2d TQFT's and partial fractions

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Topological Quantum Field Theory Club

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(joint with M.Khovanov and
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d-dim'l

<u>Definition</u> (M.Atiyah): A TQFT is a symmetric tensor functor Cob_d→Vec.

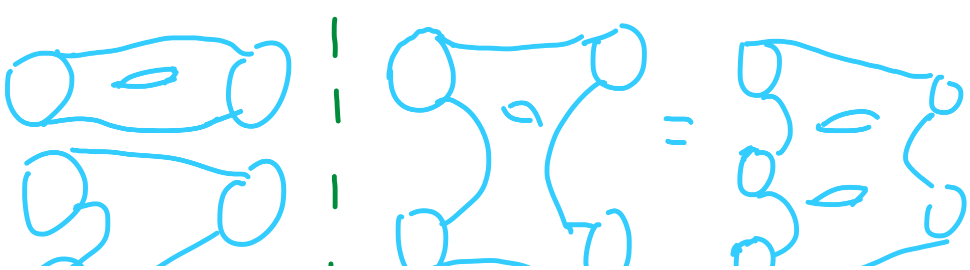
Generalization: replace Vec by a symmetric tensor category C
C-valued TQFT: symmetric tensor functor Cob_d→C.

Examples: C=Vec, sVec, Rep(G) etc

Today: d=2!

Category of cobordisms Cob_d: d=2

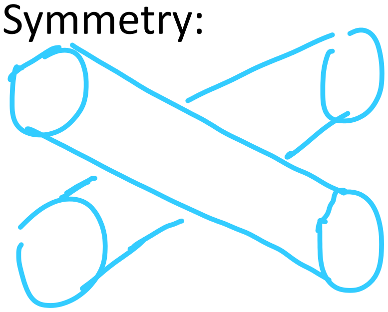
Objects: (d-1)-dimensional closed oriented manifolds
Morphisms: d-dimensional oriented cobordisms
Composition: gluing



Identity morphisms=?

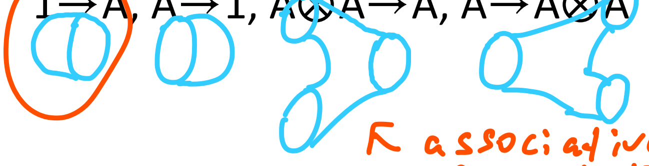
Tensor product: disjoint union
Unit object=?

Symmetry:



Some important objects of Cob_2:
empty set=1 and circle=A

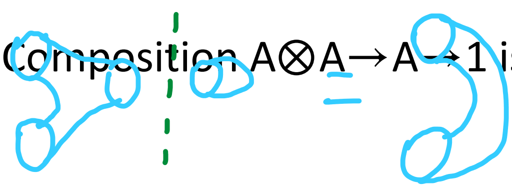
Some important morphisms:
$1 \to A$, $A \to 1$, $A \otimes A \to A$, $A \to A \otimes A$



associative
commutative

unital

Composition $A \otimes A \to A \to 1$ is non-degenerate



$A \cong A^*$

Thus A is a commutative Frobenius object in Cob_2:
commutative associative unital monoid equipped with a map $A \to 1$ such that the composition $A \otimes A \to A \to 1$ is a non-degenerate pairing.

<u>Theorem</u> (R.Dijkgraaf + folklore): Cob_2 is free category

def

generated by the commutative Frobenius object A.

Corollary: C-valued 2d TQFT's = Functors Cob_2→C = commutative Frobenius objects in C.

TQFT output: values at closed d-manifolds— elements of Hom_C(1,1)

α_0 α_1 α_2 etc

Linear setup: choose a field k
C — k-linear category
Hom_C(1,1)=k

Frobenius object = Frobenius algebra

TQFT output: sequence $\alpha_0, \ \alpha_1, \alpha_2,$ .... of elements of k

Main Question: Which sequences we will observe?

