Cyclotomic expansions for $\mathfrak{gl}_N$ knot invariants

joint w/ E. Gorsky

Plan

Ⓘ Habiro's theory for $\mathfrak{sl}_2$ + Habiro-Le

Ⓘⓘ Interpolation

Ⓘⓘⓘ Results

(I) Habiro's theory

$$U_q(sl_2) = \left\langle E, F, K \;\middle|\; \begin{array}{l} KE = qEK \\ KF = q^{-1}FK \\ EF - FE = \frac{K-K^{-1}}{v-v^{-1}} \end{array} \right\rangle \qquad v^2 = q$$

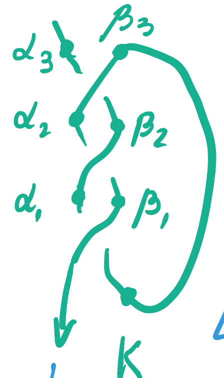
$Z(U_q sl_2)$ is generated by Casimir

$$C = (v-v^{-1})^2 FE + vK + v^{-1}K^{-1}$$

$(U_q sl_2,\ R = \alpha \otimes \beta,\ v)$ is a ribbon Hopf algebra

Given a knot K, its universal invariant

$$J_K \in Z(U_q sl_2)$$



$$J_{3_1} = \alpha_3 \beta_2 \alpha_1 K \beta_3 \alpha_2 \beta_1$$

Lemma (Habiro) For a 0-framed knot K

$$J_K \in Z(U_q sl_2) \cap U^{ev}$$

where $U^{ev} = \langle E^a K^{2b+c} F^c \rangle$

Thm (Habiro) Given a 0-framed knot K

$\exists \; a_m(K) \in \mathbb{Z}[q^{\pm 1}]$

$$J_K = \sum_{m=0}^{\infty} a_m(K)\, \sigma_m$$

If $m > n$

$$\sigma_m = \prod_{i=1}^{m} \left(C^2 - (v^i + v^{-i})^2\right) \qquad \sigma_m \big|_{V_n} = 0$$

Corollary: J_K dominates all colored Jones polynomials.

Let V_n be n-dim. irrep. Then from J_K we get

$$J_K(V_n) = \sum_{m=0}^{n} a'_m(K)\, (q^{1+n}; q)_m\, (q^{1-n}; q)_m$$

cyclotomic expansion of the colored Jones

$$(x; q)_m = (1-x)(1-xq)\cdots(1-xq^{m-1})$$

Proof:

$$C|_{V_n} = v^n + v^{-n}$$

V_n n-dim. irrep

$$\sigma_m|_{V_n} = \prod_{i=1}^{m} (v^n + v^{-n})^2 - (v^i + v^{-i})^2$$

$$= \prod_{i=1}^{m} (v^{n+i} - v^{-n-i})(v^{n-i} - v^{i-n})$$

Applications to 3-mnf

Let ξ be r^{th} root of unity

$M = S^3_{\pm 1}(K)$ a $\mathbb{Z}$-homology 3-sphere

$$F_K(\xi) = ev_\xi \left(\sum_{n=0}^{r-1} q^{\pm \frac{n^2-1}{4}} [n]^2 J_K(V_n) \right) = J_K(\Omega_\pm)$$

U is the unknot

$$WRT_M(\xi) = \frac{F_K(\xi)}{F_{U_\pm}(\xi)}$$

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}$$

Witten-Reshetikhin-Turaev invariant

Thm (Habiro) $\forall$ M $\mathbb{Z}HS$

$\exists!$ unified invariant $I_M \in \widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/(q;q)_n$

- $ev_\xi\, I_M = WRT_M(\xi)$

$\Rightarrow$ $WRT_M(\xi) \in \mathbb{Z}[\xi]$ integrality of WRT

- $T: \widehat{\mathbb{Z}[q]} \hookrightarrow \mathbb{Z}[[1-q]]$ Taylor expansion

$T(I_M)$ is Ohtsuki series

$\widehat{\mathbb{Z}[q]} \ni f = \sum f_k (q;q)_k$

$\Rightarrow$ relation with perturbative LMO invariant based on Kontsevitch integral

Example: Poincare sphere $M = S^3_{-1}(3_1)$

$$I_M = \frac{1}{1-q} \sum_{k=0}^{\infty} q^k (q^{k+1};q)_{k+1}$$

Proof (B. - Le)

$$J_K(V_n) \in \mathbb{Z}[q^{\pm 1}, q^n]$$

$$ev_\xi \sum_{n=0}^{r-1} q^{\frac{n^2-1}{4}} q^{an} = ev_\xi (q^{-a^2}) \gamma_\ell$$

$$\gamma_\ell = ev_\xi \sum_{n=0}^{r-1} q^{\frac{n^2-1}{4}}$$

Laplace transform $\mathcal{L}(q^{an}) = q^{-a^2}$

- use q-binomial formula
- apply Laplace to the sum
- use Rogers–Ramanujan identity to factorize

