-Negut-Rasmussen] did it!

Once one knows the sheaf $b_\lambda$, expect

$$H_K^{k\bullet\bullet} \simeq \mathrm{Hom}^\bullet(\Lambda^k \mathrm{Taut}, b_\lambda) \quad \text{in } D^b\mathrm{Coh}(\mathrm{Hilb}_n \mathbb{C}^2)$$

derived Hom, ie. Ext

| | | $\mathbb{C}_x \times$ | $\mathbb{C}_y$ |
|---|---|---|---|
| q-degree | $\sim$ | $1$ | $-1$ |
| coh degree mixes with | $\sim$ | $2$ | $0$ |

Example: unknot $n=1$ $\quad \mathrm{Hilb}_1(\mathbb{C}^2) = \mathbb{C}_x \times \mathbb{C}_y$

$$\Lambda^k \widetilde{\mathrm{Taut}} = \begin{cases} \mathcal{O}_{\mathbb{C}_x \times \mathbb{C}_y} & k = 0,1 \\ 0 & k \geq 2 \end{cases}$$

$$\lambda(\theta) = \text{constant} \qquad b_\lambda = \mathcal{O}_{\mathbb{C}_x}$$

$$H^{k\bullet\bullet}_{\text{unknot}} \simeq \mathrm{Hom}^\bullet(\mathcal{O}, \mathcal{O}_{\mathbb{C}_x}) \qquad k = 0,1$$

$$\simeq \mathbb{C}[x] \qquad x \quad \text{q-degree } 1, \text{ coh degree } 2$$

character $\dfrac{1}{1-q}$

cf. $\quad P_{\text{unknot}} = (\ )\dfrac{1-a}{1-q}$

$\boxed{A}$ 3d theory is <u>also</u> $G = GL_n$, $T^*V$ w/ $V = \mathfrak{gl}_n \times \mathbb{C}^n$

<u>Coulomb branch</u> resolves to $\mathrm{Hilb}_n \mathbb{C}^2$

e.g. by Braverman-Finkelberg-Nakajima

$$Z(S^1) \simeq \text{D-mod}\left( V((z)) / G((z)) \right)$$

[TD-Garner-Geracie-Hilburn, Hilburn-Yoo]

really: a quotient of this, corresponding to the resolution of the Coulomb branch [Webster]

objects are labelled by $L/H$

where $L \subset V((z))$ subspace

$H \subset G((z))$ subgroup that preserves $L$

( the corresponding D-module is $\mathcal{O}_{L/H}$, pushed-forward to $V((z))/G((z))$ )

physics: $\ell_k$ has $L = \left\{ \left( \dfrac{z\,\mathbb{C}[[z]]}{\mathbb{C}[[z]]} \right)^k_{n-k} \times \mathbb{C}^n[[z]] \right\} \subset \mathfrak{gl}_n((z)) \times \mathbb{C}^n((z))$

$$Z(S^1) \simeq D\text{-mod}\left( V((z)) / G((z)) \right)$$

objects are labelled by $L/H$

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$H \subset G((z))$ subgroup that preserves $L$

physics: $\ell_k$ has $L = \left\{ \left( \frac{z\,\mathbb{C}[[z]]}{\mathbb{C}[[z]]} \right)^{k}_{n-k} \right\} \times \mathbb{C}^n[[z]] \subset \mathfrak{gl}_n((z)) \times \mathbb{C}^n((z))$

$H = \mathrm{Iwa}_k$ : Iwahori subgp of $GL_n[[z]]$ that preserves $L$

$b_\lambda$ is a "skyscraper" D-module $H = 1$

$$L = \{ \gamma(z),\ 1^n \} \subset \mathfrak{gl}_n((z)) \times \mathbb{C}^n((z))$$

where $\gamma(z) \in \mathfrak{gl}_n((z))$ is any (fixed) matrix whose eigenvalues $(\gamma_1(re^{i\theta}), \ldots, \gamma_n(re^{i\theta}))$ converge for suff. small $r$, and agree with $(\lambda_1(\theta), \ldots, \lambda_n(\theta))$.

Not clear it is always possible to find such a $\gamma(z)$.

It is possible for "positive algebraic links" E.g. $(2,p)$ torus link $\gamma(z) = \begin{pmatrix} 0 & 1 \\ z^p & 0 \end{pmatrix}$

$\gamma(z) \in \mathfrak{gl}_n[z]$

Putting this together...

<u>Conjecture</u> $\mathcal{H}_K^{k\bullet\bullet} \simeq \mathrm{Hom}^\bullet_{Z(S^1)}(\ell_k, b_d)$

$$= H^{BM}_\bullet \left\{ g \in GL_n(\!(z)\!) \text{ st. } g\cdot(\gamma(z), 1^n) \in \left(\frac{z\,\mathbb{C}[\![z]\!]}{\mathbb{C}[\![z]\!]}\right)^k_{n-k} \times \mathbb{C}^n[\![z]\!] \right\} \Big/ \mathrm{Iwa}_k$$

$q$-degree $d$ $\sim$ stratification of $GL_n(\!(z)\!)$ by $\pi_1(GL_n) = \mathbb{Z}$

ie. $[GL_n(\!(z)\!)]_d = \{ g(z) \text{ st. } \det g(z) \in z^d\, \mathbb{C}[\![z]\!] \}$

