TOPOLOGICAL LINKS & QUANTUM ENTANGLEMENT

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Outline

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1) Introduction and motivation
2) A formalism to study links
3) From states to links

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4) From links to states
5) Applications
6) Conclusions and future work

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Introduction and motivation

- Qubit state:

$$|\psi\rangle = c_1 |0\rangle + c_2 |1\rangle$$

Not a coin flip!

Quantum superposition leads to several phenomena

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- Is a state entangled or separable?

If $\hat{\rho}_{AB} = \sum_i p_i \; \hat{\rho}_i^A \otimes \hat{\rho}_i^B$ state is separable.

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Much of quantum information revolves around the following:

Given a quantum state, how much entangled is it?

How to usefully quantify/classify entanglement?

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Given a desired entanglement property, how to find a state associated to it?

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$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$_a\langle 0|\mathrm{GHZ}\rangle = |\mathrm{Sep.}\rangle$$

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Measuring a qubit $\Leftrightarrow$ Cutting a ring

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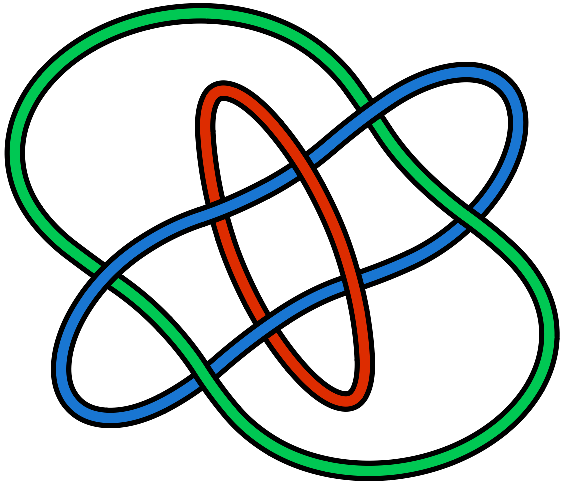
$|\mathrm{GHZ}\rangle \Rightarrow$



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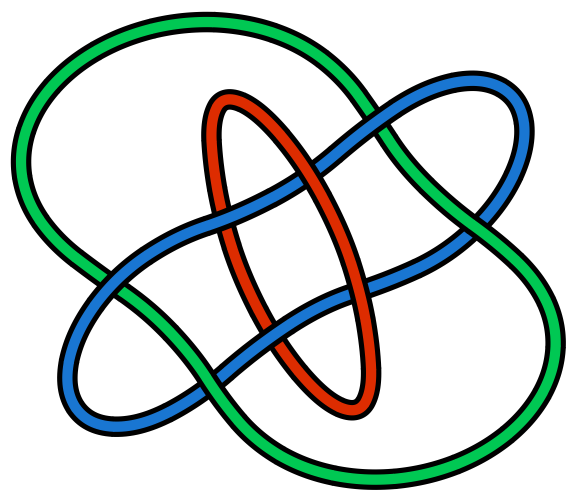
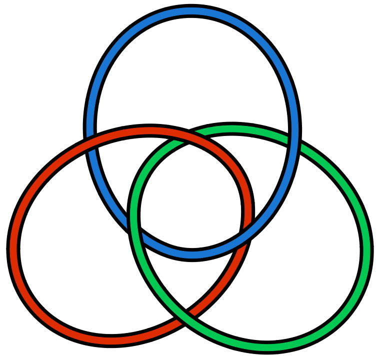


(1997)

P.K. ARAVIND

BORROMEAN ENTANGLEMENT OF THE GHZ STATE*

In this paper, I will point out some curious connections between entangled quantum states and classical knot configurations. In particular, I will show that the



$$_a\langle 0|\mathrm{W}\rangle = |\mathrm{Ent.}\rangle$$

$$_b\langle 0|\mathrm{W}\rangle = |\mathrm{Ent.}\rangle \qquad |\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

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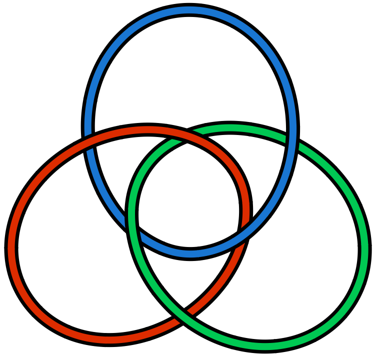

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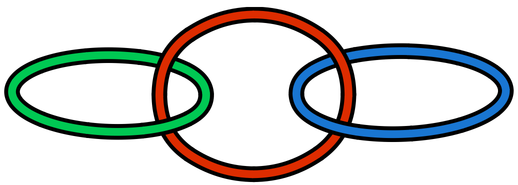

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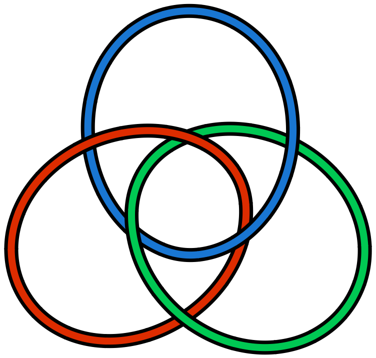

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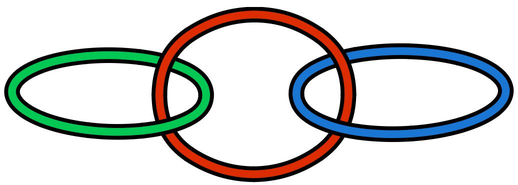

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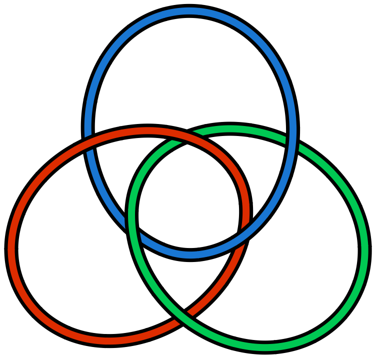
P. K. Aravind (1997)

Problems:

- Basis dependent
- Qubits > 3 too difficult



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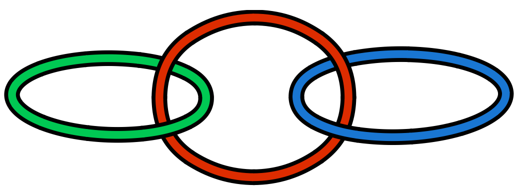

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P. K. Aravind (1997)
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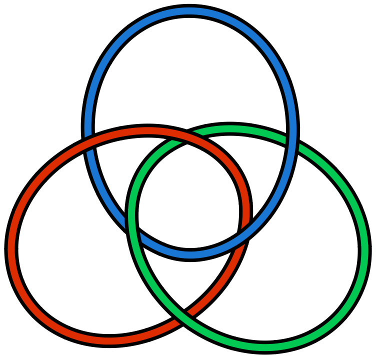
Sugita (2006)

Borromean Entanglement Revisited

Ayumu Sugita *
Osaka City University
Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan



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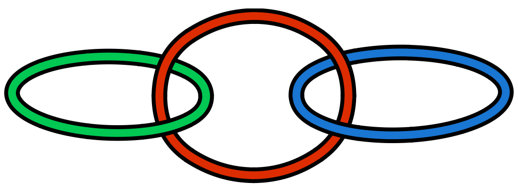

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Partial trace over subsystem B

$$\hat{\rho}_{AB} = \sum_{ijkl} \rho_{ijkl} \left| ik \right\rangle \left\langle jl \right|$$

$$\hat{\rho}_{A} \equiv \mathrm{Tr}_B[\hat{\rho}_{AB}] = \sum_{ijkl} \left\langle k | l \right\rangle \rho_{ijkl} \left| i \right\rangle \left\langle j \right|$$

A formalism to study links

Main idea

Classify entanglement using links: different entanglement classes are associated to different link classes.

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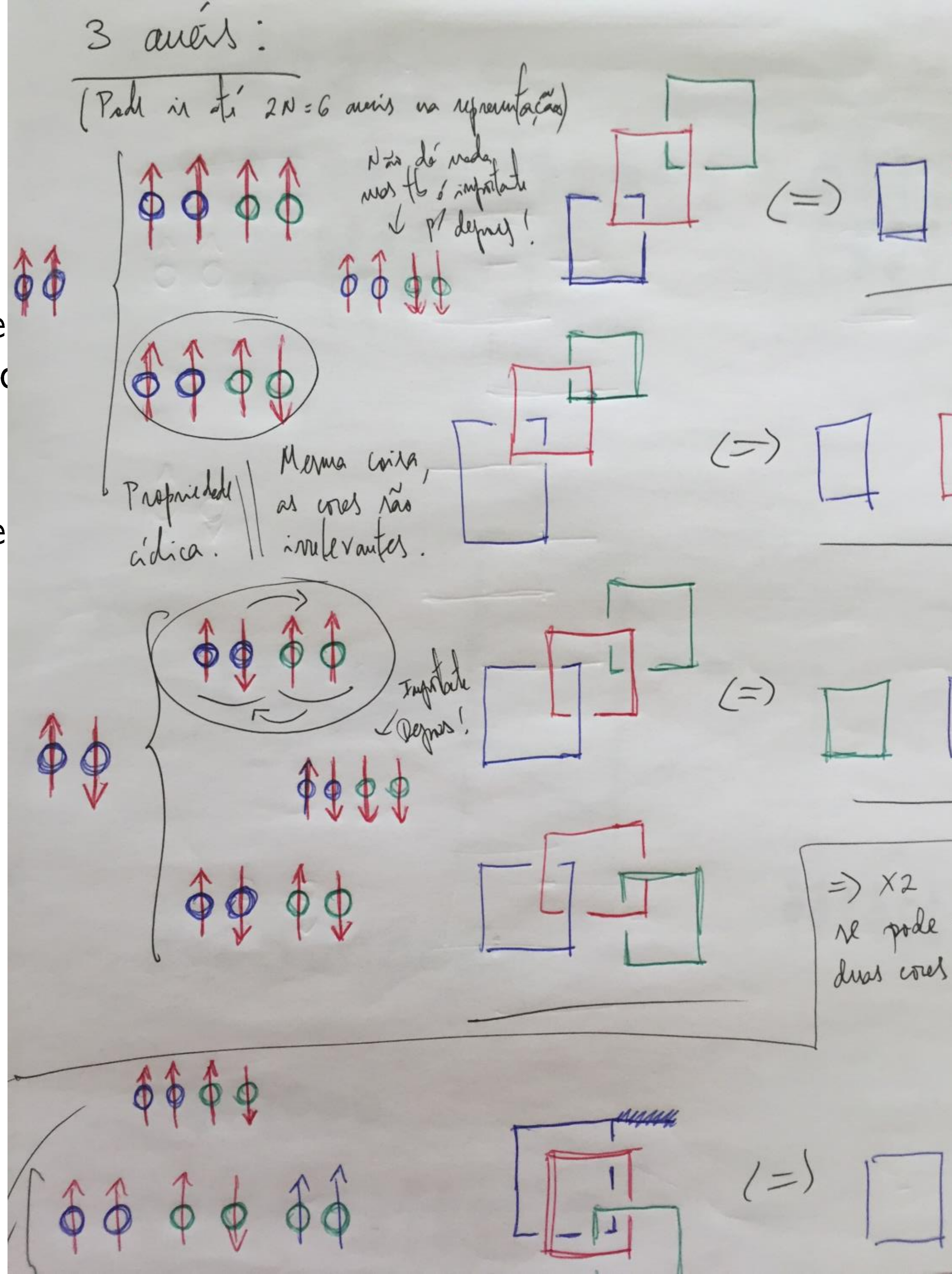
Definition: … ring cuts are the…



