A CATEGORIFICATION OF THE TUBE ALGEBRA

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Based around work
in Arxiv:2006.06536
w. Delcamp

Motivation

* Application of higher category theory to physics
  - Classification of phases of matter beyond Landau-Ginzburg symmetry breaking
  - phenomenology of such systems
  - Applications to quantum information/computation

Topological phase of matter

* Equivalence class of gapped, local quantum many-body systems.

* Equivalence relation $\Rightarrow$ Two systems in same topological phase if they share a common TQFT description of far infra-red limit.

Physically:

* Example
  - Fractional quantum hall effect, Chern-Simons effective field theory

* Topological excitations
  - Anyons
  - Candidate physics for engineering fault-tolerant quantum computer. Topological protection from errors.

TQFT in a nutshell

Atiyah, TQFT is a symmetric monoidal functor

$$Z : (n+1)\,\mathrm{Cob} \to \mathrm{Vect}$$

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Essentially a set of rules

$Z : M^n \mapsto \mathcal{H}[M^n]$ ← state-space

$Z : C : M^n \to M'^n \mapsto Z(C) : \mathcal{H}[M^n] \to \mathcal{H}[M'^n]$

↖ depends only on diffeomorphism class.

TQFT in a nutshell

Pros: nice concise defn ✓

Cons: lots of data required ✗

lack of phenomenology ✗

- only tells us $\mathcal{H}[M^n]$ as rep of $MCG(M^n)$
- nothing about excitations
- real materials have boundaries, only tells us about closed spatial materials

No 1-1 between TQFT and physically realisable TPM ✗

- not necessarily local.

TQFT in a nutshell

* Extended TQFT - addresses some of problems

$Z(M^{n+1}) \in \mathbb{C}$

$Z(M^n) \in \mathrm{Vect}$

$Z(M^{n-1}) \in 2\mathrm{Vect}$

$\vdots$

$Z(*) \in n\mathrm{Vect}$

↖ provides notion of locality

Idea: swap topological complexity for algebraic data

TQFT in a nutshell

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$Z(M^{n+1}) \in \mathbb{C}$

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$Z(M^{n-1}) \in 2\mathrm{Vect}$

$\vdots$

$Z(*) \in n\mathrm{Vect}$ ← cobordism hypothesis

≃ can reconstruct rest of theory from point

TQFT in a nutshell

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$\vdots$

$Z(*) \in n\mathrm{Vect}$

<u>However</u>: what do we assign to lower dim manifolds

cobordism hypothesis

$\simeq$ can reconstruct rest of theory from point

HAMILTONIAN MODELS OF TPM

Local, lattice Hamiltonian scheme

defn: pick spatial dimension n, then for all triangulated n-manifolds M^n_Δ

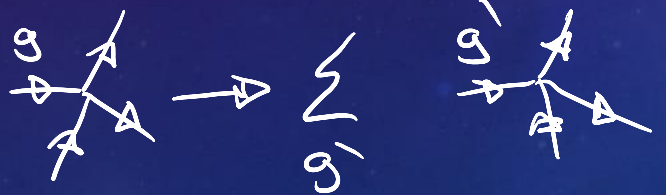
$g_{ij} \in h$

1) $\{s : M^n_\Delta \to h\}$ ← set of classical field configurations

2) $H = \sum_{\Delta^0 \subseteq Int(M^n_\Delta)} H_{\Delta^0}$ ← Hamiltonian

where $H : \mathrm{Span}_{\mathbb{C}}\{s : M^n_\Delta \to h\} \longrightarrow \mathrm{Span}_{\mathbb{C}}\{s : M^n_\Delta \to h\} \equiv V[M^n_\Delta]$

H_{Δ_i} :



$\phi(g', \dots)$ ← phase factor!

– changes field configuration in local neighbourhood of a vertex

Exactly solvable LLHS

$V[M^n_\Delta] := \mathrm{Span}_{\mathbb{C}}\{s : M^n_\Delta \to L\}$, $H = \sum_{\Delta^\circ \subseteq \mathrm{Int}(M^n_\Delta)} H_{\Delta^\circ}$

- Exactly solvable $\Longrightarrow$ $H_{\Delta^\circ} \cdot H_{\Delta^\circ} = H_{\Delta^\circ}$ $\leftarrow$ projector

$\forall \Delta^\circ \subseteq M^n_\Delta$

$[H_{\Delta^\circ}, H_{\Delta'^\circ}] = 0$ $\leftarrow$ mutually commuting projectors!

