# Klein TQFT and real Gromov-Witten theory

Penka Georgieva

Institut de mathématiques de Jussieu – Paris Rive Gauche
Sorbonne Université

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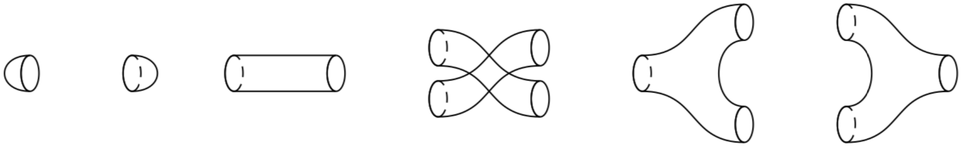
# Overview

- 2d TQFT and 2d Klein TQFT
- Semi-simplicity
- Extension to 2**SymCob**$^L$
- GW and RGW invariants
- Local theory and its KTQFT

- The category 2**Cob** :
  - Objects : disjoint unions of oriented circles
  - Morphisms : oriented surfaces with boundary
- Generators :

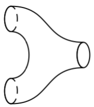
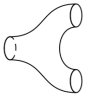


The elementary cobordisms: cap, cup, tube, twist and the pairs of pants

# 2d TQFT

- A *2d TQFT with values in a commutative ring R* is a symmetric monoidal functor from the category of oriented 2-dimensional cobordisms to the category of $R$-modules.

$$\mathcal{F} : 2\textbf{Cob} \longrightarrow Rmod$$

- A 2d TQFT $\mathcal{F}$ is equivalent to a commutative Frobenius algebra $H$ over $R$.
  - $\mathcal{F}(S^1) = H$
  - unit $\mathcal{F}\left( \right.$  $\left. \right) = 1 \in H$
  - product $\mathcal{F}\left( \right.$  $\left. \right) : H^{\otimes 2} \longrightarrow H$
  - coproduct $\mathcal{F}\left( \right.$  $\left. \right) : H \longrightarrow H^{\otimes 2}$
  - counit $\mathcal{F}\left( \right.$  $\left. \right) : H \longrightarrow R$

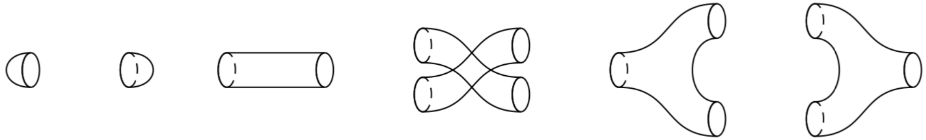
# 2d Klein TQFT

(Forget orientations/orientability)

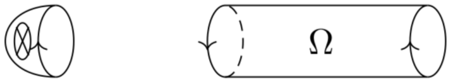
- The category 2**KCob** :
  - Objects : disjoint unions of circles
  - Morphisms : surfaces with boundary (possibly non-orientable)

- 2**Cob** $\subset$ 2**KCob**

- Generators :



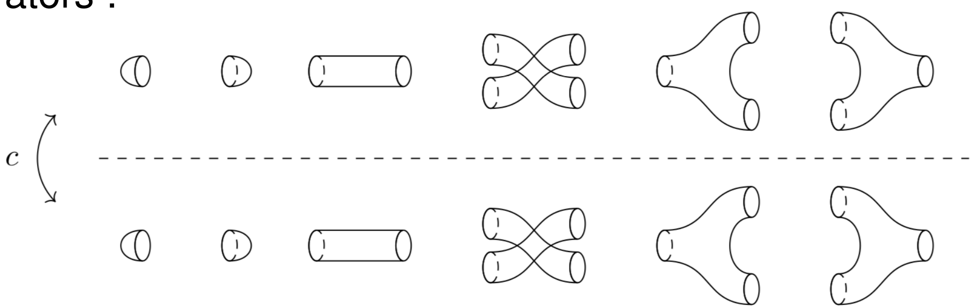
together with the cross-cap (Möbius band) and the involution



# 2d KTQFT

(A useful perspective : consider the orientation double cover of both object and morphisms)