Answer 1: all sequences appear

Take C = VCob_α:= linearized Cob_2 modulo relations

etc

Hom' are fin dim'l

Hom(A,A) =

Realization of sequence $\alpha = (\alpha_0, \ \alpha_1, \alpha_2,$ ....) : pair (C,A) such that the corresponding TQFT outputs α.

Realization is finite if Hom spaces in C are finite dimensional.

Answer 2 (M.Khovanov): sequence α admits a finite realization if and only if it is eventually recurrent ~~linearly recursive,~~ that is the generating function Z(T)= $\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \ldots$ is rational.

$X = F(\ldots) \in Hom_C(A, A)$

★ $1, X, X^2, X^3, \ldots$ are linearly dependent

e.g. $X^3 - 2X^4 + 3X^5 = 0$ negligible

hence $X^n - 2X^{n+1} + 3X^{n+2} = 0$ for $n \geq 3$

hence $\alpha_n - 2\alpha_{n+1} + 3\alpha_{n+2} = 0$ for $n \geq 3$

$\alpha_n = F(\ldots) = F(\ldots)\, X^n\, F(\ldots)$

Khovanov - Sazdanovic

Existence: some quotients SCob_α of linearized VCob_2.

★ Realization is abelian if C is abelian, rigid, with finite dimensional Hom spaces.

<u>Answer 3</u> (M.Khovanov, Y.Kononov, V.O.): sequence α admits an abelian realization if and only if it is

1) linearly recursive, Z(T)=$\frac{P(T)}{Q(T)}$ with relatively prime P(T) and Q(T). alg closure of K

2) Q(T) has no ~~multiple~~ roots (in K) and deg P $\leq$ deg Q+1. γ_i are distinct and in char $K=0$

Thus Z(T)=$\delta_0 + \delta_1$T+ $\sum_i \frac{\beta_i}{1 - \gamma_i T}$

3) If char k=p>0 then δ_1 and $\beta_i \gamma_i$ (for all i) are in the prime

subfield $F_p \subset$ k

Remark: conditions 2) and 3) can be expressed in terms of 1-form Z(T)$\dfrac{dT}{T^2}$:

2. all its poles are simple except, possibly, at T=0
3. All its residues are in F_p

Examples:

$C = Vec$, $A = K$

$A \to K$ id

1) α=1,1,1,1,... Thus Z(T)=$\dfrac{1}{1-T}$ and α has an abelian realization over any field.

2) α=1,2,3,4,5,... Thus Z(T)=$\dfrac{1}{(1-T)^2}$ and α has no abelian realization over any field

3) α=1,2,1,2,1,2,... Thus Z(T)=$\dfrac{1+2T}{1-T^2} = \dfrac{\frac{3}{2}}{1-T} + \dfrac{-\frac{1}{2}}{1+T}$, so α has an abelian realization over any field of characteristic not 2.

4) α=1,1,2,3,5,8,13,... Lucas Thus Z(T)=$\dfrac{1}{1-T-T^2}$

α has an abelian realization: p=0,11,19,29,31,41,

α has no abelian realization: p=2,3,5,7,13,17,23,37

$\leftarrow t \in \mathbb{C}$

Note that: α=-1,2,1,3,4,7,11,... has an abelian realization over any field.

1, 1, 0, 0, 0, 0, ...

dim (A)

Remark: abelian realization in characteristic p>0 implies a realization with C=Vec. This is NOT the case in characteristic 0. Deligne categories (like Rep(S_t)) of super-exponential growth are needed.

Example:

1) α=1,2,1,2,1,2,... so Z(T)=$\dfrac{\frac{3}{2}}{1-T}+\dfrac{-\frac{1}{2}}{1+T}$; realization of α requires Rep(S_t) with t=3/2 and t=1/2.

2) α=3,1,3,1,3,1,... Here Z(T)=$\dfrac{2}{1-T}+\dfrac{1}{1+T}$; realization of α requires Rep(S_{-1}).