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Method: Build a polynomial that is uniquely associated to each link class.

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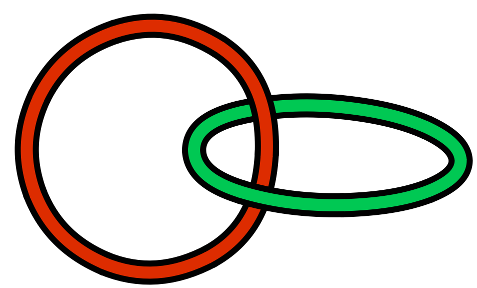
Method: Build a polynomial that is uniquely associated to each link class.

- 1 letter to each ring
- Cutting a ring = set variable to 0
- Product of variables = rings are linked
- No linked rings = polynomial is 0

$$\begin{tikzcd} & 0 \\ \mathcal{P}(2^1) = ab \arrow[ur, "a \to 0"] \arrow[dr, "b \to 0"'] & \\ & 0 \end{tikzcd}$$

Example: 2 rings



Link class 2^1

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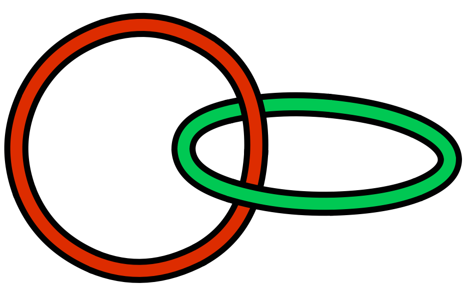
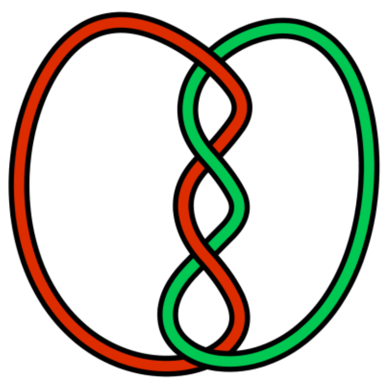
Hopf link





... ... $\in$ Link class 2^1

Rules to construct polynomials:

1) Only powers of 1. (aab is superfluous)
2) Each variable must appear at least once. (we want all rings linked)
3) No first order terms. (same as above)
4) Relabeling is irrelevant.
5) An n-variable term M is irrelevant if all variables already appear as a smaller n-ring link polynomial. (E.g. abc+ab+ac is the same as ab+ac since setting each variable to 0 gives the same outcomes.)

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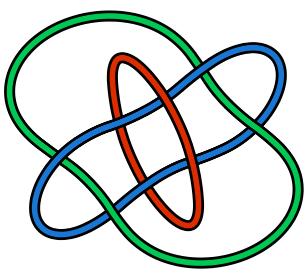
Full 3 ring case a b c

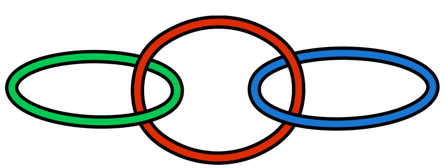
$$\mathcal{P}(3^1) = abc$$

$$3^1 \xrightarrow{3} 0$$



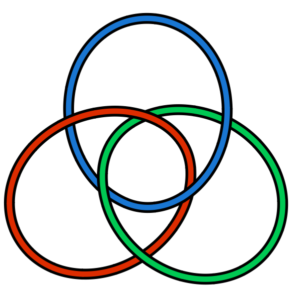
$$\mathcal{P}(3^3) = ab + ac$$

$$3^3 \xrightarrow{1} 0 \qquad 3^3 \xrightarrow{2} 2^1$$



$$\mathcal{P}(3^4) = ab + ac + bc$$

$$3^4 \xrightarrow{3} 2^1$$



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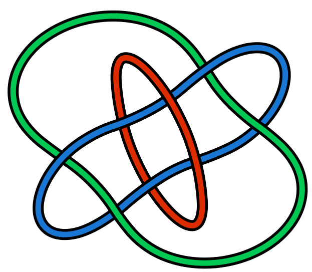
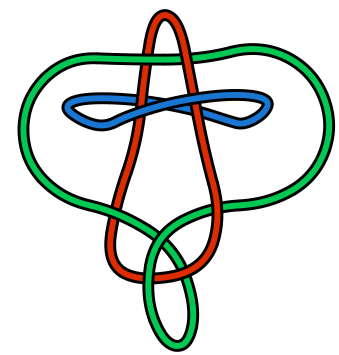


$$\mathcal{P}(3^2) = abc + ab$$

$$\begin{tikzcd} 3^2 \arrow[r, "2"] \arrow[dr, bend right, "1"] & 0 \\ & 2^1 \end{tikzcd}$$



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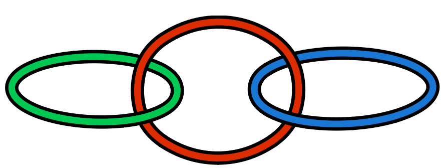
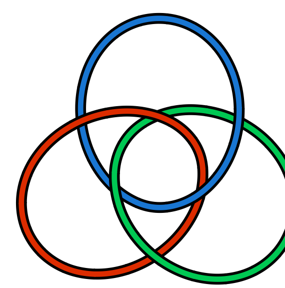


$$\mathcal{P}(3^4) = ab + ac + bc$$

$$3^4 \xrightarrow{3} 2^1$$



Three ring classes constructed from the basis: $\{abc,\, ab,\, ac,\, bc\}$

$\mathcal{P}(3^1) = abc$, $\mathcal{P}(3^2) = abc + ab$, $\mathcal{P}(3^3) = ab + ac$, $\mathcal{P}(3^4) = ab + ac + bc$

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Four ring basis: $\{abcd,\, abc,\, abd,\, acd,\, bcd,\, ab,\, ac,\, ad,\, bc,\, bd,\, cd\}$

$$\#\,\mathrm{basis} = 2^N - N - 1$$

Three ring classes constructed from the basis: $\{abc,\, ab,\, ac,\, bc\}$

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40 Link classes for 4 rings

$\mathcal{P}(4^1) = abcd\,,$
$\mathcal{P}(4^2) = abcd + abc\,,$
$\mathcal{P}(4^3) = abcd + abc + ab\,,$
$\mathcal{P}(4^4) = abcd + ab\,,$
$\mathcal{P}(4^5) = abcd + ab + ac\,,$
$\mathcal{P}(4^6) = abcd + ab + cd\,,$
$\mathcal{P}(4^7) = abcd + ab + ac + bc\,,$
$\mathcal{P}(4^8) = abc + abd\,,$
$\mathcal{P}(4^9) = abc + abd + acd\,,$
$\mathcal{P}(4^{10}) = abc + abd + acd + bcd\,,$
$\mathcal{P}(4^{11}) = abc + ad\,,$
$\mathcal{P}(4^{12}) = abc + ab + ad\,,$
$\mathcal{P}(4^{13}) = abc + ad + bd\,,$
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$\mathcal{P}(4^{20}) = abc + abd + ac\,,$
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$\mathcal{P}(4^{22}) = abc + abd + ab + cd\,,$
$\mathcal{P}(4^{23}) = abc + abd + ac + ad\,,$
$\mathcal{P}(4^{24}) = abc + abd + ac + cd\,,$
$\mathcal{P}(4^{25}) = abc + abd + ac + bd\,,$
$\mathcal{P}(4^{26}) = abc + abd + ac + ad + cd\,,$
$\mathcal{P}(4^{27}) = abc + abd + ac + bd + cd\,,$