$\boxed{/\!/\!/}$

Habiro-Le constructed unified invariants for all simple Lie algebras

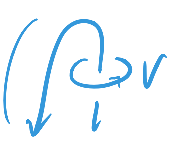
$$I_M := \langle r, J_K \rangle \in \widehat{\mathbb{Z}[q]}$$

ribbon element

$$\langle\ ,\ \rangle : \mathcal{Z} \times \mathcal{Z} \to \mathbb{Z}[v^{\pm 1}]$$

Hopf pairing or quantum Killing form (Rosso)

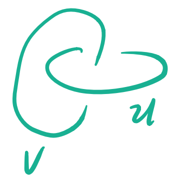
$\mathcal{D} : \mathcal{R} \xrightarrow{\cong} \mathcal{Z}$ Drinfeld map

$V \mapsto J\big($  $\big)$

representation ring

Hopf pairing

$$\langle\ ,\ \rangle : \mathcal{R}\times\mathcal{R} \to \mathbb{Z}[v^{\pm 1}]$$

$\langle V, U \rangle = J\Big($  $\Big)$

extends to the center

$$\langle \mathcal{D}(V), \mathcal{D}(U) \rangle := \langle V, U \rangle$$

Drawbacks

(1) does not provide cyclotomic expansions

(2) does not include $\mathfrak{gl}_N$

(3) quite involved

Quantum gl_N

$K_i = v^{H_i}$

$E_1, \ldots, E_{N-1}$ $\quad$ $\underbrace{K_1^{\pm 1} \ldots K_N^{\pm 1}}_{\text{Cartan}}$

$F_1, \ldots, F_{N-1}$

$$K_i E_i = v E_i K_i \qquad K_{i+1} E_i = v^{-1} E_i K_{i+1}$$

$K = K_1 \cdots K_N$

$$[E_i, F_j] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}$$

$$E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 \qquad |i-j| = 1$$

$$R = v^{-\sum_i H_i \otimes H_i} \sum_{\underline{n}} e_{\underline{n}} \otimes F_{\underline{n}}$$

Representation ring $\mathcal{R} = \{ V_\lambda \}$

partition

$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N)$

$$r|_{V_\lambda} = v^{-(\lambda, \lambda + 2\rho)}$$

Center of $U_q gl_N$

$$J_K(gl_N) \in Z(U_q\, gl_N) = Z$$

$$Sym = \mathbb{Z}[v^{\pm 1}][x_1, \dots, x_N]^{S_N} \big/ e_N(y)$$

$$x_i = K_i^2, \quad y_i = K_i$$

$$\begin{tikzcd} Z \arrow[r, "hc"] & Sym \\ & R \arrow[ul, "\mathcal{D}"] \arrow[u, "ch"'] \end{tikzcd}$$

$$\mathcal{D} = (hc)^{-1} \circ ch$$

$$ch : \mathcal{R} \to Sym$$ character map

$$V_\lambda \mapsto s_\lambda(x_1, \dots, x_N)$$

$\forall$ partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N)$

Example:

$$C_{\mathfrak{gl}_2} = (v-v^{-1})^2 FE + v K_1 K_2^{-1} + v^{-1} K_1^{-1} K_2$$

acts on V_λ for $\lambda = (\lambda_1, \lambda_2)$ by

$$v^{1+\lambda_1-\lambda_2} + v^{-1-\lambda_1+\lambda_2}$$

$$hc(z)(v^{\rho+\lambda}) = z|_{V_\lambda} \qquad \text{Harish-Chandra}$$

$$hc(C_{\mathfrak{gl}_2})(y_1, y_2) = \frac{y_1}{y_2} + \frac{y_2}{y_1} = \frac{1}{y_1 y_2}[y_1^2 + y_2^2]$$

$y_1 = v^{1/2+\lambda_1}$

$y_2 = v^{-1/2+\lambda_2}$

$\rho = (\frac{1}{2}, -\frac{1}{2})$

$$= \frac{1}{e_2} S_1(x_1, x_2)$$

Hopf pairing for $\mathfrak{gl}_N$

The clasp (or monodromy) $C = (S \otimes 1) R_{21} R$ allows to extend $\langle , \rangle$ to $\mathcal{U}_q \mathfrak{gl}_N$

$$C = \sum_i C_i \otimes C_i' \Rightarrow \langle C_i, C_j' \rangle = \delta_{ij}$$