Answer 4 (well known?) Assume p=0 and Z(T)=$\delta_0+\delta_1$T $\cancel{+\sum_i \frac{\beta_i}{1-\gamma_i T}}$ (for $\delta_1=0$ and $\delta_0=0$)

Sequence α admits an abelian realization of exponential growth if and only if δ_1 is an integer and $\beta_i\gamma_i$ (for all i) are integers >0.

This implies a realization with C=sVec.

Sequence α admits an realization with C=Vec if and only if δ_1 is an integer >~~1 and β_i~~γ_i (for all i) are integers >0.

Crucial tool: quotients by negligible morphis~~ms (“gligible”~~ quotients) aka “semisimplifications”.

Theorem of Andre-Kahn (abstract version of Jannsen theorem) implies:

true in general

Corollary: α admits an abelian realization if and only if the gligible quotient Cob_α of SCob_α is semisimple.

Important computation: compute categories Cob_α.

Results for α with abelian realization (k algebraically closed):

$p=0$

1) if Z(T)=sum of partial fractions then the gligible quotient is a product of quotients for each summand.

Thus if Z(T)=$\delta_0+\delta_1$T+$\sum_i \frac{\beta_i}{1-\gamma_i T}$ we need to consider only the following cases:

$p=0$ $2T$

2) If Z(T)=$\frac{\beta}{1-\gamma T}$ then the semisimplification of Cob_2(α) is semisimplification of Rep(S_t) with t=$\beta\gamma$ (exceptional values of t: non-negative integers)

$x^2-1=(x-1)^2$

3) if Z(T)=$\delta_0+\delta_1$T with $\delta_1\neq 0$ then the semisimplification of Cob_2(α) is semisimplification of Rep(O_t) with t=δ_1-2 (exceptional values of t: integers)

4) if Z(T)=δ_0 then the semisimplification of Cob_2(α) is Rep(osp(1|2)) or, in characteristic p, a semisimplification

Example: Z(T)=1+2T. In characteristic $\neq 2$ we get C=Vec and A=k$[x]/x^2$. In characteristic 2 we get C=Z/2Z-graded vector spaces and A=k$[x]/(x^2-1)$.

$$\sum_{n\geq 0} a_n \frac{t^n}{n!} = \exp(2\exp(t)-2)$$

What about sequences without abelian realization, e.g. 1,2,3,4,5,...?

We can start by finding dimensions of Hom spaces in the gligible quotients Cob_α, e.g. Hom(1,$A^{\otimes n}$)=:a_n.

Theorem. Assume b_2 is nonzero. Then we have

Conjecture: Assume Z(T)=$\frac{1}{(1-T)^2}$ or, more generally, Z(T)=$\frac{\beta}{(1-\gamma T)^2}$. Then the category SCob_α has no negligible morphisms, i.e. SCob_α =Cob_α . Thus the generating function

$$\sum_{n\geq 0} t^n \frac{a_n}{n!} = \exp(3t + t^2 + t^3/6)$$

$-1,-1,-1,\dots \sim Rep(S_{-1})$

$t,t,t,t,\dots \sim Rep(S_t)$

What if Z(T)=quadratic polynomial=b_0+b_1T+b_2T^2. Then the category SCob_α does have negligible morphisms, i.e. SCob_$\alpha \neq$ Cob_α .

main copy

1. Gram determinants

n	B_n	det
1	1	β
2	2	$\beta^2(\beta\gamma-1)$
3	5	$\beta^5(\beta\gamma-1)^4(\beta\gamma-2)$
4	15	$\beta^{15}\gamma(\beta\gamma-1)^{14}(\beta\gamma-2)^7(\beta\gamma-3)$
5	52	$\beta^{52}\gamma^{10}(\beta\gamma-1)^{51}(\beta\gamma-2)^{36}(\beta\gamma-3)^{11}(\beta\gamma-4)$
6	203	$\beta^{203}\gamma^{73}(\beta\gamma-1)^{202}(\beta\gamma-2)^{171}(\beta\gamma-3)^{81}(\beta\gamma-4)^{16}(\beta\gamma-5)$
7	877	$\beta^{877}\gamma^{490}(\beta\gamma-1)^{876}(\beta\gamma-2)^{813}(\beta\gamma-3)^{512}(\beta\gamma-4)^{162}(\beta\gamma-5)^{22}(\beta\gamma-6)$