$\mathcal{P}(4^{28}) = abc + abd + acd + ab\,,$
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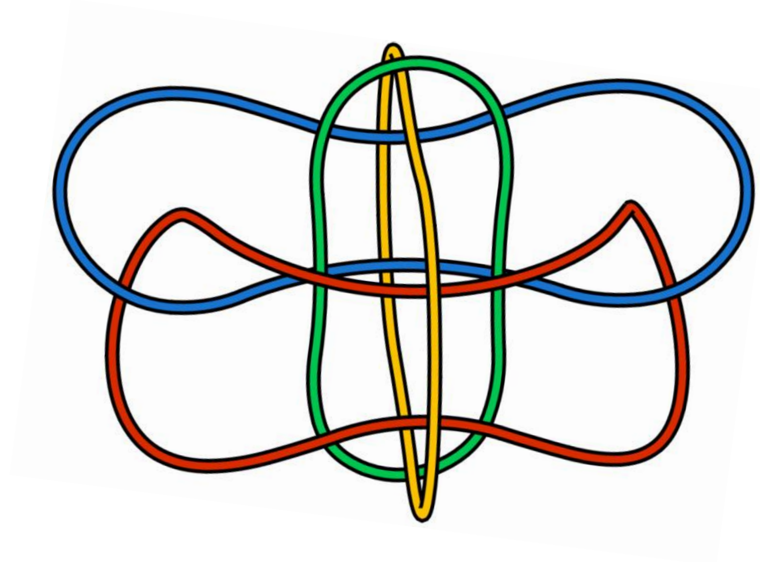
$$\mathcal{P}\left(4^1\right) = abcd$$

$$|4^1\rangle = |0000\rangle + |1111\rangle$$



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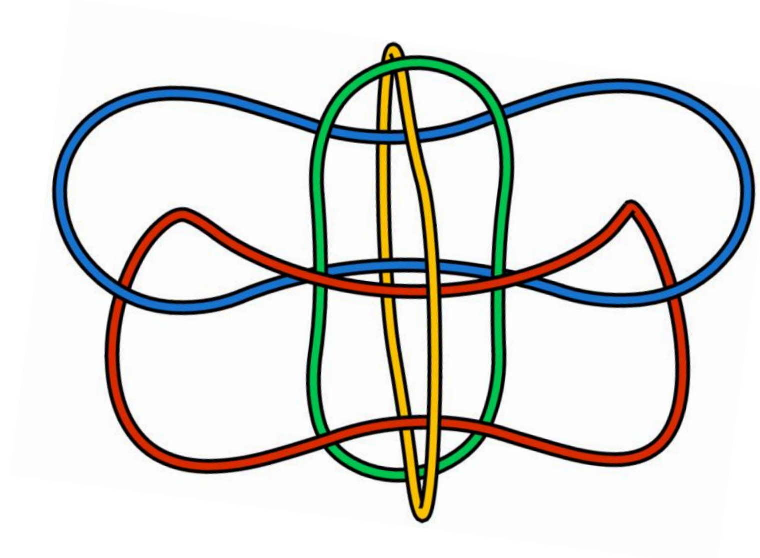

GHZ-like states

$$|2^1\rangle_{ij} = \frac{1}{\sqrt{2}}(|00\rangle_{ij} + |11\rangle_{ij}),$$

$$|3^1\rangle_{ijk} = \frac{1}{\sqrt{2}}(|000\rangle_{ijk} + |111\rangle_{ijk}),$$

$$|4^1\rangle_{ijkl} = \frac{1}{\sqrt{2}}(|0000\rangle_{ijkl} + |1111\rangle_{ijkl}),$$

$$|N^1\rangle_{ij\ldots} = \frac{1}{\sqrt{2}}(|00\ldots\rangle_{ij\ldots} + |11\ldots\rangle_{ij\ldots}),$$

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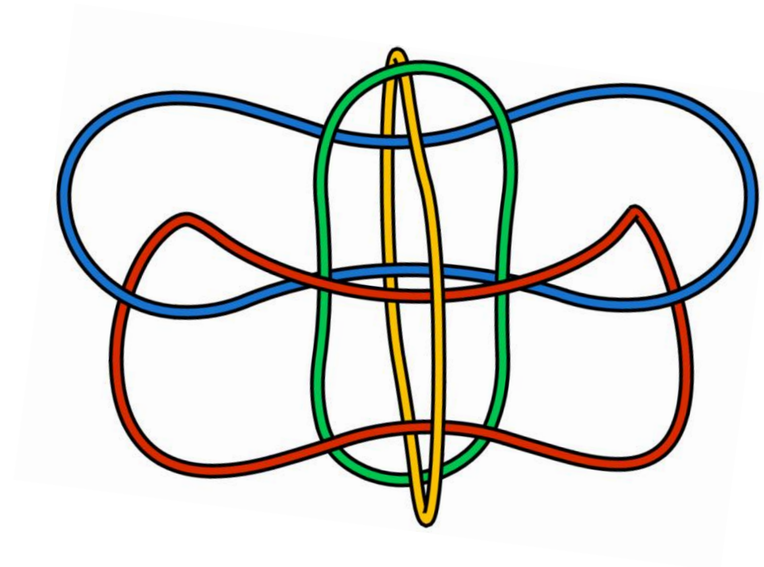

GHZ-like states

Brunnian links

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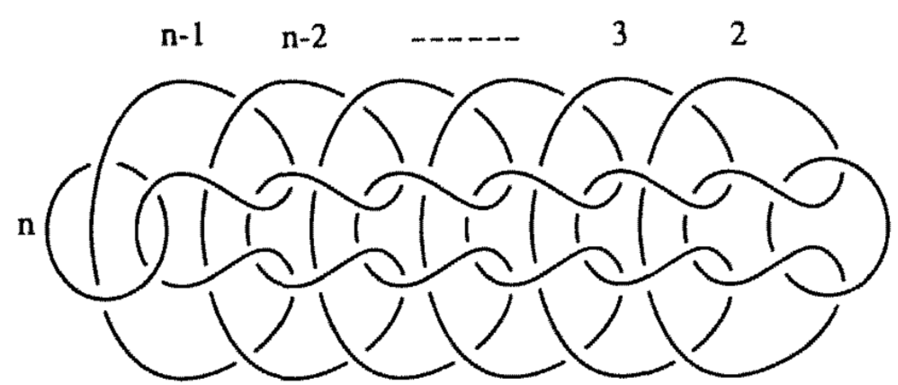
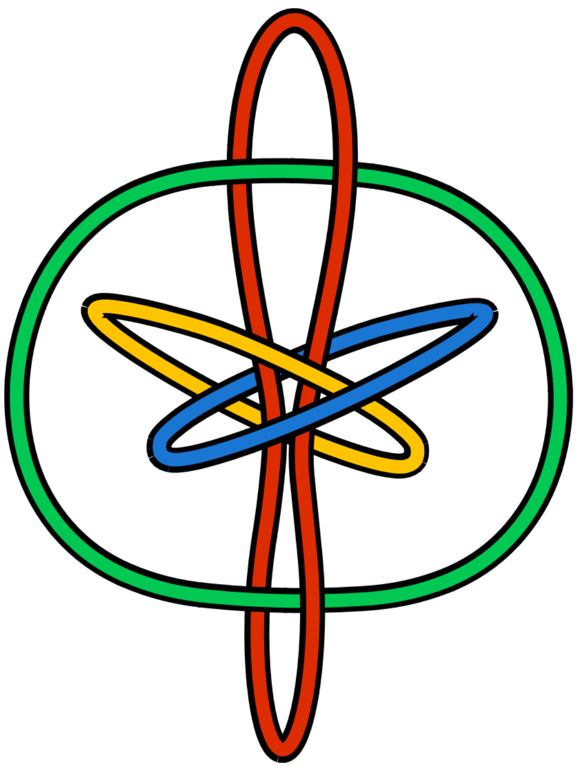