- ground state subspace $\Rightarrow \mathcal{H}[M^n_\Delta] \equiv \mathrm{Im}\left[\prod_{\Delta^\circ} H_{\Delta^\circ}\right]$

Models for TPM

* Given exactly solvabe LLHS

  if $\forall M^n$ w. triangulations M^n_Δ, M'^n_Δ s.t. $\partial M^n_\Delta = \partial M'^n_\Delta$

  $\exists$ unitary isomorphism s.t. $\prod_{\Delta^\circ \subseteq Int(M^n_\Delta)} H_{\Delta^\circ} \circ U = U \circ \prod_{\Delta^\circ \subseteq Int(M'^n_\Delta)} H_{\Delta^\circ}$ and

  $$\mathcal{H}[M^n_\Delta] \xrightarrow{U} \mathcal{H}[M'^n_\Delta] \xrightarrow{U'} \mathcal{H}[M''^n_\Delta] = \mathcal{H}[M^n_\Delta] \xrightarrow{U' \circ U} \mathcal{H}[M''^n_\Delta]$$

  – we say we have a topological lattice model

* such models expected to capture infra-red limit effective field theory of condensed matter lattice models eg. TPM.

State-Sum TQFT

Given data of topological lattice model we can construct state-sum TQFT

Roughly: ssTQFT computes TQFT on simplicial model of space-time

To find ssTQFT we use unitary isomorphisms U to define partition on local balls of spacetime and glue to evaluate a full partition function.

Continuum theory

* Using colimit over all triangulations we can define $\mathcal{H}[\hat{M}]$ for closed M^n via a colimit construction
* Such construction defines continuum theory which can be lifted to Atiyah TQFT.

Folklore : ssTQFT in 1-1-correspondence w. fully extended TQFT

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Assuming true ⇒

Question : What do our TLM / ssTQFT assign to lower dimensional manifolds

- How can we compute properties of lattice model from this vantage point?

TUBE ALGEBRAS

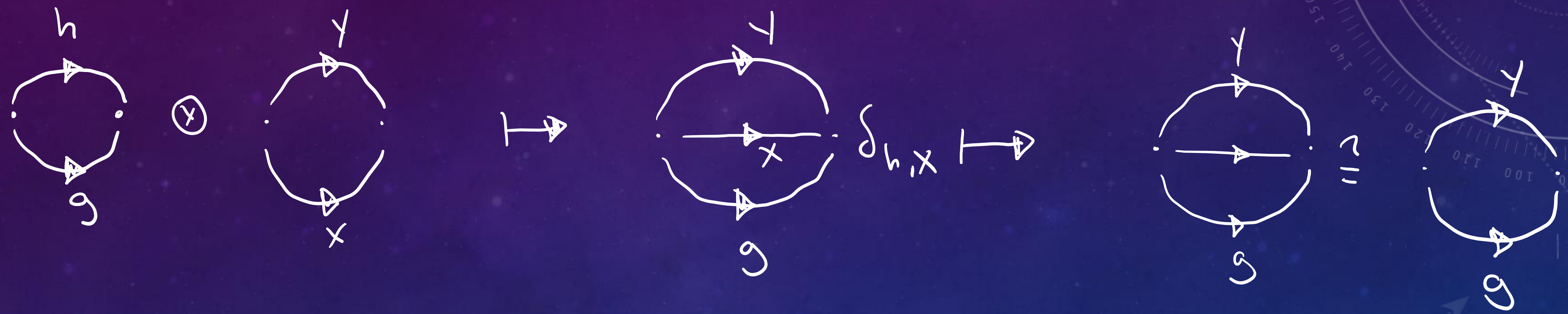
Tube algebras:

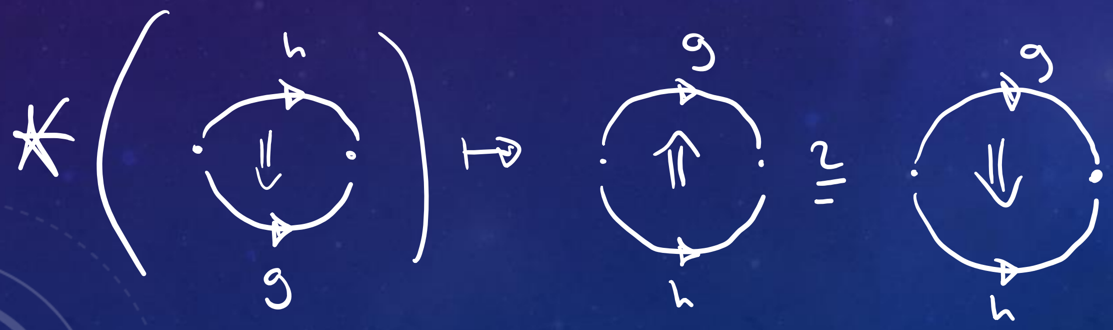
So far given a TLM we showed how to define a Hilbert space $\mathcal{H}[M_\Delta]$ to a triangulated n-manifold

- the idea of the tube algebra is to associate a "2-Hilbert space" to a triangulated (n-1)-manifold.