Table 1. Determinants of the bilinear form on $A(n)$ for the generating function $Z(T)=\frac{\beta}{1-\gamma T}$.

n	$B_n^{(2)}$	det
1	2	$-\beta^2$
2	6	$-\beta^{10}\gamma^{12}$
3	22	$-\beta^{50}\gamma^{66}$
4	94	$-\beta^{266}\gamma^{376}$
5	454	$-\beta^{1522}\gamma^{2270}$

TABLE 2. Two-colored Bell numbers and Gram determinants for the function $Z(T) = \beta/(1-\gamma T)^2$.

n	dim	det
1	2	$-(\beta_0\gamma+\beta_1)^2$
2	6	$-\gamma^2(\beta_0\gamma+\beta_1)^{10}$
3	22	$-\gamma^{16}(\beta_0\gamma+\beta_1)^{50}$
4	94	$-\gamma^{110}(\beta_0\gamma+\beta_1)^{266}$
5	454	$-\gamma^{748}(\beta_0\gamma+\beta_1)^{1522}$

TABLE 3. Dimensions and determinants for the function $Z(T) = (\beta_0 + \beta_1 T)/(1-\gamma T)^2$. The difference with the previous table is $\beta_0\gamma + \beta_1$ taking place of β.

2. POLYNOMIAL GENERATING FUNCTIONS

n	$B_n^{(2)}$	$\dim A(n)$	det
1	2	2	$-\beta_1^2$
2	6	5	$(\beta_1-2)\,\beta_1^8$
3	22	14	$-(\beta_1-2)^6\beta_1^{30}$
4	94	43	$(\beta_1-3)^2(\beta_1-2)^{27}\beta_1^{113}$
5	454	142	$-(\beta_1-3)^{20}(\beta_1-2)^{110}\beta_1^{440}$
6	2430	499	$(\beta_1-4)^5(\beta_1-3)^{134}(\beta_1-2)^{435}\beta_1^{1774}(\beta_1+2)$
7	14214	1850	$-(\beta_1-4)^{70}(\beta_1-3)^{756}(\beta_1-2)^{1722}\beta_1^{7406}(\beta_1+2)^{14}$

TABLE 4. Determinants of the bilinear form on $A(n)$ for the generating function $Z(T) = \beta_0 + \beta_1 T$. Notice the appearance of the term $\beta_1 + 2$ in the last two lines.

Prediction for $n = 8$ (determinant of size 7193):

$$(\beta_1-5)^{14}(\beta_1-4)^{630}(\beta_1-3)^{3912}(\beta_1-2)^{6937}(\beta_1-1)^{14}\beta_1^{31931}(\beta_1+2)^{133}(\beta_1+4).$$

n	$B_n^{(3)}$	dim	det
0	1	1	1
1	3	3	$-\beta_2^3$
2	12	11	$-\beta_2^{20}$
3	57	46	β_2^{118}
4	309	213	β_2^{696}
5	1866	1073	$-\beta_2^{4225}$

TABLE 5. Computation of dimensions and the determinant for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$.

n	$B_n^{(4)}$	dim	det
0	1	1	1
1	4	4	β_3^4
2	20	19	$-\beta_3^{35}$
3	116	102	$-\beta_3^{266}$
4	756	604	β_3^{2007}
5	5428	3884	β_3^{15540}

TABLE 6. Computation of dimensions and the determinant for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2 + \beta_3 T^3$.