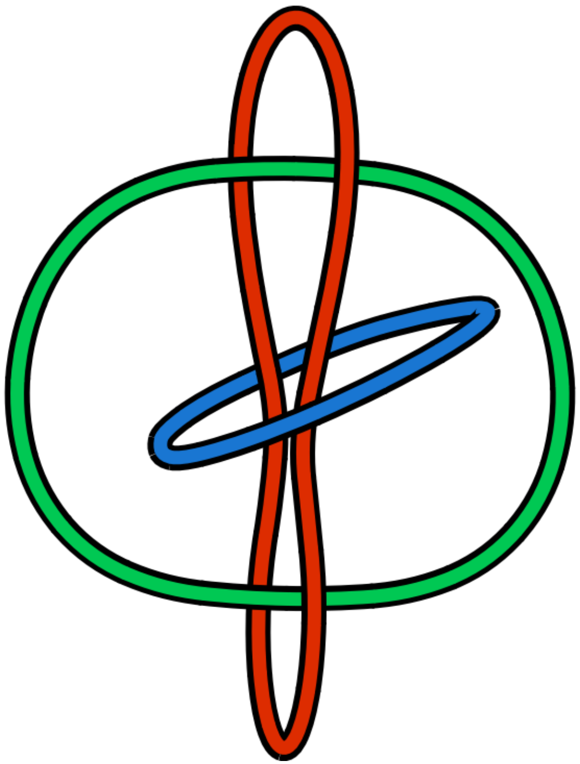
Liang & Mislow 1997

$$\{abcd,\ abc,\ abd,\ acd,\ bcd,\ ab,\ ac,\ ad,\ bc,\ bd,\ cd\}$$



$$\mathcal{P}\left(4^{21}\right) = abcd +$$

$$\{abcd,\ abc,\ abd,\ acd,\ bcd,\ ab,\ ac,\ ad,\ bc,\ bd,\ cd\}$$



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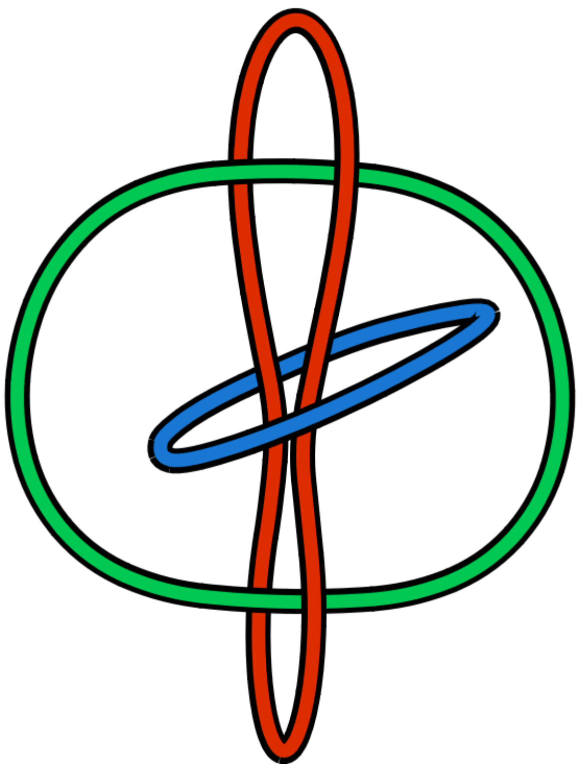
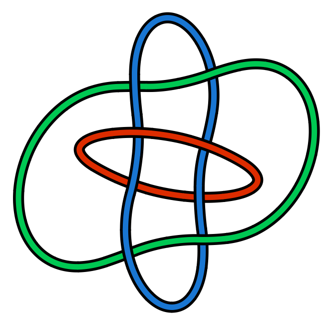
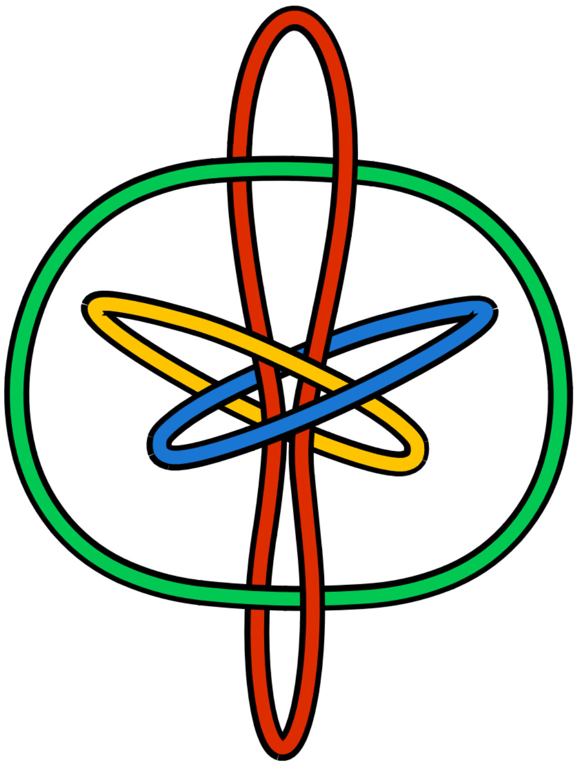


$$\mathcal{P}\left(4^{21}\right) = abcd + abc +$$



$$\mathcal{P}(3^{1}) = abc$$

$$\{abcd,\ abc,\ abd,\ acd,\ bcd,\ ab,\ ac,\ ad,\ bc,\ bd,\ cd\}$$



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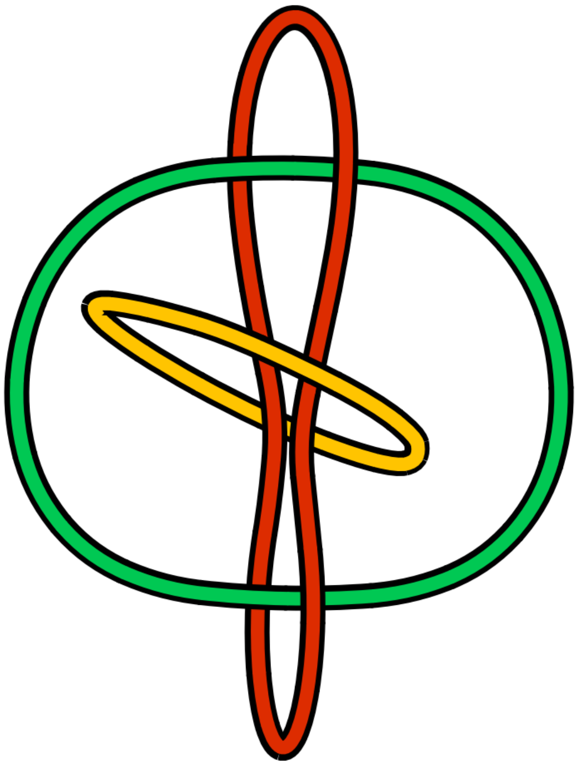

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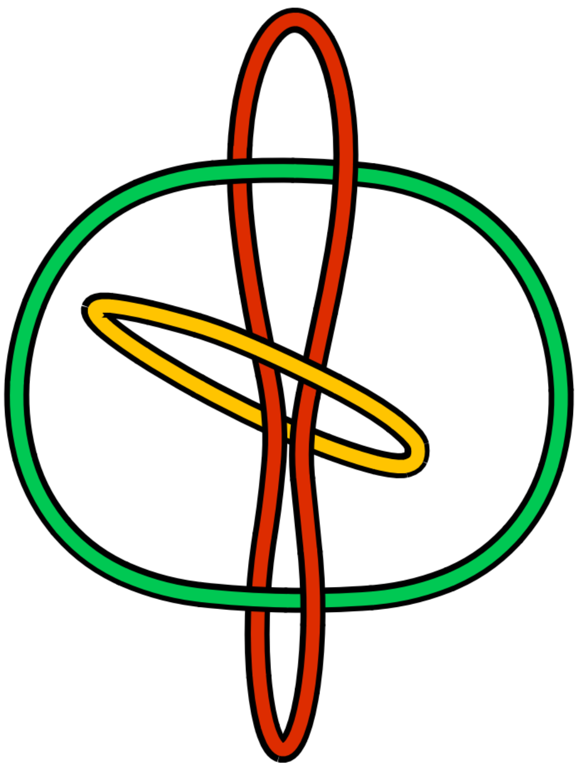
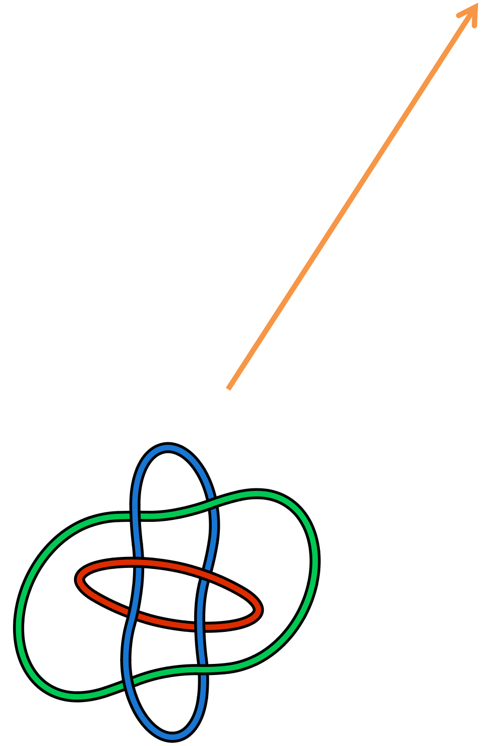
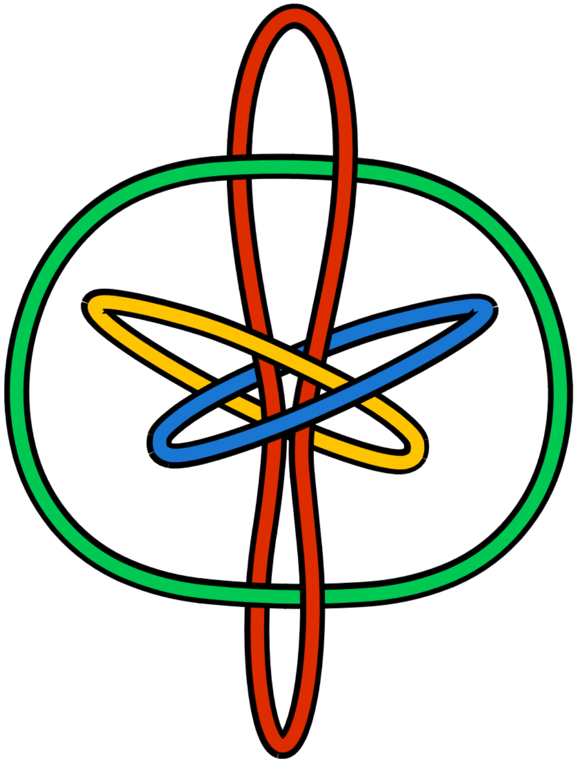


$$\mathcal{P}\left(4^{21}\right) = abcd + abc + abd +$$



$$\{abcd,\ abc,\ abd,\ acd,\ bcd,\ ab,\ ac,\ ad,\ bc,\ bd,\ cd\}$$



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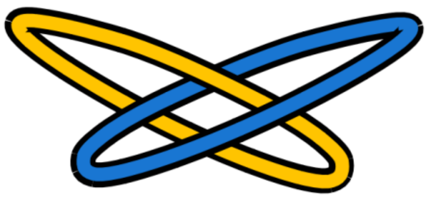