* in the following a (finite dim) 2-Hilbert space := semisimple $\mathbb{C}$-linear Abelian category.
  (I won't discuss categorified inner product structure but can be added!)

Tube algebra





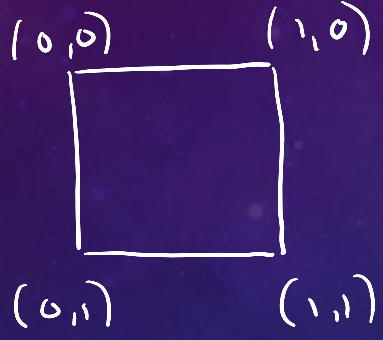
(* - structure)

Tube algebras

Let N_Δ be a triangulated $(n-1)$-manifold and ΔN_Δ a triangulation of

$N \times_p I = \dfrac{N \times I}{\sim} \qquad (n,i)\sim(n,j) \quad \forall\, (n,i),(n,j) \in \partial N \times I$

eg: $I \times_p I =$



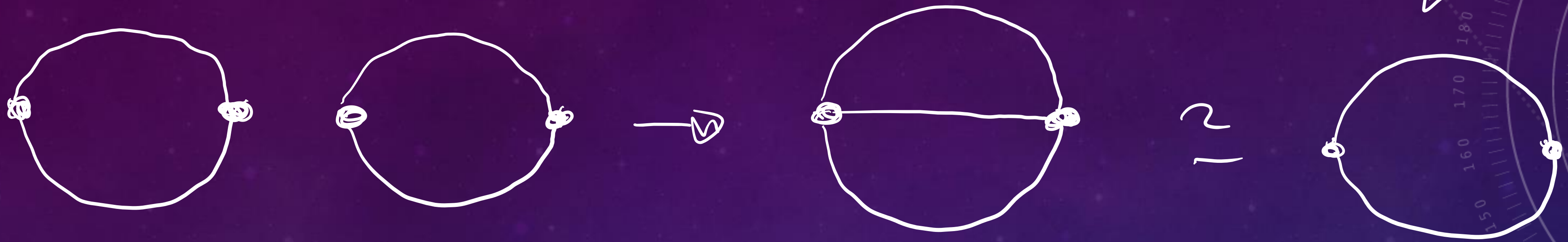
$\Big/ \;\; (0,i)\sim(0,i')$
$\quad\;\; (1,j)\sim(1,j')$

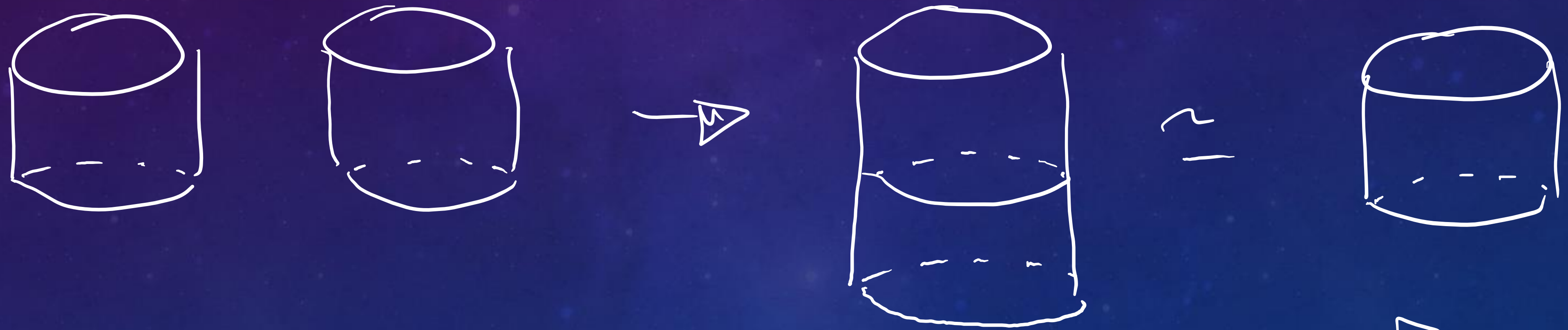
$=$

eg: $S^1 \times_p I = S^1 \times I =$

Tube algebras

$N_a = I$





$${}_aN_a \sqcup {}_aN_a \to {}_aN_a \cup_{N_a} {}_aN_a \cong {}_aN_a$$

$N_a = S^1$

Tube algebra

Given the tube algebra on $\mathcal{H}^*[{}_\Delta N_\Delta]$ we define semisimple Abelian category $\mathrm{Mod}(N_\Delta)$ as category of $\mathcal{H}^*[{}_\Delta N_\Delta]$ modules.