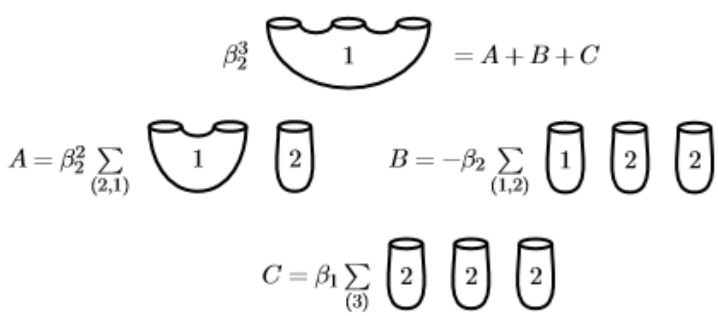


FIGURE 2.0.1. Relation in $A(3)$ for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$. Numbers 1 and 2 show the number of handles (dots) on the component. Summation means symmetrization with respect to permutations of boundary components parametrized by cosets of the stabilizer of the surface in S_3. Sums A, B, C have 3, 3, 1 terms respectively (7 terms in the right hand side in total).

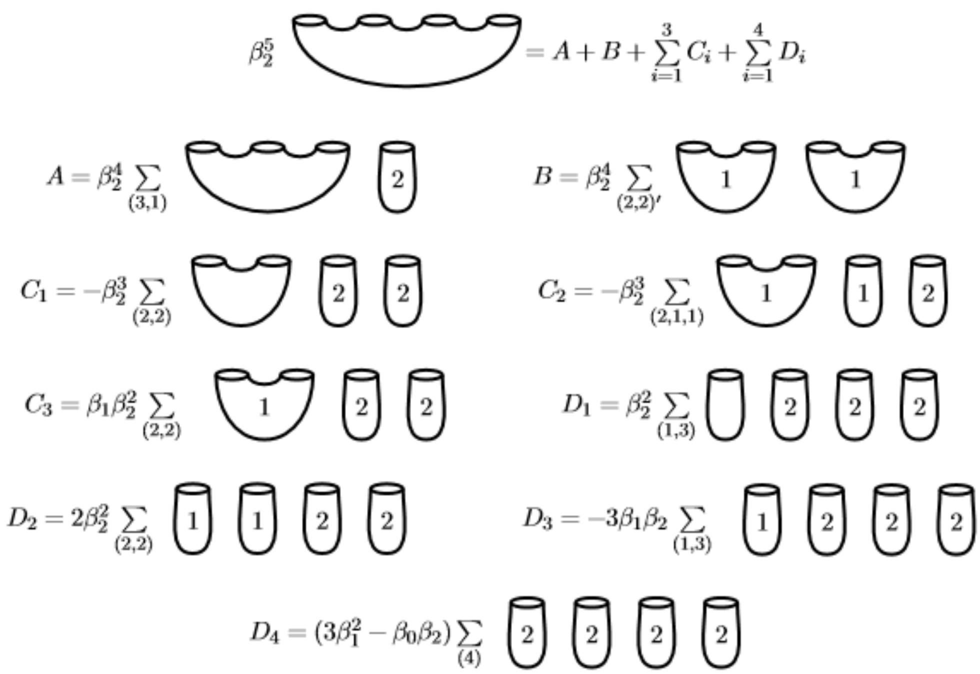


FIGURE 2.0.2. Relation in $A(4)$ for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$. Numbers 1 and 2 show the number of handles (dots) on the component. Summation means symmetrization with respect to permuting the boundary components, as described in the proof. Sums $A, B, C_1, C_2, C_3, D_1, D_2, D_3, D_4$ have 4, 3, 6, 12, 6, 4, 6, 4, 1 terms respectively (46 terms in the right hand side in total).

FIGURE 2.0.3. Relations in $A(2)$ and $A(1)$ for $Z(T) = \beta_0 + \beta_1 T + \beta_2 T^2$.

Numbers 2 and 3 show the number of handles (dots) on the component.