- The category 2**SymCob** :
  - Objects : disjoint unions of pairs circles $\mathcal{S} = (S^1 \sqcup \overline{S}^1, \epsilon)$
  - Morphisms : oriented surfaces with boundary equipped with an orientation reversing involution *c* covering $\epsilon$ (no fixed points).
- 2**KCob** $\cong$ 2**SymCob**
- Generators :

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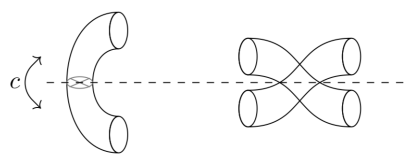
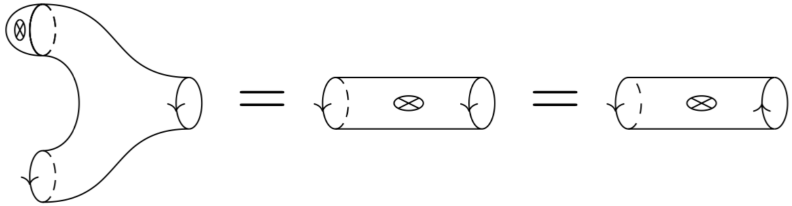
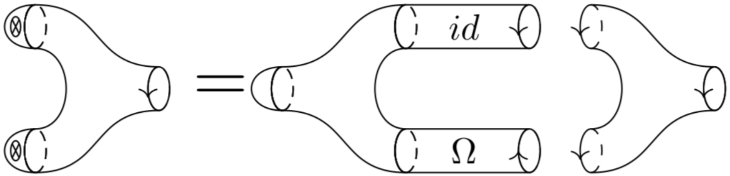


## Special relations

- the involution acts trivially on the product of the cross-cap with another element



- decomposition of the Klein bottle



- A *2d Klein TQFT with values in a commutative ring R* is a symmetric monoidal functor

$$\mathcal{F} : 2\mathbf{KCob} \longrightarrow Rmod$$

(or equivalently $\widetilde{\mathcal{F}} : 2\mathbf{SymCob} \longrightarrow Rmod$).

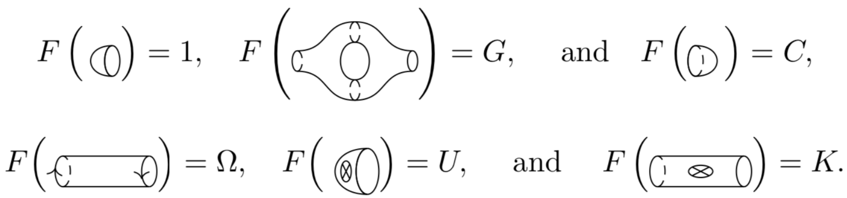
- A 2d KTQFT $\mathcal{F}$ is equivalent to a commutative Frobenius algebra $H = \mathcal{F}(S^1)$ together with two extra structures :
  - an involutive (anti)-automorphism $\Omega$ of the Frobenius algebra $H$, denoted $x \mapsto x^*$.
  - an element $U \in H$ such that

$$(aU)^* = aU \text{ for all } a \in H \text{ and}$$

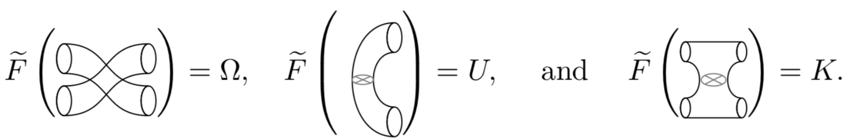
$$U^2 = m(id \otimes \Omega)(\Delta(1)) = \sum_i \alpha_i \beta_i^* , \quad \text{where the coproduct } \Delta(1) = \sum_i \alpha_i \otimes \beta_i$$

The automorphism $\Omega$ corresponds to the involution in 2**KCob** and the element $U$ to the cross-cap.

