Iterated Spans & Classical TFTs

29 July 2020

Defn (Atiyah): A TQFT is a symm. mon. functor

$$\mathrm{Bord}_{n-1,n}^{(?)} \xrightarrow{Z} \mathrm{Vect}$$

closed $(n-1)$-mfd M $\longmapsto$ $Z(M)$ v. sp. of "quantum states" on M

n-dim. cobordism X from M to M' $\longmapsto$ $Z(X): Z(M) \to Z(M')$

$(\partial X \cong M \amalg M')$ "evolution of states" along X

Basic picture of classical TFT:

- mfd. M $\longmapsto$ "space of fields on M" $Z(M)$
  (space — could be fancy things – (derived) stacks, ...)
- X mfd. w/ bd.ry – can restrict fields to bd.ry
$$Z(X) \to Z(\partial X)$$
- bordism X from M to M'
$$Z(M) \leftarrow Z(X) \rightarrow Z(M')$$
- "Locality" $X = X_1^n \cup_M X_2^n$ $\rightsquigarrow$ field on X = fields on X_1, X_2 that agree on M

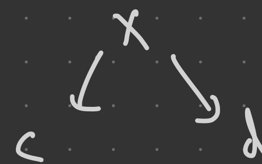
i.e. $Z(X) = Z(X_1) \times_{Z(M)} Z(X_2)$

• $Z(X_1 \amalg X_2) = Z(X_1) \times Z(X_2)$

<u>Defn.</u>: $\mathcal{C}$ a cat. w/ finite limits.

$\mathrm{Span}_1(\mathcal{C})$ has ob.s = ob.s of $\mathcal{C}$

- mor. from c to d =



[isom. classes to get an ordinary cat.]

- compose by pullback:

- id. of c

- symm. mon. via $\times$

$\rightsquigarrow$ A classical TFT is a symm. mon. functor

$$\mathrm{Bord}_{n-1,n} \longrightarrow \mathrm{Span}_1(\mathcal{C})$$

- Often $Z(X) = \mathrm{Map}(X, T)$ ~ "σ-model w/ target T"
- T typically has symplectic/Poisson str.

<u>Defn.</u> (Baez–Dolan, Freed, Lurie, ...):

An extended n-dim'l TQFT is a symm. mon. functor of (∞, n)-cat.s

$$\mathrm{Bord}^{(?)}_{(0,n)} \longrightarrow \mathcal{C}$$

for some symm. mon. (∞, n)-cat.

- Additional data for mfd.s of dim. < n-1 relates to "defects" in QFT

- Heuristically, an (∞, n)-cat.s is a str. with
  - ob.s, mor.s, 2-mor.s, ...
  - i-mor.s invertible for $i > n$
  - composition is only associative up to coherent choice of inv.ble higher mor.s

∞-cat. = $(\infty, 1)$-cat.

$\mathrm{Bord}_{(0,n)}$ is an (∞,n)-cat. w/

- objs are compact 0-mfld.s
- 1-mor.s are 1-dim'l cobordisms
- 2-mor.s are 2-dim'l cobordisms w/ corners

...

- n-mor.s are n-dim'l cobordisms w/ corners of all codim.s
- (n+1)-mor.s = diffeomor.s

...

- symm. mon. via $\amalg$ [Precise defn. Calaque-Scheimbauer]

Idea: Extended classical TFTs should be symm. mon. ftr.s

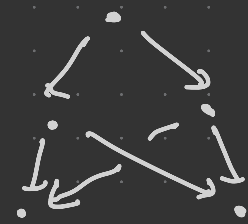
$$\mathrm{Bord}_{(0,n)} \longrightarrow \mathrm{Span}_n(\mathcal{C})$$

($\mathcal{C}$ ∞-cat. of "spaces")

- ∞-cat. w/ finite limits

where $\mathrm{Span}_n(\mathcal{C})$ has

- ob.s = ob.s of $\mathcal{C}$
- mor.s = spans $\bullet \leftarrow \bullet \rightarrow \bullet$
- 2-mor.s = spans of spans



...

- n+1 = eq.ces of n-fold spans

- symm. mon. in $\times$

Pretend we're happy with ∞-cat.s

$\mathcal{S}$ = ∞-cat. of spaces / ∞-groupoids / ht.py types

Δ = cat. of ordered sets $[n] = \{0 < \dots < n\}$

<u>Defn.</u> (Rezk): A <u>Segal space</u> is a functor $X : \Delta^{op} \to \mathcal{S}$

s.t. $X_n \xrightarrow{\sim} X_1 \times_{X_0} \dots \times_{X_0} X_1$ via $[1] \to [n]$, $0 \mapsto i-1$, $1 \mapsto i$; $[0] \to [n]$

Captures <u>algebraic</u> str. of an ∞-cat.:

X_0 = space of obs, X_1 = space of mor.s

$X_1 \overset{d_1}{\underset{d_0}{\rightrightarrows}} X_0$ - source & target $s_0 : X_0 \to X_1$ - identities

$$\underbrace{X_1 \times_{X_0} X_1}_{\text{space of composable pairs of mor.s}} \xleftarrow{\sim} X_2 \xrightarrow{d_1} X_1 \quad \text{– composition}$$

Makes sense in any ∞-cat. w/ finite limits

$\Rightarrow$ can iterate

<u>Defn</u>: An <u>n-uple Segal space</u> is a Segal ob. in $(n-1)$-uple Segal spaces.