$\mathrm{Mod}(N_\Delta) \equiv$ 2-Hilbert space we associate to triangulated $(n-1)$-manifold N_Δ !

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$\mathrm{Mod}(N_\Delta) \equiv$ 2-Hilbert space we associate to triangulated $(n-1)$-manifold N_Δ !

* Importantly $\mathrm{Mod}(N_\Delta) \simeq \mathrm{Mod}(N_{\Delta'})$ (equivalence of ss. Abelian categories) for all $N_\Delta, N_{\Delta'}$ s.t. $\partial N_\Delta = \partial N_{\Delta'}$

- this follows from Morita equivalence of $\mathcal{H}^*[{}_\Delta N_\Delta]$ and $\mathcal{H}^*[{}_{\Delta'} N_{\Delta'}]$

Morita equivalence:

two equivalent definitions: two algebras A, B are Morita equivalent iff

1) there exists an A-B-bimodule ${}_AQ_B$ and a B-A-bimodule ${}_BP_A$
s.t. ${}_AQ_B \otimes_B {}_BP_A \cong A$ ← isomorphic as A-A-bimodules
${}_BP_A \otimes_A {}_AQ_B \cong B$ B-B-bimodules

2) $\mathrm{Mod}(A)$ is equivalent to $\mathrm{Mod}(B)$

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${}_BP_A \otimes_A {}_AQ_B \cong B$ $\qquad\qquad$ B-B-bimodules

2) $\mathrm{Mod}(A)$ is equivalent to $\mathrm{Mod}(B)$

To see $\mathcal{H}^*[{}_\Delta N_\Delta]$ is Morita to $\mathcal{H}^*[{}_{\dot\Delta} N_{\dot\Delta}]$ we make following observations:

* $\mathcal{H}[\ ]$ defines a right $\mathcal{H}^*[\ ]$ and left $\mathcal{H}^*[\ ]$ module!

* $\mathcal{H}[\ ]$

$\otimes_{\mathcal{H}^*[\ ]} \cong \mathcal{H}^*[\ ]$ as $\mathcal{H}^*[\ ]$ bimodules and similarly in other

$\mathcal{H}[\ ]$ direction!

Crossing with circle

defn: the dimension of a category $\equiv \mathrm{Nat}(\mathrm{id}, \mathrm{id})$

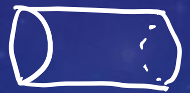
facts: * for an algebra A $\quad \dim[\mathrm{Mod}(A)] \cong Z(A)$ as commutative algebras

* if A is Morita B $\quad Z(A) \cong Z(B)$

$$\dim \mathrm{Mod}(N_\Delta) \cong \dim \mathrm{Mod}(N_{\Delta'}) \cong \mathcal{H}[N_\Delta \times S^1]$$

↑ isomorphism of Hilbert spaces

for all closed $(n-1)$-manifolds N

$\mathcal{H}[$  $] \subseteq \mathcal{H}[$  $]$

↗ identify boundary + H operators on "gluing seam"

CATEGORIFIED TUBE ALGEBRAS

(and some physics . . .)

Algebra

Given a monoidal category $\mathcal{C}$ an algebra (A, p) is an object $A \in \mathcal{C}^0$ and morphism $p : A \otimes A \to A$ s.t. following commutes

$$
\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "p \otimes A"] \arrow[d, "\alpha_{A,A,A}"] & A \otimes A \arrow[dr, "p"] & \\
 & & A \\
A \otimes (A \otimes A) \arrow[r, "A \otimes p"'] & A \otimes A \arrow[ur, "p"'] &
\end{tikzcd}
$$

2-Algebra

Given a monoidal bicategory $\mathcal{B}$ a 2-algebra (A, P, Q) is an object $A \in \mathcal{B}^0$
a morphism $P : A \otimes A \to A$ and 2-morphism

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "P \otimes A"] \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \arrow[dr, "P"] \arrow[d, phantom, "\swarrow Q"] & \\
 & & A \\
A \otimes (A \otimes A) \arrow[r, "A \otimes P"'] & A \otimes A \arrow[ur, "P"'] &
\end{tikzcd}$$

satisfying some coherence data...