These elements play a special role :




and equivalently in 2**SymCob**



- A *semi-simple* (Klein) TQFT is a (Klein) TQFT whose associated Frobenius algebra is semi-simple.
- A semi-simple TQFT is determined by the structure constants $\lambda_\rho$, i.e. the coefficients of the co-multiplication $\Delta(v_\rho) = \lambda_\rho v_\rho \otimes v_\rho$ in the idempotent basis $v_\rho$.
- If $\mathcal{F}$ is a semi-simple KTQFT :

(i) $G(v_\rho) = \lambda_\rho v_\rho$ *and* $C(v_\rho) = \lambda_\rho^{-1}$.
(ii) $\Omega$ *defines an involution on the idempotent basis* $\Omega(v_\rho) = v_{\rho^*}$.
(iii) *If* $U = \sum_\rho U_\rho v_\rho$ *then* $U_\rho^2 = \lambda_\rho$ *if* $\rho = \rho^*$, *and* $U_\rho = 0$ *if* $\rho \neq \rho^*$.
(iv) $K(v_\rho) = U_\rho v_\rho$.

If $\Sigma$ is a closed symmetric surface of genus $g$, considered as a morphism in 2**SymCob** from the ground ring to the ground ring, we have

$$\widetilde{\mathcal{F}}(\Sigma) = \sum_{\rho=\rho^*} U_\rho^{g-1} .$$

and for a doublet $(\Sigma \sqcup \overline{\Sigma})$

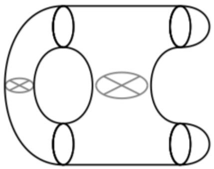
$$\widetilde{\mathcal{F}}(\Sigma \sqcup \overline{\Sigma}) = \sum_{\rho} \lambda_\rho^{g-1}$$

recovering the classical theory.
In particular,

$$\widetilde{F}(\mathbb{P}^1) = CU = \sum_{\rho} \lambda_\rho^{-1} U_\rho = \sum_{\rho=\rho^*} \lambda_\rho^{-1} U_\rho = \sum_{\rho=\rho^*} U_\rho^{-1}$$

# 2d KTQFT

$$\widetilde{F}(T^2) = \widetilde{F}\Bigg($$



$$\Bigg) = CKU$$

$$\widetilde{F}(\Sigma) = CK^gU = \sum_{\rho} \lambda_\rho^{-1} U_\rho^g U_\rho = \sum_{\rho=\rho^*} U_\rho^{g-1}$$

$$\widetilde{F}(\Sigma \sqcup \overline{\Sigma}) = CG^g(1) = \sum_{\rho} \lambda_\rho^{g-1}$$

Motivation : we are interested in the real Gromov-Witten theory of $Tot(L \oplus c^*\overline{L} \longrightarrow \Sigma)$.

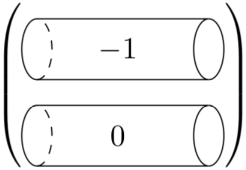
- The category $2\textbf{SymCob}^L$ :
  - Objects : disjoint unions of pairs circles $\mathcal{S} = (S^1 \sqcup \bar{S}^1, \epsilon)$
  - Morphisms : oriented surfaces with boundary equipped with an orientation reversing involution $c$ covering $\epsilon$ (no fixed points) together with a complex line bundle $L$ trivialized over the boundaries.

  The Euler class of $L$ determines the bundle up to isomorphism and is the only additional data that we record.
- Generators : same as before together with

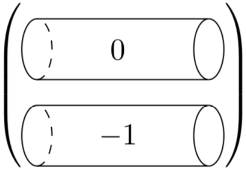
- A 2d Klein TQFT on 2**SymCob**$^L$ is a symmetric monoidal functor

$$\widetilde{\mathcal{F}} : 2\textbf{SymCob}^L \longrightarrow Rmod.$$

It is completely determined by the level 0 theory together with the level-decreasing operators

$A = F$ 


and

$\bar{A} = F$ 

If the theory is semi-simple with idempotent basis $\{v_\rho\}$ then

$$A(v_\rho) = \eta_\rho v_\rho \quad \text{and} \quad \overline{A}(v_\rho) = \overline{\eta}_\rho v_\rho.$$