$$\mathcal{P}\left(4^{21}\right) = abc + abd + cd$$

$\{abcd,\ abc,\ abd,\ acd,\ bcd,\ ab,\ ac,\ ad,\ bc,\ bd,\ cd\}$

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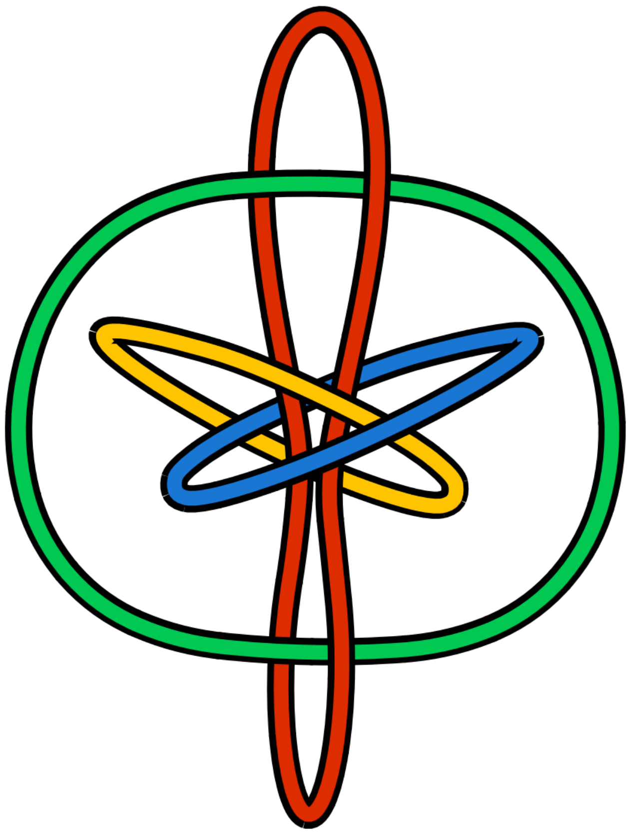
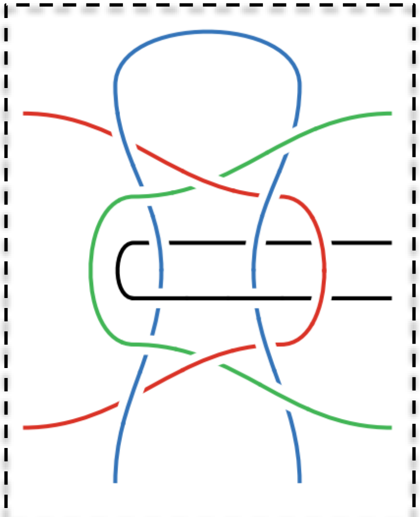
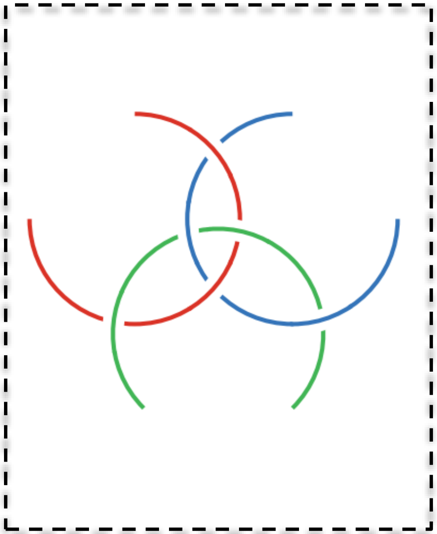


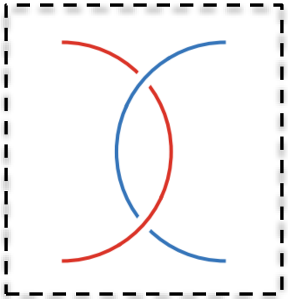
Link Representation



a b c d



a b c



a c

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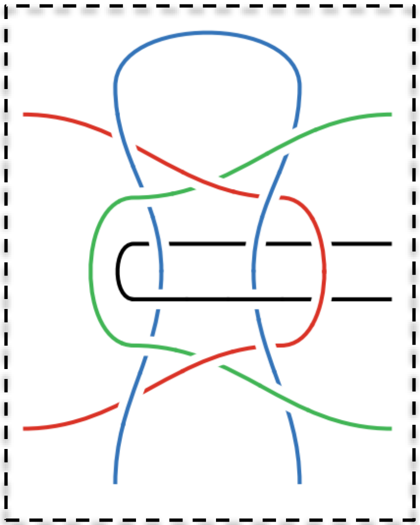
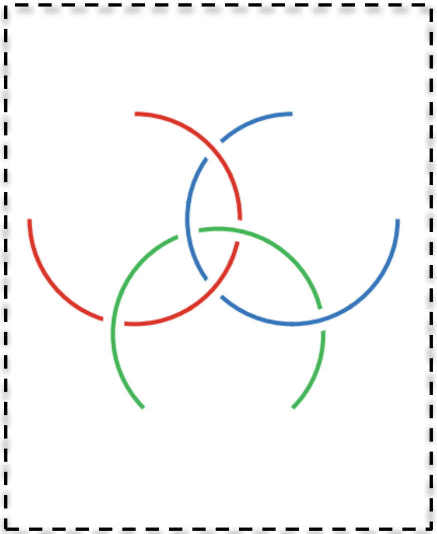
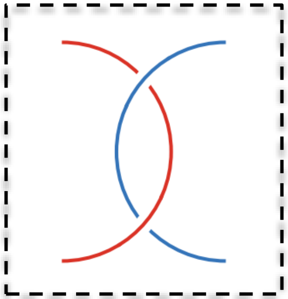


a b c d

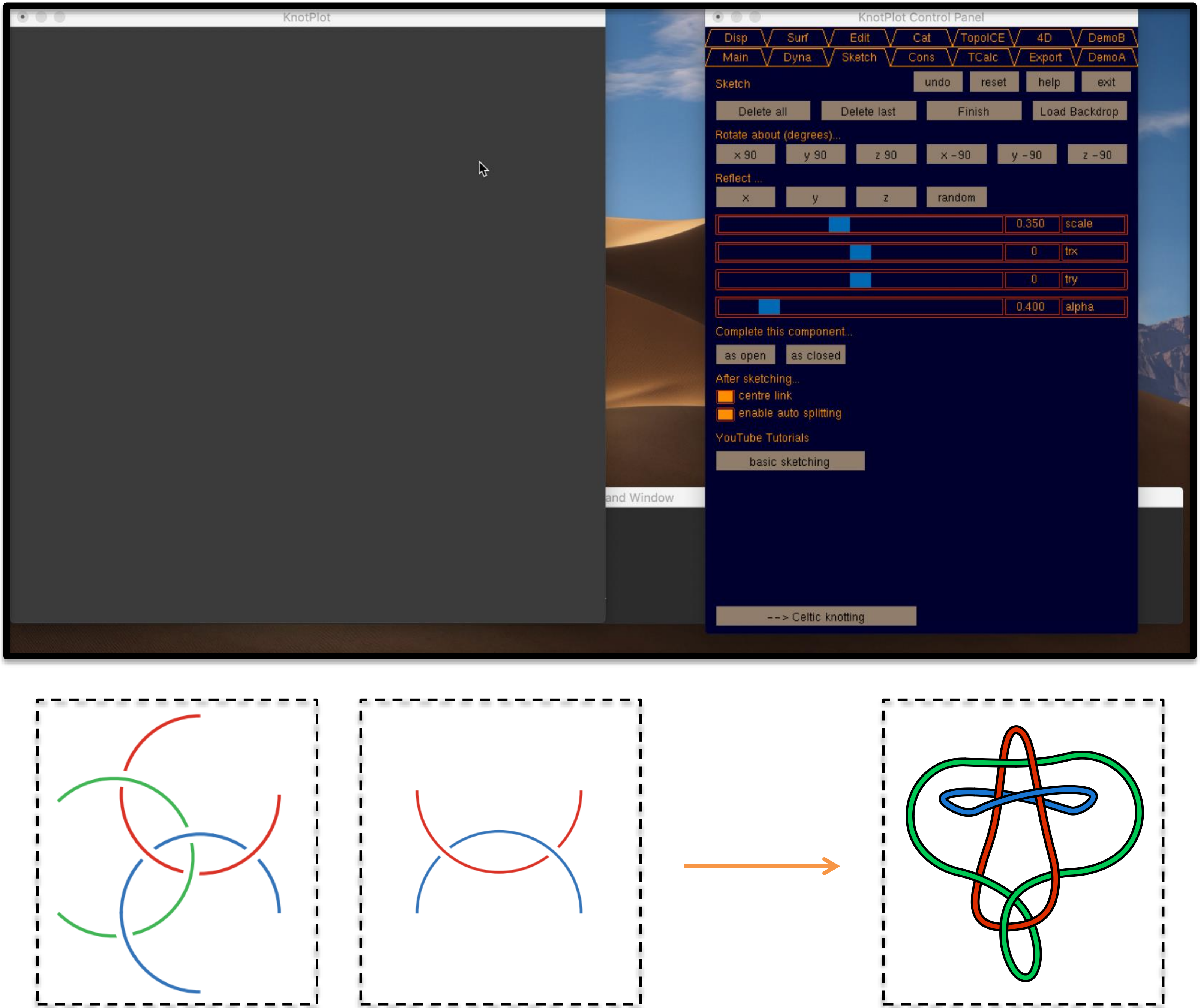


a b c



a c

Just connect the ends with no additional intersections!


KnotPlot
KnotPlot Control Panel
Disp Surf Edit Cat TopoICE 4D DemoB
Main Dyna Sketch Cons TCalc Export DemoA
Sketch
undo reset help exit
Delete all
Delete last
Finish
Load Backdrop
Rotate about (degrees)...
x 90 y 90 z 90 x -90 y -90 z -90
Reflect ...
x y z random
0.350 scale
0 trx
0 try
0.400 alpha
Complete this component...
as open
as closed
After sketching...
centre link
enable auto splitting
YouTube Tutorials
basic sketching
--> Celtic knotting
and Window

Number of rings	Number of terms	Number of classes
2	1	1
3	4	4
4	11	40
5		
N		

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N	$2^N - N - 1$	

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Rules to construct polynomials:

1) Only powers of 1. (aab is superfluous)
2) Each variable must appear at least once. (we want all rings linked)
3) No first order terms. (same as above)
4) Relabeling is irrelevant.
5) An n-variable term M is irrelevant if all variables already appear as a smaller n-ring link polynomial. (e.g. abc+ab+ac is the same as ab+ac since setting each variable to 0 gives the same outcomes.)