2-Algebra

Given a monoidal bicategory $\mathcal{B}$ a 2-algebra (A, P, Q) is an object $A \in \mathcal{B}^0$ a morphism $P : A \boxtimes A \to A$ and 2-morphism

$$\begin{tikzcd}
(A \boxtimes A)\, A \arrow[r, "P \boxtimes A"] \arrow[d, "\alpha_{A,A,A}"] & A \boxtimes A \arrow[dr, "P"] \arrow[d, phantom, "\Swarrow Q"] & \\
 & {} & A \\
A \boxtimes (A \boxtimes A) \arrow[r, "A \boxtimes P"'] & A \boxtimes A \arrow[ur, "P"'] &
\end{tikzcd}$$

satisfying some coherence data...

- * example in 2Vect (Bicategory of Vect-Module categories) are tensor categories
- * semisimple 2-algebras in 2Vect are multifusion categories

see eg. EGNO, Douglas+Reutter 1812.11933

Categorified tube algebras

* semisimple 2-algebra in 2Vect, multifusion categories.
* Let O_Δ be a closed triangulated $(n-2)$-manifold eg $*$ $n=2$
* Can define $\mathrm{Mod}(O_\Delta \times I)$ eg $\mathrm{Mod}(\bullet\!-\!\bullet)$ $n=2$
* want to define linear monoidal structure

$$\otimes : \mathrm{Mod}(O_\Delta \times I) \boxtimes \mathrm{Mod}(O_\Delta \times I) \longrightarrow \mathrm{Mod}(O_\Delta \times I)$$

$\Rightarrow$ Categorified tube algebra for O_Δ !

Categorified tube algebra for $*$ in 2+1D

induced from Morita equiv.

$$\mathrm{Mod}(\bullet\!-\!\bullet) \boxtimes \mathrm{Mod}(\bullet\!-\!\bullet) \to \mathrm{Mod}(\bullet\!-\!\bullet\!-\!\bullet) \simeq \mathrm{Mod}(\bullet\!-\!\bullet)$$

$*$ action on morphisms similarly defined

Categorified tube algebra for * in 2+1D

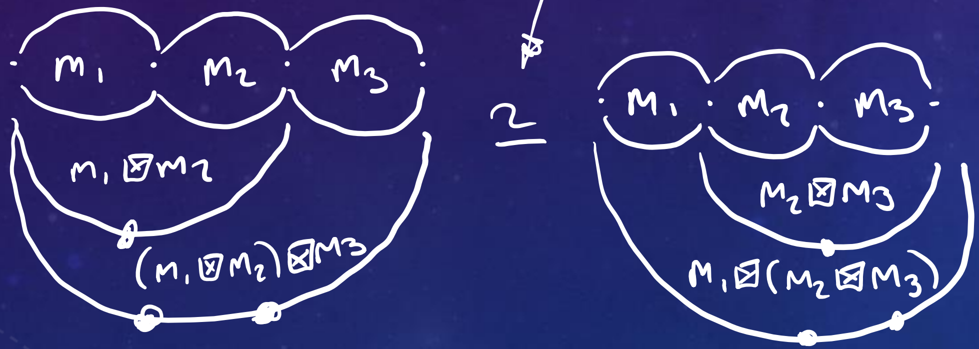
induced from Morita equiv.

$\mathrm{Mod}(\bullet\!-\!\bullet) \boxtimes \mathrm{Mod}(\bullet\!-\!\bullet) \to \mathrm{Mod}(\bullet\!-\!\bullet\!-\!\bullet) \simeq \mathrm{Mod}(\bullet\!-\!\bullet)$

* action on morphisms similarly defined

- Now we have a natural choice for associator

Triangulation change defines module intertwiner. Triangulation of partition function guarantees solution to pentagon equation.



* unit + dualisability are consequences of existence of 2-inner product

Examples

$O_\Delta = *$ in 2+1D THGT with $\mathcal{G} = (\partial : E \to G, \triangleright)$

let $\bar{\mathcal{G}}$ denote corresponding monoidal groupoid

$[\bar{\mathcal{G}}, \mathrm{Vect}]^{\otimes}$ = multifusion cat of monoidal functors + monoidal nat trans

$O_\Delta = *$ in 2+1D TGT theory $\simeq \mathrm{Vect}_G^{\otimes}$, multifusion cat of G-graded vector spaces

Now we define

$MOD(O_\Delta)$ = bicategory of $\mathrm{Mod}^{\otimes}({}_\Delta O_\Delta)$-module categories, nat-trans, modifications

$\equiv$ 3-Hilbert space assigned to N_Δ

$\Rightarrow MOD(*)$ in 2+1D THGT $\simeq 2\mathrm{Rep}(\mathcal{G})$

and some physics....