where $\{C_i\}, \{C_i'\}$ are topological bases of $\mathfrak{gl}_N$

$q = \exp h$

$$\prod_{i=1}^{N} q^{-H_i \otimes H_i} = \prod_{i=1}^{N} \sum_n (-1)^n \frac{h^n}{n!} H_i^n \otimes H_i^n$$

$$\Rightarrow \langle H_i^n, H_j^m \rangle = \delta_{ij} \delta_{nm} (-1)^n \frac{n!}{h^n}$$

$$\langle K_i^{2a}, K_j^{2b} \rangle = \delta_{ij} q^{-ab}$$

$$\langle x_i^a, x_j^b \rangle = \delta_{ij} q^{-ab}$$

Let Y be a root latice, $\Gamma = Y/2Y$

$\Gamma = \mathbb{Z}_2^N$ for gl_N

Thm (B.-Gorsky) The universal gl_N invariant of any evenly framed link is Γ-invariant.

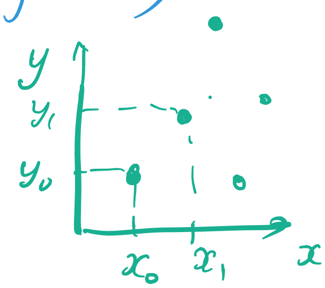
Remark: For sl_N $\Gamma = \mathbb{Z}_2^{N-1}$

For sl_N link invariant Γ-invariance holds only if all coefficients of the linking matrix are zero.

(II) Interpolation

One variable

Suppose we know $y_0 = f(x_0)$
$\vdots$
$y_k = f(x_k)$



can we reconstruct f?

Answer:

$$f = y_0 + a_1(x-x_0) + \dots + a_k(x-x_0)\dots(x-x_{k-1})$$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = [y_0, y_1] \qquad a_2 = [y_0, y_1, y_2] = \frac{[y_0, y_2] - [y_0, y_1]}{x_2 - x_0}$$

Consider $f_m(x) = (x; q)_m = (1-x)\dots(1-xq^{m-1})$

Bilinear form $\quad \langle x^k, x^m \rangle := q^{-km}$

$$\langle f_m(x), x^k \rangle = f_m(q^{-k}) = 0$$

if $k < m$

Lemma: $\{f_n\}_{n \geq 0}$ is an orthogonal basis of $\mathbb{Z}[q^{\pm 1}][x]$ wrt $\langle \, , \rangle$

Proof: $\langle f_n(x), f_m(x) \rangle = \delta_{nm} (-1)^m q^{-m} (q;q)_m$ $\square$

$$f(x) = \sum_{m \geq 0} a_m f_m$$

Can we find a_m?

$$a_m = \frac{\langle f, f_m \rangle}{\langle f_m, f_m \rangle} \qquad \langle f, f_m \rangle = \sum_{j=0}^{m} \binom{m}{j}_q f(q^{-j})$$

For more variables $\quad \underline{k} = (k_1, \dots, k_N) \in \mathbb{N}^N$

$$f_{\underline{k}}(\underline{x}) = \prod_i f_{k_i}(x_i)$$

Given a partition $\lambda = (\lambda_1, \dots \lambda_N)$

$$F_\lambda(\underline{x};q) = \frac{\det\left(f_{\lambda_i+N-i}(x_i)\right)}{\prod_{i<j}(x_i - x_j)}$$

Macdonald interpolation polynomials

Properties:

- F_λ is symmetric of degree $|\lambda| = \sum \lambda_i$
- $F_\lambda\left(q^{-\mu_i - N + i}\right) =: C_{\lambda\mu}(q) = 0$ unless $\lambda \subset \mu$
- $F_\lambda\left(q^{-\lambda_i - N + i}\right) = \text{coeff} \prod_{\square \in \lambda}\left(1 - q^{-h(\square)}\right)$

Interpolation problem : Find

$$f = \sum a_\lambda F_\lambda \qquad a_\lambda = ?$$

$$f\left(q^{-\mu_i - N - i}\right) = \sum a_\lambda F_\lambda\left(q^{-\mu_i - N + i}\right) = \sum_{\lambda \subset \mu} C_{\lambda\mu}(q)\, a_\lambda$$

Thm (Okounkov) $\exists\ \mathcal{D} = C^{-1}$ $C = (c_{\lambda\mu})$

$\mathcal{D} = (d_{\lambda\mu})$

$$d_{\lambda\mu}(q) = (-1)^{|\mu| - |\lambda|} q^{\mathrm{cont}(\lambda) - \mathrm{cont}(\mu)} \frac{c_{\lambda\mu}(q^{-1})}{c_{\mu\mu}(q)\, c_{\lambda\lambda}(q^{-1})}$$

Solution to interpolation problem

$$a_\mu = \sum_{\lambda \subset \mu} d_{\lambda\mu} f\left(q^{-\lambda_i - N + i}\right)$$