Plug rules into a code

Number of rings	Number of terms	Number of classes
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Plug rules into a code

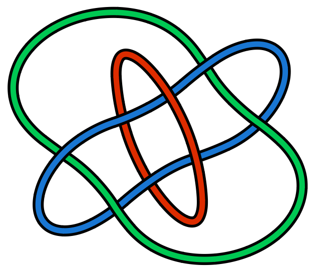
From states to links

Full 3 ring case

a b c

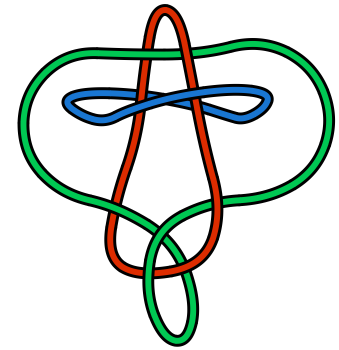
$$\mathcal{P}(3^1) = abc$$

$$3^1 \xrightarrow{3} 0$$



$$\mathcal{P}(3^2) = abc + ab$$

$$\begin{tikzcd} 3^2 \arrow[r, "2"] \arrow[dr, bend right, "1"] & 0 \\ & 2^1 \end{tikzcd}$$



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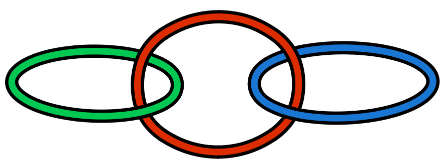
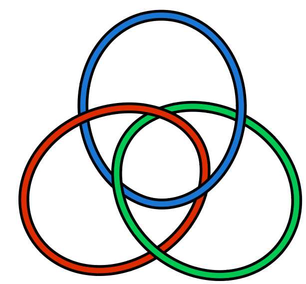


$$\mathcal{P}(3^4) = ab + ac + bc$$

$$3^4 \xrightarrow{3} 2^1$$



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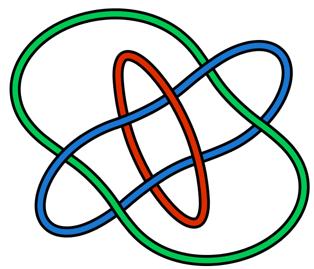

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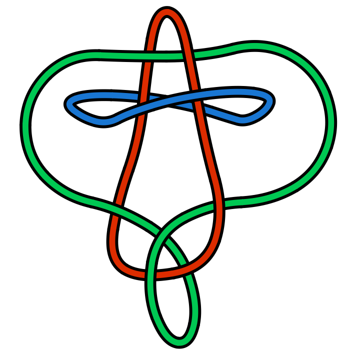

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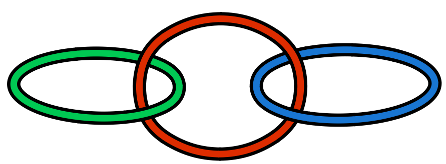

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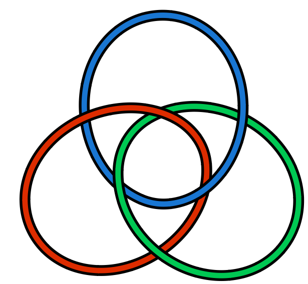

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle_{abc} + |111\rangle_{abc} + |001\rangle_{abc}\right)$$

What link class does this state belong to?

From states to links

A density matrix is separable if $\hat{\rho}_{AB} = \sum_i p_i \; \hat{\rho}_i^A \otimes \hat{\rho}_i^B$

A necessary and sufficient criterion for 2x2 and 2x3 systems:

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Separability Criterion for Density Matrices

Asher Peres*

Department of Physics, Technion–Israel Institute of Technology, 32 000, Haifa, Israel

(Received 8 April 1996)

PRL 77.1413 (1996)

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Separability of mixed states: necessary and sufficient conditions

Michał Horodecki [a], Paweł Horodecki [b], Ryszard Horodecki [c]

[a] *Department of Mathematics and Physics, University of Gdańsk, 80-952 Gdańsk, Poland*

[b] *Faculty of Applied Physics and Mathematics, Technical University of Gdańsk, 80-952 Gdańsk, Poland*

[c] *Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland*

Received 24 June 1996; accepted for publication 9 September 1996

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PLA 223.1 (1996)

Peres-Horodecki criterion ⟷ Positive Partial Transposition criterion (**PPT**)

From states to links

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$\hat{\rho}_{abc} = |\psi\rangle\langle\psi| \xrightarrow{\text{PPT}}$ Ent. $\longrightarrow$ All 3 rings are linked. $\longrightarrow$ abc

$\hat{\rho}_{ab} = \mathrm{tr}_c[\hat{\rho}_{abc}] \longrightarrow$ Ent. $\longrightarrow$ a and b are linked. $\longrightarrow$ ab

$\hat{\rho}_{ac} = \mathrm{tr}_b[\hat{\rho}_{abc}] \longrightarrow$ Sep. $\longrightarrow$ a and c are **not** linked. $\longrightarrow$ 0

$\hat{\rho}_{bc} = \mathrm{tr}_a[\hat{\rho}_{abc}] \longrightarrow$ Sep. $\longrightarrow$ b and c are **not** linked. $\longrightarrow$ 0

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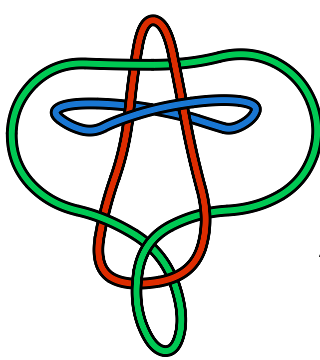

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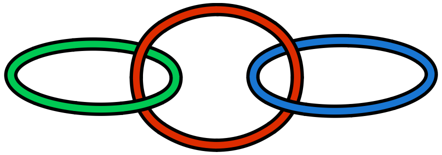
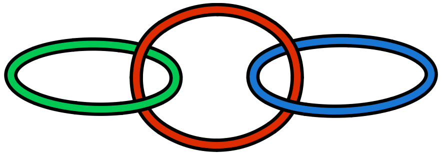
From links to states



Starting from the polynomial, how can we obtain a representative state? (“qubit map”)

From links to states



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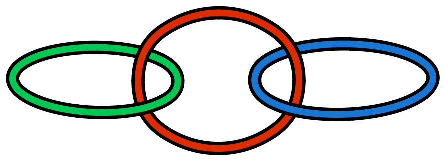
Ideally, we would want a pure state map. For now, we only have a mixed state (incomplete) one.

From links to states



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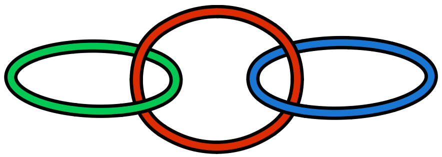
$$\vdots$$

$$\mathcal{P}(3^3) = ab + ac$$

$$|2^1\rangle_{ab}\,|q_1\rangle_c\,|0\rangle_d \qquad |2^1\rangle_{ac}\,|q_2\rangle_b\,|1\rangle_d$$

From links to states



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$$\mathcal{P}(3^3) = ab + ac$$

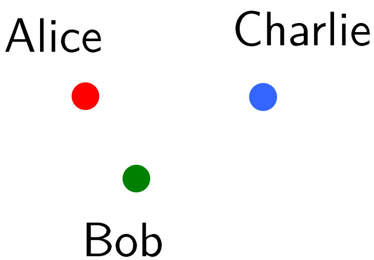
$$|\psi_3\rangle = c_1\,|2^1\rangle_{ab}\,|q_1\rangle_c\,|0\rangle_d + c_2\,|2^1\rangle_{ac}\,|q_2\rangle_b\,|1\rangle_d$$

$$\hat{\rho}_{abc}(3^3) = \frac{\mathrm{tr}_d\left[|\psi_3\rangle\langle\psi_3|\right]}{\sqrt{\langle\psi_3|\psi_3\rangle}}$$

Plug in a code and find the free parameters.

Applications

What is the physical advantage of using link classification for entanglement?



$\hat{\rho}_{abc}$ entangled

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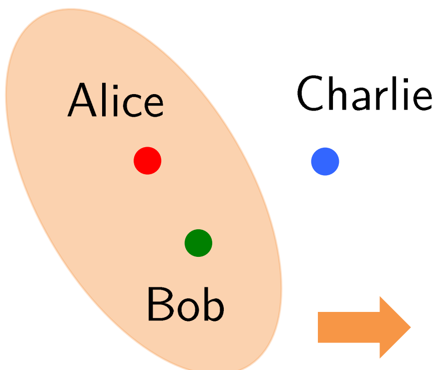

$\hat{\rho}_{abc}$ entangled

If $\hat{\rho}_{ab} = \mathrm{tr}_c[\hat{\rho}_{abc}]$ is separable, Alice and Bob will never be able to perform protocols without the help of Charlie.

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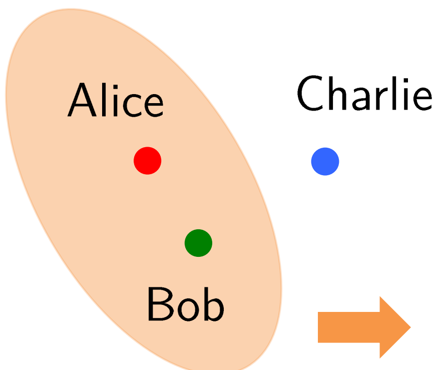


$\hat{\rho}_{abc}$ entangled

If $\hat{\rho}_{ab} = \mathrm{tr}_c[\hat{\rho}_{abc}]$ is separable, Alice and Bob will never be able to perform protocols without the help of Charlie.