– How can we interpret MOD(*) ?

2-morphisms define fusion

Defn: Dimension of bicategory $\equiv$ braided monoidal category of pseudo-natural transformations of identity bifunctor

$$\dim \mathrm{MOD}(\mathcal{O}_\Delta) \simeq \mathrm{Mod}^{R,\otimes}(\mathcal{O}_\Delta \times S^1) \simeq Z\left[\mathrm{Mod}^{\otimes}(\mathcal{O}_\Delta \times I)\right]$$

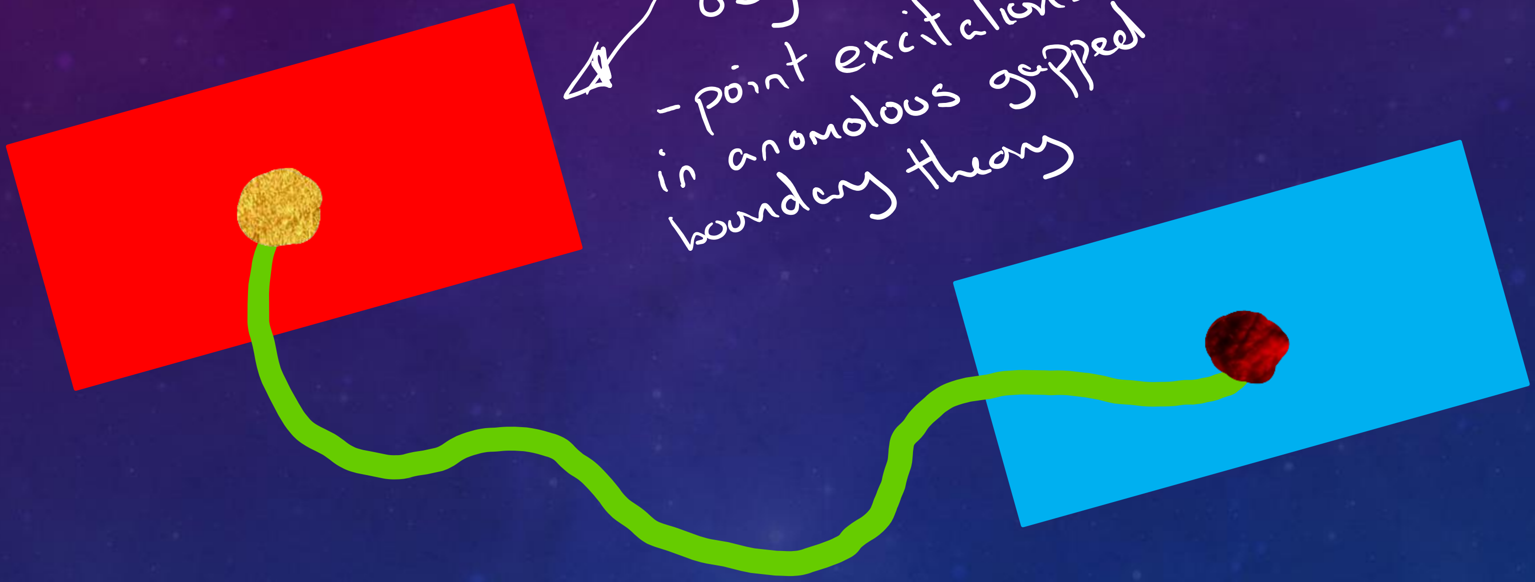
$$\dim \mathrm{MOD}(*) \simeq Z\left[\mathrm{Mod}^{\otimes}(\,*\,)\right] \leftarrow \text{Drinfeld center}$$

$\Rightarrow$ algebraic data of anyons w. fusion + braiding and some more physics ....

In 3+1D $\mathrm{MOD}(S^1)$

[for DW theory see 2006.06536 w. Delcamp]