Moreover, the basis $\{F_\lambda\}_\lambda$ of symmetric functions is orthogonal:

$$\langle F_\lambda, F_\nu \rangle = \delta_{\lambda\nu} \text{ coeff}$$

Example 9.23. For $N = 2$ and $\lambda = (3, 2)$ we have

$F_{(3,2)} = q^2(1 - x_1)(1 - qx_1)(1 - x_2)(1 - qx_2)(q^3(x_1 + x_2) - (1 + q)) =$
$q^7 s_{3,2} - q^6(1 + q)s_{3,1} - q^4(1 + q + q^2 + q^3)s_{2,2} + q^6 s_{3,0} + q^3(1 + q + q^2 + q^3)(1 + q)s_{2,1} -$
$q^3(1 + q + q^2 + q^3)s_{2,0} - q^2(1 + q + q^2 + q^3)(1 + q)s_{1,1} + (q^5 + q^4 + 2q^3 + q^2)s_{1,0} - (q^3 + q^2)$.

Also

$$(F_{3,2}, F_{3,2}) = -q^{-5}(1 - q^4)(1 - q^3)(1 - q^2)^2(1 - q)$$

Therefore the interpolation coefficient for $\lambda = (3, 2)$ and $\mu = (1, 0)$ equals

$$d_{(3,2),(1,0)} = (q^5 + q^4 + 2q^3 + q^2)\frac{s_{1,0}(q^{-1}, 1)}{(F_{3,2}, F_{3,2})} =$$

$$-\frac{(q^5 + q^4 + 2q^3 + q^2)(1 + q^{-1})}{q^{-5}(1 - q^4)(1 - q^3)(1 - q^2)^2(1 - q)} = -\frac{q^6 + q^4 - q^3 - q^2}{q^{-4}(1 - q^4)(1 - q^3)(1 - q^2)(1 - q)^3}.$$

$F(x, t)$

III Main results

$\exists$ a basis of $Z(U_q gl_N)$

$$\sigma_\lambda := hc^{-1}(F_\lambda(v^{N-1}x_1, \dots, v^{N-1}x_N; q))$$

$$\sigma_\lambda |_{V_\mu} = F_\lambda(q^{\mu_i + N - i}) = 0 \text{ unless } \lambda \subset \mu$$

$\exists$ a basis P_μ of $R = K_0(Rep(U_q gl_N))$

$$\langle P_\mu, \sigma_\lambda \rangle = \delta_{\mu\lambda}$$

$$P_\mu = \sum_{\lambda \subset \mu} d_{\mu\lambda}(q^{-1}) \frac{V_\lambda}{\dim_q V_\lambda}$$

<u>Thm</u> (B. - Gorsky) For any evenly framed knot K, $\exists\ a_\lambda(K) \in \mathbb{Z}[v, v^{-1}]$

F_∞ Swiller

$$J_K(\mathfrak{gl}_N; q) = \sum_\lambda a_\lambda(K)\, \sigma_\lambda$$

cyclotomic expansion of $\mathfrak{gl}_N$ knot invariant

$$a_\lambda(K) = J_K(P'_\lambda) \qquad P'_\lambda = \sum_{\mu \subset \lambda} d_{\lambda\mu}(q^{-1}) V_\mu$$

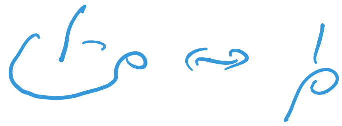
Applications to 3-mnf invariants

<u>Thm</u> (B.-Gorsky) $\omega_\pm = \sum_\lambda v^{\mp(\lambda, \lambda+2\rho)} P'_\lambda$

is a universal Kirby color for (± 1)-surgeries,

i.e. $\forall\ x \in \mathcal{R}$

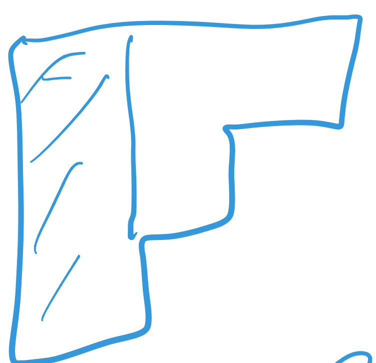
$$\langle \omega_\pm, x \rangle = J_{U_\mp}(x, q) = \langle r^{\pm 1}, \mathcal{D}(x) \rangle$$



Thm (B.-Gorsky) Let $M_\pm = S^3(K_{\pm 1})$

$\exists !$ $I_{M_\pm} := J_K(\omega_\pm)$ unified invariant

- $I_M \in \widehat{\mathbb{Z}[q]}$
- $ev_\xi I_M = WRT_M(\xi)$

λ_N 

$e_N^{\lambda_N}$