SLOCC classification

States of a class can perform the same protocols.

Link classification

States of a class have the same restrictions regarding successful protocol performance.

Great for managing communications in qubit networks.

Applications

Example: Alice, Bob, Charlie and Diana share a four-partite state.

We want to allow only the following parties to communicate:

- Alice, Bob and Charlie;
- Alice, Bob and Diana;
- Charlie and Diana.

Applications

Example: Alice, Bob, Charlie and Diana share a four-partite state.

We want to allow only the following parties to communicate:

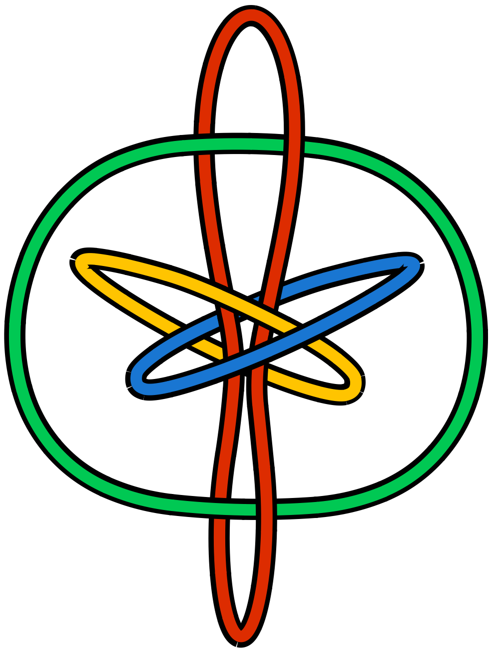
- Alice, Bob and Charlie; $\longrightarrow$ $\hat{\rho}_{abc}$ Ent. $\longrightarrow$ abc
- Alice, Bob and Diana; $\longrightarrow$ $\hat{\rho}_{abd}$ Ent. $\longrightarrow$ abd
- Charlie and Diana. $\longrightarrow$ $\hat{\rho}_{cd}$ Ent. $\longrightarrow$ cd

Applications

Example: Alice, Bob, Charlie and Diana share a four-partite state.

We want to allow only the following parties to communicate:

- Alice, Bob and Charlie; $\longrightarrow$ $\hat{\rho}_{abc}$ Ent. $\longrightarrow$ abc
- Alice, Bob and Diana; $\longrightarrow$ $\hat{\rho}_{abd}$ Ent. $\longrightarrow$ abd
- Charlie and Diana. $\longrightarrow$ $\hat{\rho}_{cd}$ Ent. $\longrightarrow$ cd



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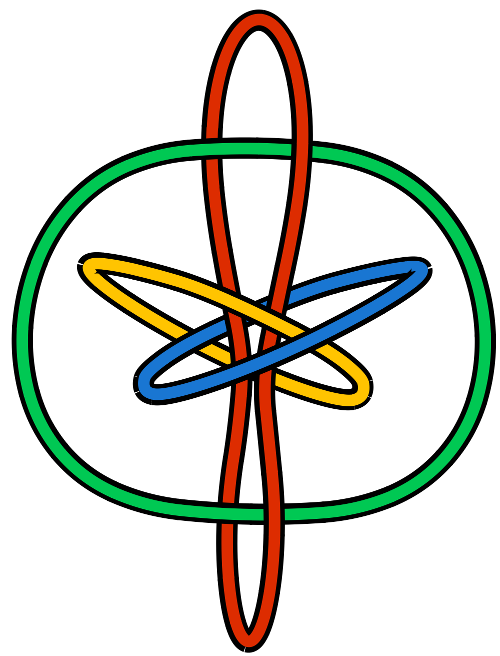

$$|\psi_{21}\rangle = |3^1\rangle_{abc}\,|1\rangle_d\,|0\rangle_e + |3^1\rangle_{abd}\,|0\rangle_c\,|1\rangle_e + \\ + |2^1\rangle_{cd}\,|00\rangle_{ab}\,|2\rangle_e$$

$$\hat{\rho}_{abcd}(4^{21}) = \frac{\mathrm{tr}_e\left[|\psi_{21}\rangle\langle\psi_{21}|\right]}{\sqrt{\langle\psi_{21}|\psi_{21}\rangle}}$$

This method bypasses a **lot** of calculations.

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(**Phys. Rev. A 100, 062329 (2019)**)

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➡ In terms of polynomials, this gives the classes:

N=4	**N=5**
k=1: abcd	**k=1:** abcde
k=2: abc + abd + acd + bcd	**k=2:** abcd + abce + ... + bcde
k=3: ab + ac + ... + cd	**k=3:** abc + abd + ... + cde
	k=4: ab + ac + ... + de

We found a complete map for **m**-resistant <u>mixed states</u> of **N** qubits:

$$|E_m\rangle_{ijk\ldots} = (m+1)\,|0\ldots0\rangle_{ijk\ldots} + |1\ldots1\rangle_{ijk\ldots}$$

$$\hat{\rho}(4,m) = \frac{\mathrm{tr}_e\left[|\psi_{4,m}\rangle\,\langle\psi_{4,m}|\right]}{\sqrt{\langle\psi_{4,m}|\psi_{4,m}\rangle}}$$

k=1: $\mathcal{P} = abcd$

$$|\psi_{4,0}\rangle = |E_0\rangle_{abcd}\,|0\rangle_e$$

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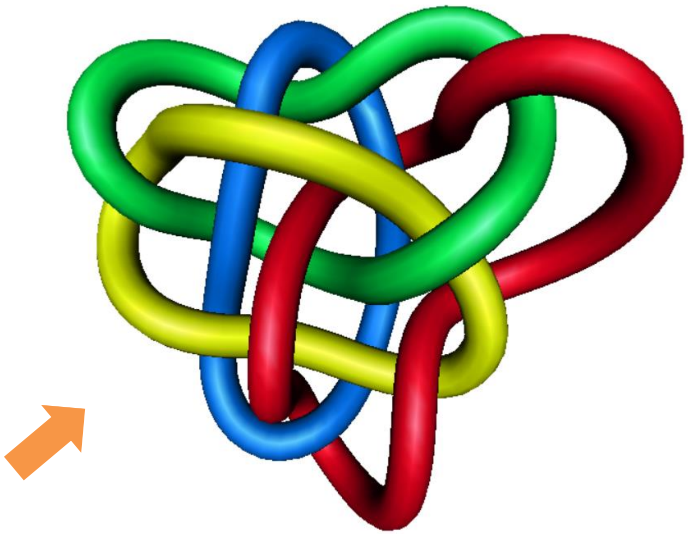

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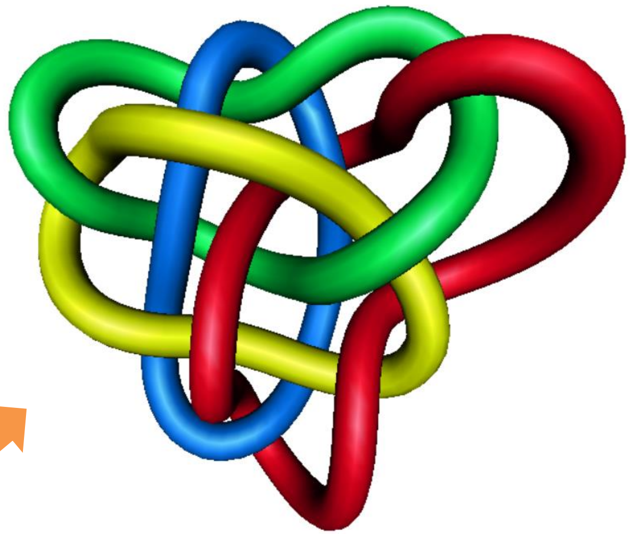

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k=3:

$$\mathcal{P} = ab + ac + ad + bc + bd + cd$$

$$|\psi_{4,2}\rangle = |E_1\rangle_{ab}\,|00\rangle_d\,|0\rangle_e + \dots + |E_1\rangle_{cd}\,|00\rangle_{ab}\,|5\rangle_e$$

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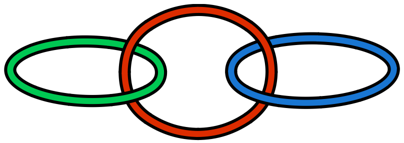
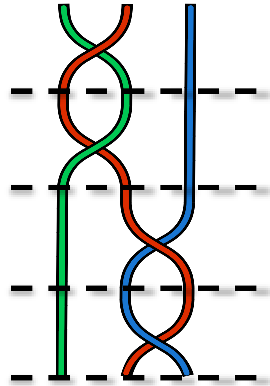


$\mathcal{P}(3^3) = ab + ac$



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Use the **braid group** to build a direct connection to quantum states, via the generators σ_i.

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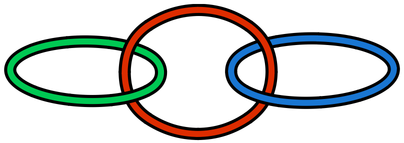

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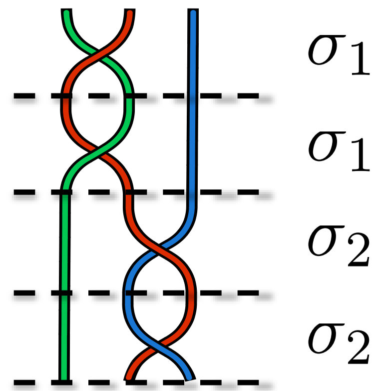

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Use the **braid group** to build a direct connection to quantum states, via the generators σ_i.