objects
- point excitations in anomolous gapped boundary theory

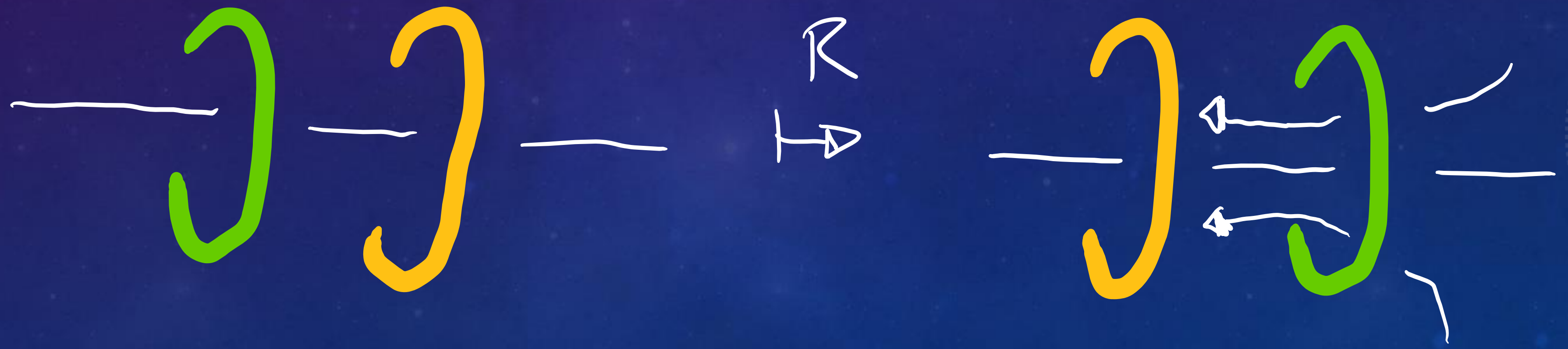


morphisms
- open string excitations

and some more physics....

In 3+1D $\quad MOD(S^1)$ $\qquad$ (coming soon w. Delcamp)

$$\dim[MOD(S^1)] \simeq \mathrm{Mod}^{R,\otimes}(S^1 \times S^1)$$ - describes closed loop excitations w. braiding and fusion!



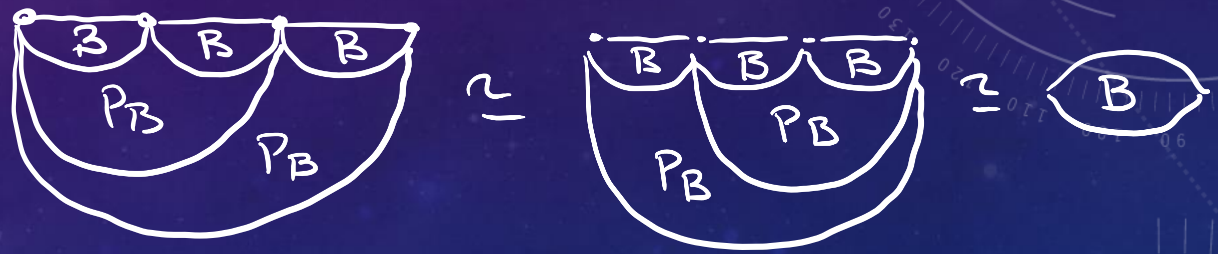
Thanks

For

Listening!

Algebras and chunks of space

$$\begin{tikzcd}
(B\otimes B)\otimes B \arrow[r, "P_B\otimes B"] \arrow[d, "\alpha"] & B\otimes B \arrow[dr, "P_B"] & \\
& & B \\
B\otimes(B\otimes B) \arrow[r, "B\otimes P_B"'] & B\otimes B \arrow[ur, "P_B"'] &
\end{tikzcd}$$



Modules and chunks of space

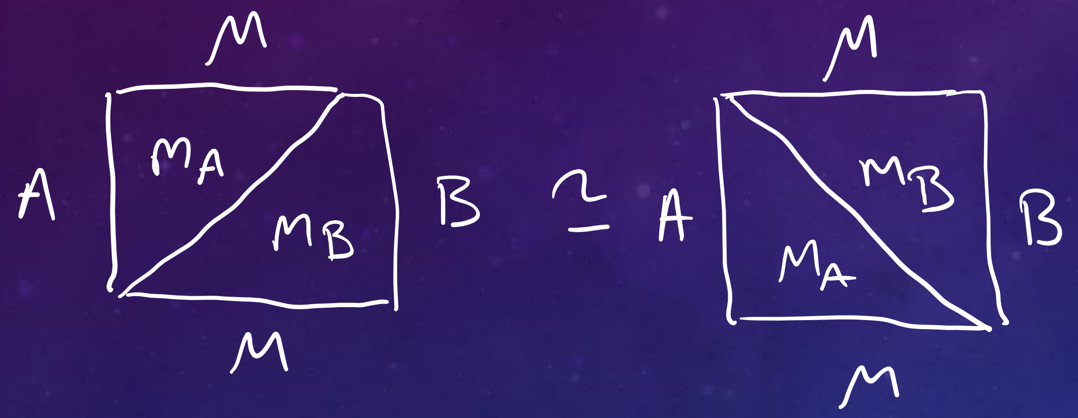
$$
\begin{tikzcd}
(M \otimes B) \otimes B \arrow[r, "m_B"] \arrow[d, "\alpha_{M,B,B}"] & M \otimes B \arrow[dr, "B_B"] & \\
 & & M \\
M \otimes (B \otimes B) \arrow[r, "M \otimes P_B"'] & M \otimes B \arrow[ur, "m_B"'] &
\end{tikzcd}
$$