If $\Sigma$ is a genus $g$ symmetric surface and $L \longrightarrow \Sigma$ a complex line bundle of Chern class $k$

$$\widetilde{\mathcal{F}}(\Sigma|L) = CA^{-k}K^g U = \sum_{\rho=\rho^*} U_\rho^{g-1}\eta_\rho^{-k}.$$

The value of $\mathcal{F}$ on a $g$-doublet $\Sigma \sqcup \overline{\Sigma}$ with a line bundle $L_1 \sqcup L_2$ is similarly equal to

$$\widetilde{\mathcal{F}}(\Sigma \sqcup \overline{\Sigma}|L_1 \sqcup L_2) = CA^{-k_1}\overline{A}^{-k_2}G^g(1) = \sum_\rho \lambda_\rho^{g-1}\eta_\rho^{-k_1}\overline{\eta}_\rho^{-k_2},$$

where $k_i$ denotes the Chern class of $L_i$.

# Symplectic manifolds

- A symplectic manifold $(X, \omega)$ is a smooth manifold $X$ together with a closed non-degenerate 2-form $\omega$.
  *Examples:* Cotangent bundles $T^*M$, Kähler manifolds (e.g. $\mathbb{CP}^n$).
- On a symplectic manifold $(X, \omega)$ there exists an infinite dimensional family of almost-complex structures $J$ on $X$ compatible with $\omega$. That is

$$J : TX \longrightarrow TX, \quad J^2 = -Id$$

  and

$$g(u, v) = \omega(u, Jv)$$

  is a Riemannian metric.
- A pseudo-holomorphic map is a map from a Riemann surface $(\Sigma_g, j)$ to $X$

$$u : (\Sigma_g, j) \longrightarrow (X, J), \qquad J \circ du = u \circ j.$$

Let $(X, \omega)$ be a symplectic manifold and $J$ a compatible almost complex structure.

- Moduli space of stable marked pseudo-holomorphic maps

$$\overline{\mathcal{M}}_{g,l}(X, A) = \{u : (\Sigma_g, j, x_1, .., x_l) \to X, [u] = A \in H_2(X), J \circ du = du \circ j\} \, / Diff.$$

- There are natural maps

$$ev_i : \overline{\mathcal{M}}_{g,l}(X, A) \longrightarrow X, \quad [u, j, x_1, .., x_l] \mapsto u(x_i)$$

and

$$\mathfrak{f} : \overline{\mathcal{M}}_{g,l}(X, A) \longrightarrow \overline{\mathcal{M}}_{g,l}, \quad [u, j, x_1, .., x_l] \mapsto [j, x_1, .., x_l].$$

- The Gromov-Witten invariants are defined as

$$GW_g^{X,A}(\alpha_1, \dots, \alpha_l) = \int_{\overline{\mathcal{M}}_{g,l}(X,A)} \wedge_i ev_i^* \alpha_i,$$

where $\alpha_i$ representatives of cohomology classes on $X$.

# GW generating series

- The Gromov-Witten potential is a formal power series in formal variables keeping track of the genus, the homology class of the map, and the cohomology classes of the constraints.
- Examples :
  - Generating function of a Calabi-Yau 3-fold :
    $$GW(u,q) = \sum GW_g^{X,A} q^A u^{2-2g} \quad \text{and} \quad Z = \exp(GW(q,u)).$$
  - Generating function $GW_{g=0}(\mathbb{CP}^2) = \sum_d N_d q^d$ - generating series of the counts $N_d$ of genus 0 degree $d$ curves in $\mathbb{CP}^2$ passing through $3d-1$ points.
    $$N_1 = 1 \quad N_2 = 1 \quad N_3 = 12 \quad N_4 = 620 \quad N_5 = 87304 \quad \ldots$$

- $(X, \omega, \phi)$ a symplectic manifold with an involution $\phi : X \to X$ satisfying $\phi^*\omega = -\omega$.
  *Example*: $\mathbb{CP}^n$ with the standard conjugation.
- The maps invariant under $\phi$ form the real moduli space $\overline{\mathcal{M}}_g^{\phi}(X, A)$.