Braid group

$$B_3 = \{\sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$$

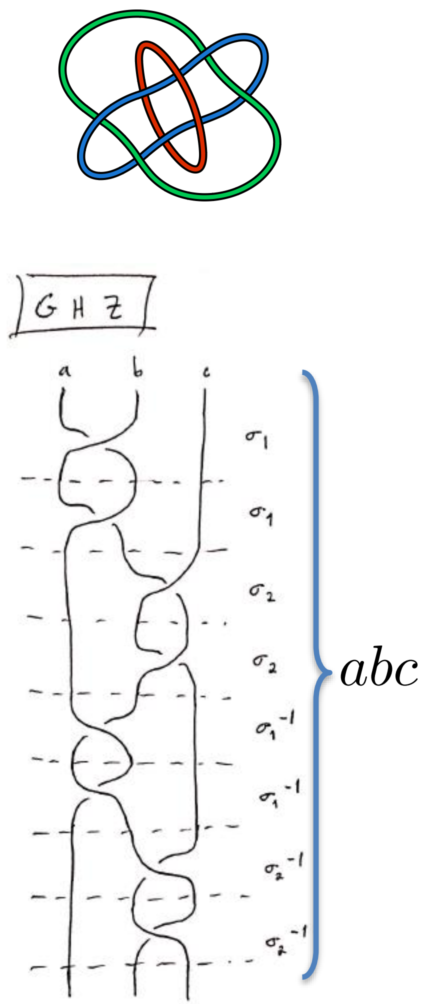
$$\sigma_1 \to R \otimes 1$$
$$\sigma_2 \to 1 \otimes R$$

<u>Yang-Baxter Eq.</u>

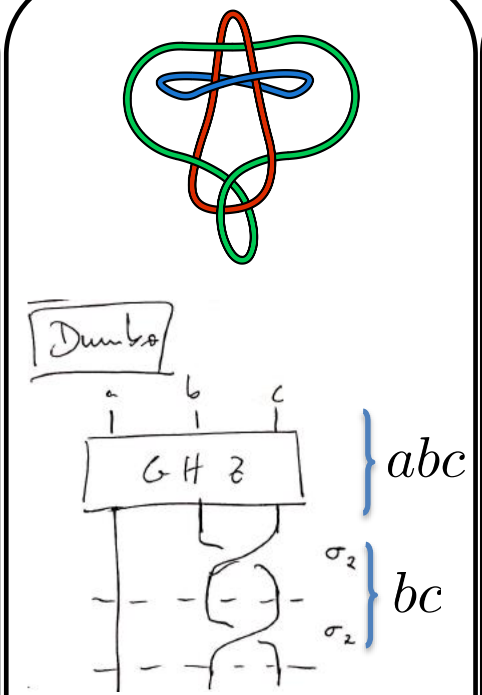
$$(1 \otimes R)(R \otimes 1)(1 \otimes R) = (R \otimes 1)(1 \otimes R)(R \otimes 1)$$

Work in progress...

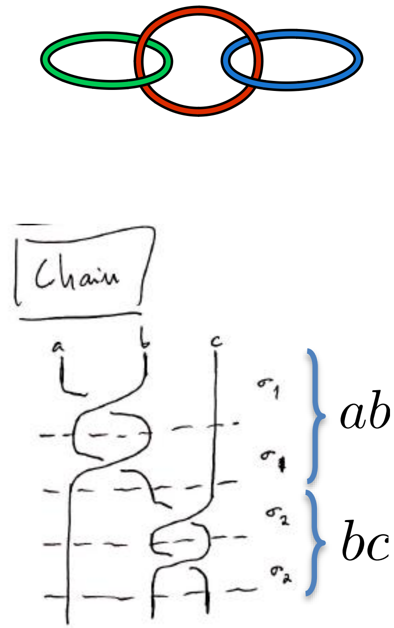
If the idea is correct, we must start by getting right the case for 3 qubits:



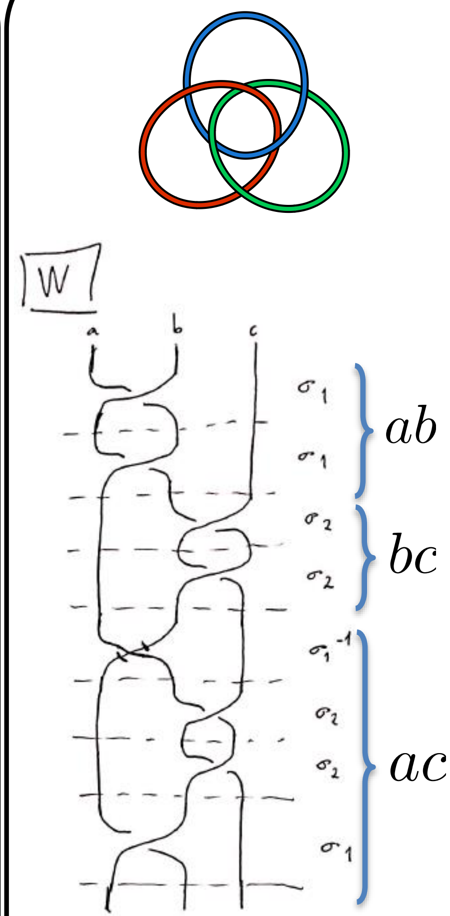
$\mathcal{P}(3^1) = abc$



$\mathcal{P}(3^2) = abc + ab$

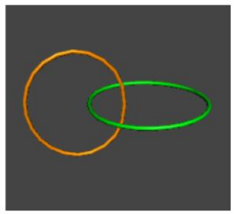
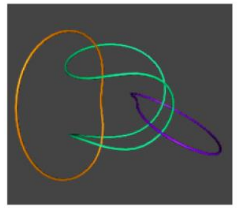
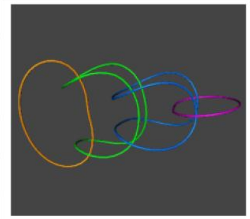
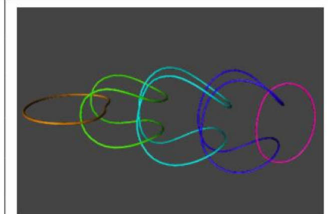
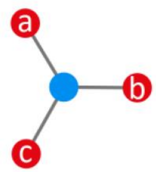
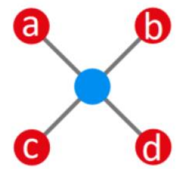
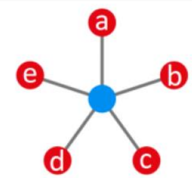


$\mathcal{P}(3^3) = ab + ac$



$\mathcal{P}(3^4) = ab + ac + bc$

A very recent work was published (**arXiv:2007.02882 [quant-ph]**) which uses the link formalism to derive a representation of the entanglement flow and dynamics in quantum teleportation using graphs.

Monomials	*ab*	*abc*	*abcd*	*abcde*
Links				
Graphs				

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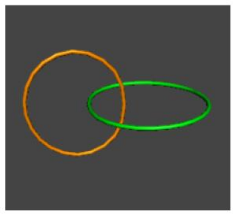
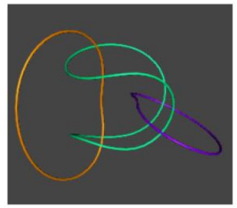
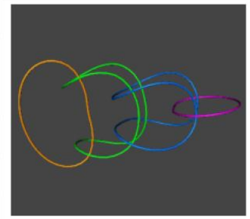
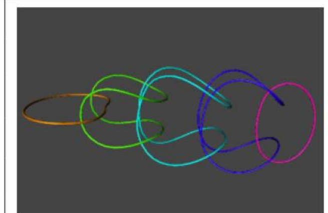
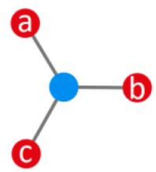
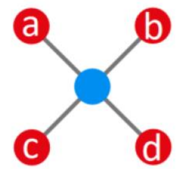
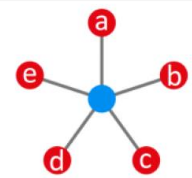
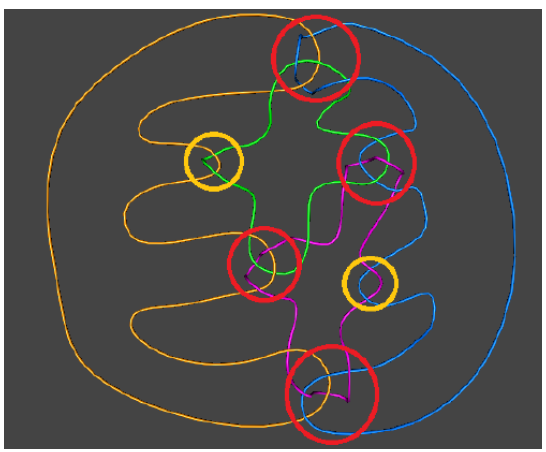
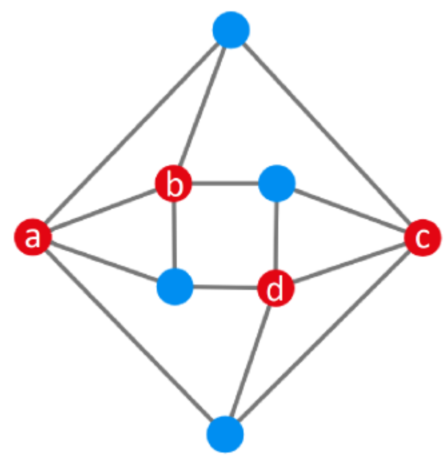
Links				
Graphs				





$$\mathcal{P}(3^{34}) = abc + abd + acd + bcd + ab + cd$$

Conclusions and future work

- New classification scheme: **link classes** $=$ **entanglement classes.**
- Intuitive as a cataloging system $+$ has computational advantages.
- Physical applications in qubit networks: who gets to communicate with who.
- States from polynomials show problem-solving potential.

Future work:

- Complete pure map (**abcd**+**ab**+**ac**);
- Protocols based on links;
- Consider details of the links (such as crossing numbers)?
- Finding the number of classes for N rings;
- A rigorous connection with known mathematical formalisms.