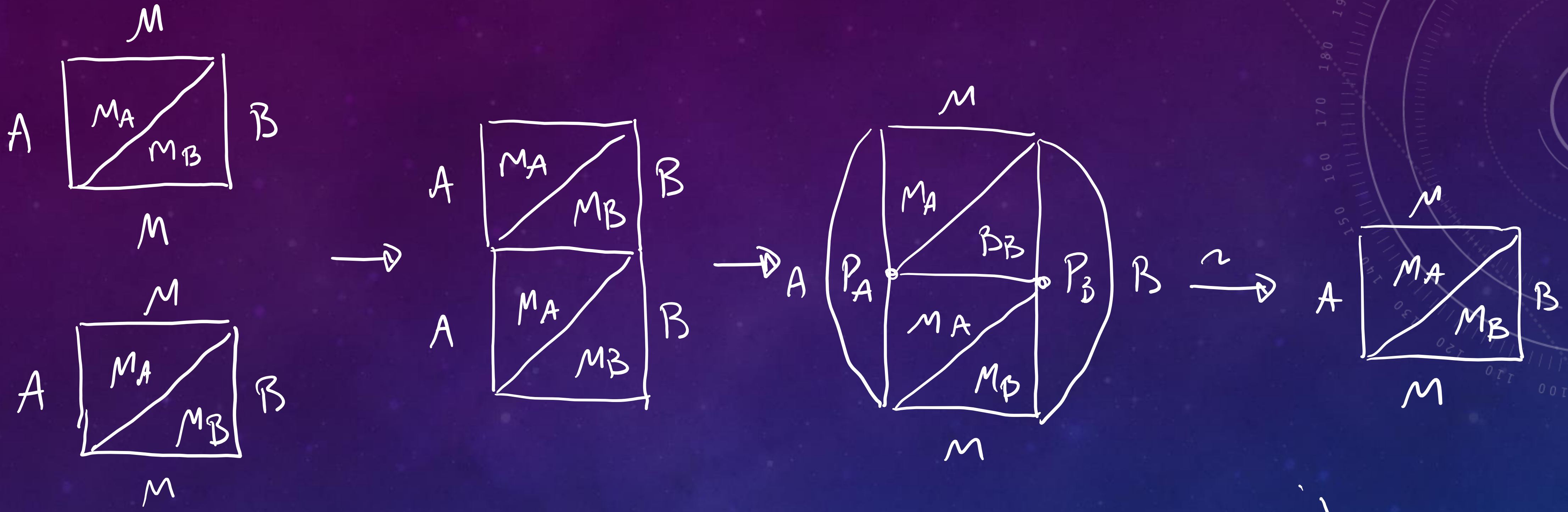
Bimodules and chunks of space

$$
\begin{tikzcd}
A\otimes([A\otimes(M\otimes B)]\otimes B) \arrow[rr, "A\otimes(\phi_{A,B}\otimes B)"] \arrow[d, "A\otimes\alpha_{A,M\otimes B,B}"] & & A\otimes(M\otimes B) \arrow[dr, "\phi_{A,B}"] & \\
A\otimes(A\otimes[(M\otimes B)\otimes B]) \arrow[d, "A\otimes(A\otimes\alpha_{M,B,B})"] & & & M \\
A\otimes(A\otimes[M\otimes(B\otimes B)]) \arrow[d, "A\otimes(A\otimes(M\otimes\rho_B))"] & & & \\
A\otimes(A\otimes(M\otimes B)) \arrow[r, "\alpha^{-1}_{A,A,M\otimes B}"] & (A\otimes A)\otimes(M\otimes B) \arrow[rr, "m_A\otimes(M\otimes B)"] & & A\otimes(M\otimes B) \arrow[uu, "\phi_{A,B}"']
\end{tikzcd}
$$

Bimodules and chunks of space

$$\begin{tikzcd}
(A \otimes M) \otimes B \arrow[r, "m_A \otimes B"] \arrow[d, "\alpha_{A,M,B}"] & M \otimes B \arrow[dr, "m_B"] & \\
A \otimes (M \otimes B) \arrow[r, "A \otimes m_B"'] & A \otimes M \arrow[r, "m_A"'] & M
\end{tikzcd}$$





- boundary tube algebra

describes renormalisation properties for boundary excitation!

- Rep ⇒ excite