$$\overline{\mathcal{M}}_g^{\phi}(X, A) = \bigcup_{\sigma} \overline{\mathcal{M}}_g^{\phi,\sigma}(X, A),$$

  where $\sigma$ is an orientation-reversing involution on the domain and

$$\overline{\mathcal{M}}_g^{\phi,\sigma}(X, A) = \{u : \Sigma_g \to X, [u] = A \in H_2(X), J \circ du = du \circ j, d\phi \circ u = u \circ \sigma\} / Diff.$$

- The real Gromov-Witten invariants are defined as

$$RGW_g^{X,A}(\alpha_1, \ldots, \alpha_l) = \int_{\overline{\mathcal{M}}_{g,l}^{\phi}(X,A)} \wedge_i ev_i^* \alpha_i.$$

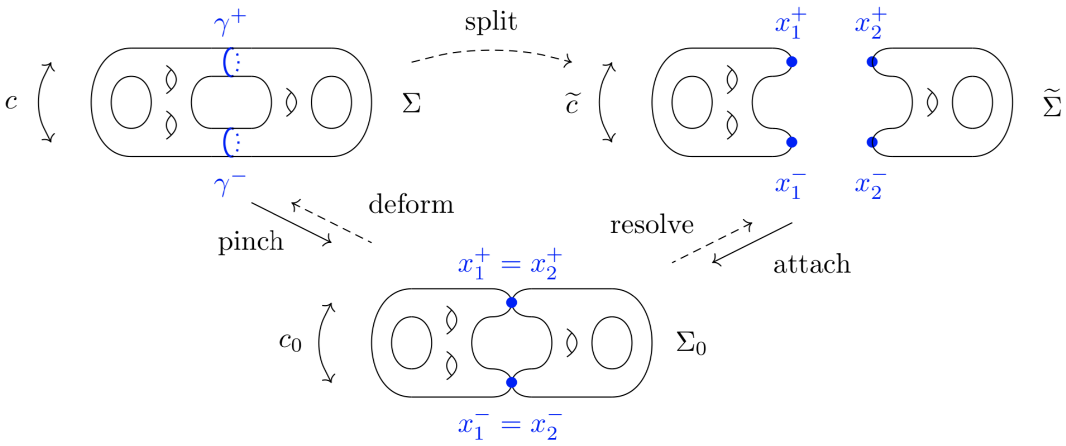
# Local theory

- Let $(\Sigma_g, c)$ be a genus $g$ symmetric surface, $L \to \Sigma_g$ a complex line bundle, and $(X, \phi) = Tot(L \oplus c^*(\bar{L}))$.
- The real GW theory of such targets is called local theory.
- The disconnected partition function $\mathbb{R}Z_d(X) = exp(RGW_d(X) + \frac{1}{2}GW_d(X))$ satisfies

$$\mathbb{R}Z_d(X) = \sum_{\lambda \vdash d} \mathbb{R}Z_d(X_1)_\lambda \mathbb{R}Z_d(X_2)^\lambda,$$

where $\mathbb{R}Z_d(X)_\lambda$ denotes the restricted count of maps with given ramification profile fixed by a partition $\lambda$ and $\mathbb{R}Z_d(X)^\lambda = \zeta(\lambda)\mathbb{R}Z_d(X)_\lambda$.

This comes from degeneration of the target and a splitting formula for the invariants



Let $R = \mathbb{C}(t)((u))$.