<u>Example</u> Unknot $n = 1$ $\quad G = GL_1 = \mathbb{C}^*$ $\quad V = \mathbb{C}_{(0)} \times \mathbb{C}_{(1)}$

$\ell_0 = \mathbb{C}[\![z]\!] \times \mathbb{C}[\![z]\!] \subset \mathbb{C}(\!(z)\!) \times \mathbb{C}(\!(z)\!)$ $\quad \mathrm{Iwa}_0 = GL_1[\![z]\!] = \mathbb{C}[\![z]\!]^*$

$\gamma(z) = (1)$

$$\mathcal{H}^{0\bullet\bullet}_{\text{unknot}} = H_\bullet \{ g \in \mathbb{C}(\!(z)\!)^* \text{ st. } g(1)g^{-1} \in \mathbb{C}[\![z]\!] \text{ and } g\cdot 1 \in \mathbb{C}[\![z]\!] \} / \mathbb{C}[\![z]\!]^*$$

$$= H_\bullet \{ \mathbb{C}[\![z]\!] \text{ (nonzero)} \} / \mathbb{C}[\![z]\!]^*$$

$$= H_\bullet \{ 1, z, z^2, z^3, z^4, \dots \}$$

$$= \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \dots, \qquad \text{character} = \frac{1}{1-q}$$

**Thm** [Garner-Kivinen]

For $\gamma(z) \in \mathfrak{gl}_n[z]$, the moduli spaces that appear here are Hilbert schemes of spectral curves,

e.g. $k=0$ $\quad \left[\{ g \in GL_n((z)) \text{ s.t.} \dots \} / GL_n[[z]] \right]_d \simeq \mathrm{Hilb}_d\left( \det(\gamma(z) - w) = 0 \right).$

Then our construction reproduces a proposed construction of HOMFLY-PT homology by [Oblomkov-Rasmussen-Shende].

$\rightsquigarrow$

# Summary

$\{\lambda_i(\theta)\}_{i=1}^{n}$

M-theory on $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$ w/ M5 on tight-braid Lagrangian $\widetilde{N^*K}$
M2 on disc $D$ $\quad\partial D\subset\widetilde{N^*K}$

$\rightsquigarrow$ two 3d $\mathcal{N}=4$ theories on $D\times\mathbb{R}_t$

B twist $\xleftrightarrow[\text{3d MS}]{}$ A twist

HOMFLY-PT homology $\simeq$ $\mathrm{Hom}_{Z(S^1)}(\ell_K, b_\lambda)$

B : $Z(S^1)=D^bCoh(\mathrm{Hilb}_n\,\mathbb{C}^2)$ nice!

but object $b_\lambda$ is hard to describe

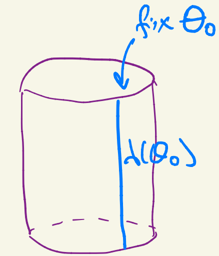
A : $Z(S^1)\simeq D\text{-mod}\left(\mathfrak{gl}_n((z))\times\mathbb{C}^n((z))/GL_n((z))\right)$ eep!

but object $b_\lambda$ is relatively simple,

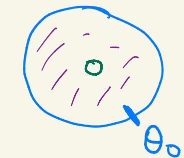
at least for algebraic links — braid is manifest.

## Can we do better?

Yes, in principle. For fixed $\lambda \in \mathbb{C}^n$,
(e.g. fixed $\theta = \theta_0$)




or




$b_{\lambda(\theta_0)}$ defines an object of the 2-category $Z(pt)$

- $\mathrm{End}_{Z(pt)}\, b_{\lambda(\theta_0)}$ is a monoidal category w/ a rep of the braid gp $Br_n$

- The derived center $HH^\bullet Z(pt) \simeq Z(S^1)$.

Thus, given a braid word $\beta \in Br_n$ representing $\lambda_i(\theta)$ for $\theta \in [\theta_0, \theta_0 + 2\pi]$, get an object $\beta \in \mathrm{End}_{Z(pt)}\, b_{\lambda(\theta_0)}$

$$\xrightarrow[\text{image in derived center}]{} \quad b_\lambda \in Z(S^1).$$

I.e. there should be a construction of $b_\lambda$ from braid words.

→

[Oblomkov-Rozansky] implemented this in the 3d B-model,
generalizing constructions of [Kapustin-Rozansky-Saulina]

In particular, $\mathrm{End}_{Z(pt)}\, b_{\lambda(\mathcal{O}_0)} \simeq MF_{GL_n}\big(\underset{\mu}{T^*Flag} \times \underset{X}{\mathfrak{gl}_n} \times \mathbb{C}^n \times \underset{\mu'}{T^*Flag},\ W = \mathrm{Tr}[X(\mu-\mu')]\big)$

$$\uparrow$$
$$Br_n$$

used this to get the sheaf $b_\lambda \in D^bCoh(\mathrm{Hilb}_n \mathbb{C}^2)$ !

There should be a 3d mirror of this on the A side

$\rightsquigarrow$ work in progress!

Thank you