$(n=2)$ $X : \Delta^{2,op} \to \mathcal{S}$ X_{ij} decomposes as a limit of

X_{00} – space of ob.s

X_{10} – "horizontal" mor.s

X_{01} – "vertical" mor.s

X_{11}

$$\begin{array}{ccc} X_{10} & \leftleftarrows & X_{11} \\ \downarrow\downarrow & & \downarrow\downarrow \\ X_{00} & \leftleftarrows & X_{01} \end{array} \quad X_{11} \sim \text{"squares"}$$

A double Segal space is an ∞-version of a double cat.

A 2-cat. is a double cat. where all mor-s in one direction are identities

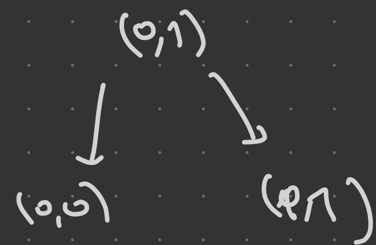
Defn. (Barwick): An *n-fold Segal space* X is an n-uple Segal space s.t.

- X_0 is constant
- X_1 is an $(n-1)$-fold Segal space

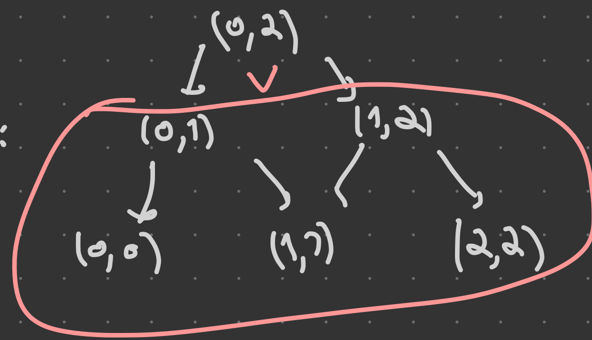
Defn.: Σ^n = partially ordered set of pairs (i,j) w/ $0 \le i \le j \le n$

& $(i,j) \le (i',j') \iff i \le i' \le j' \le j$

Σ^1:



Σ^2:



$\varphi : [n] \to [m]$ in Δ $\rightsquigarrow$ $\Sigma^n \longrightarrow \Sigma^m$

$(i,j) \longmapsto (\varphi i, \varphi j)$

$\mathcal{C}$ ∞-cat. w/ finite limits

$$\mathrm{Map}(\Sigma^n, \mathcal{C}) \supset \mathrm{Map}_{cart}(\Sigma^n, \mathcal{C})$$

subsp. of $\Sigma^n \to \mathcal{C}$

that take all squares in Σ^n to pullbacks in $\mathcal{C}$

Then $\mathrm{Span}_1(\mathcal{C}) = \mathrm{Map}_{cart}(\Sigma^\bullet, \mathcal{C})$ is a Segal space

$$\mathrm{SPAN}_n(\mathcal{C}) = \mathrm{Map}_{cart}(\Sigma^\bullet \times \cdots \times \Sigma^\bullet, \mathcal{C})$$

– n-uple Segal sp.

$\rightsquigarrow$ $\mathrm{Span}_n(\mathcal{C})$ "underlying" n-fold Segal sp.

Variants:

- H. – Melani – Safronov: derived Poisson stacks & iterated coisotropic correspondences
  ∃ symm. (∞, n)-cat. of
- Calaque – H. – Scheimbauer: symplectic derived stacks & iterated Lagrangian corr.s
  - all obj.s fully dualizable
  - explicitly define oriented TQFTs using AKSZ constr.

$$T \in \mathcal{S}$$

$$X \quad \mathrm{Bord}^{un}_{0,n} \longrightarrow \mathrm{Span}_n(\mathcal{S})$$

$$X \longmapsto \mathrm{Map}(X, T)$$

$$\mathrm{Bord}^{un}_{0,n} \longrightarrow \mathrm{Cospan}_n(\mathcal{S})^{fin}$$

$$\mathrm{Cospan}_n(\mathcal{S})^{fin} \longrightarrow \mathrm{Span}_n(\mathcal{C})$$

$$X \longmapsto C^X = \lim_X C$$

$$T = BG \qquad \mathrm{Span}_n(\mathcal{C})(x,y) = \mathrm{Span}_{n-1}(\mathcal{C}_{/x \times y})$$