$\mathbf{RGW_d} : 2\mathbf{SymCob}^L \longrightarrow Rmod$

$\mathbf{RGW_d}(S^1 \sqcup \overline{S}^1) = H = \bigoplus_{\alpha \vdash d} Re_\alpha$

To a cobordism $(\Sigma, L) \in 2\mathbf{SymCob}$ from $n$-copies of $\mathcal{S}$ to $m$-copies of $\mathcal{S}$, $\mathbf{RGW_d}$ associates $R$-module homomorphism

$\mathbf{RGW_d}(\Sigma, L) : H^{\otimes n} \longrightarrow H^{\otimes m}$

$e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_n} \mapsto \sum_{\mu_1,\ldots,\mu_m} \mathbb{R}Z_d(\Sigma_g, L)^{\mu_1,\ldots,\mu_m}_{\lambda_1,\ldots,\lambda_n} e_{\mu_1} \otimes \cdots \otimes e_{\mu_m}.$

The $\mathbf{RGW_d}$ theory is semi-simple with idempotent basis

$$v_\rho = \frac{\dim \rho}{d!} \sum_\alpha (-t)^{\ell(\alpha)-d} \chi_\rho(\alpha) e_\alpha.$$

Furthermore, the structure constants and the level-decreasing operators are given by

$$\lambda_\rho = t^{2d} \left( \frac{d!}{\dim \rho} \right)^2, \quad \eta_\rho = t^d Q^{c_\rho/2} \left( \frac{\mathrm{dimh}_Q \rho}{\dim \rho} \right), \quad \overline{\eta}_\rho = t^d Q^{-c_\rho/2} \left( \frac{\mathrm{dimh}_Q \rho}{\dim \rho} \right)$$

where $Q = e^u$, $c_\rho$ is the total content of the Young diagram associated to $\rho$, and

$$\mathrm{dimh}_Q \rho = d! \prod_{\square \in \rho} \left( 2 \sinh \frac{h(\square) u}{2} \right)^{-1} = d! \prod_{\square \in \rho} \left( Q^{\frac{h(\square)}{2}} - Q^{-\frac{h(\square)}{2}} \right)^{-1}$$

where $h(\square)$ denotes the hooklength.

- The involution $\Omega$ is given by

$$\Omega(e_\alpha) = (-1)^{d-\ell(\alpha)} e_\alpha \quad and \quad \Omega(v_\rho) = v_{\rho'}$$

where $\rho'$ denotes the conjugate representation.

- The crosscap $U$ is given by

$$U = \sum_{\substack{\rho \vdash d \\ \rho = \rho'}} (-1)^{(d-r(\rho))/2} \, t^d \frac{d!}{\dim \rho} v_\rho$$

where $r(\rho)$ is the length of the main diagonal of the Young diagram of $\rho$.

This is very non-trivial - it uses

– a signed Frobenius-Schur indicator which recognizes self-conjugate representations

– Weyl formula for $B_n$

Let $(\Sigma, c)$ be a genus $g$ symmetric surface, $L \to \Sigma$ a line bundle of degree $g-1$ and $(X, \phi) = Tot(L \oplus c^*(\bar{L}))$.

**Theorem (G.-Ionel)**

$$\mathbb{R}Z(X) = \sum_{\rho \vdash d, \rho = \rho^*} \left( \epsilon(\rho) \prod_{\square \in \rho} 2\sinh\left(\frac{h(\square)u}{2}\right) \right)^{g-1},$$

*where* $\epsilon(\rho) = (-1)^{\sum_{\alpha\, odd} \frac{\chi^\rho(\alpha^2)}{\zeta(\alpha)}} = (-1)^{\frac{|\rho| - r(\rho)}{2}}$.

Coincides with the computation of V. Bouchard, B. Florea, and M. Marinõ of the partition function of SO/Sp Chern-Simons theory on $S^3$.

# Real Gopakumar-Vafa formula

Let $(\Sigma, c)$ a genus $g$ symmetric surface, $L \to \Sigma$ a line bundle of degree $g-1$ and $(X, \phi) = Tot(L \oplus c^*(\bar{L}))$.

**Theorem (G.-Ionel)**

$$\sum_d RGW_{d,h} q^d u^{h-1} = \sum_{d \neq 0, h \geq 0} n^{\mathbb{R}}_{d,h}(g) \sum_{k>0, k\ odd} \frac{1}{k}\left(2\sinh\left(\frac{ku}{2}\right)\right)^{h-1} q^{kd},$$

*where* $n^{\mathbb{R}}_{d,h}(g) \in \mathbb{Z}$.

Merci!