Some Categorical Considerations in Extending TQFT via Higher Gauge Theory

Jeffrey C. Morton

SUNY Buffalo State

TQFT Club Thematic Seminar, July 2020

Abstract: In this informal talk, I will look at some considerations that show up when extending gauge-theoretic construction of Topological Quantum Field Theory to connections on gerbes and higher structures. In particular, I will mention some contexts where higher categories with cubical or other more complex shapes of higher morphisms seem to recur, and suggest a few questions this raises.

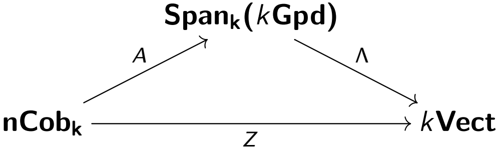
- Construction of TQFT from Gauge Theory
- Higher Gauge Theory and Double Categories of Connections
- Double Categories of Cobordisms
- Some Questions

A Extended TQFT is a k-functor between k-categories:

$$Z : \mathbf{nCob_k} \to k\mathbf{Vect}$$

By $\mathbf{nCob_k}$, we mean a k-category whose objects are $(n-k)$-dimensional manifolds (possibly with some structure) and whose morphisms are cobordisms of manifolds with dimension up to n.
By $k\mathbf{Vect}$ (sometimes called $k\mathbf{Alg}$) we mean a suitable k-category which is a suitable generalization of **Vect** (or **Hilb**) in the case of $k = 1$. (In particular, it should be Abelian, have adjoints for all morphisms, and some other properties.)
This is intended to extend Atiyah's definition of aTQFT, which is the case where $k = 1$.

The Freed-Hopkins-Lurie-Teleman program for constructing Extended TQFT's is to obtain Z as a composite of two k-functors, which I'll write as:



The k-category $\mathbf{Span_k}(k\mathbf{Gpd})$ has objects which are k-groupoids (that is, k-categories where everything is invertible), and all morphisms are *spans* of the corresponding morphisms from $k\mathbf{Gpd}$. A is *classical field theory* valued in k-groupoids, and Λ is a *quantization* k-functor.

One type of TQFT uses *gauge theory* as the classical field theory:

- Basic objects: classically, "fields" are *connections* on *principal G-bundles*
- These define parallel transport along curves in associated vector bundles
- Connections are related by *gauge transformations*.
- Groupoid: $A(\Sigma)$ has connections on principal bundles over Σ as objects, and gauge transformations as morphisms

In some cases, we are interested mainly in *flat* connections, where we have the result:

$$A(\Sigma) = Conn(\Sigma) /\!\!/ Gauge(\Sigma) \simeq Hom(\Pi_1(\Sigma), G)$$

The first form is the *transformation groupoid* of the action of the group $Gauge(\Sigma)$ of all gauge transformations, on the *fine moduli space* $Conn(\Sigma)$ of connections. It has:

- Objects: connections on Σ
- Morphisms: pairs (g, γ) where $\gamma : g \to g'$ is a gauge transformation

This is a general construction which can be done for any group action:

$$\begin{array}{ccc} G \times G \times X & \longrightarrow & G \times X \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array}$$

Generalization: We want to do the same for *higher gauge theories* based on 2-groups:

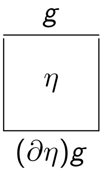
A **2-group** $\mathcal{G}$ is a 2-category with a unique object $\star$, and all morphisms and 2-morphisms invertible. This is equivalent to a group object in **Gpd** (up to the existence of the object $\star$).

2-groups are classified by *crossed modules* $(G, H, \triangleright, \partial)$, where G and H are groups, $G \triangleright H$ is an action of G on H by automorphisms and $\partial : H \to G$ is a homomorphism, satisfying some natural equations.

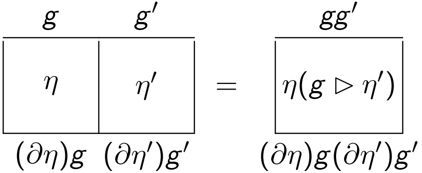
The 2-group **G** given by $(G, H, \triangleright, \partial)$ has:

- **Objects**: elements of G
- **Morphisms**: $G \times H$, with $(g, h) : g \to (\partial h)g$

We can depict the horizontal composition of a 2-group like this:

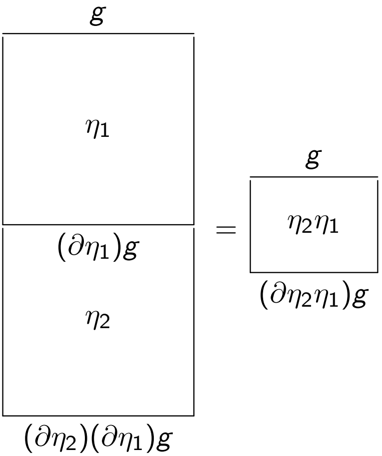


With horizontal composition:



(Using that $(\partial\eta)g(\partial\eta')g' = \partial(\eta g \triangleright \eta')$ by the basic axioms of crossed modules.)

The vertical compsition would be drawn like:



(Which only uses that ∂ is a homomorphism.)

Actions of 2-Groups on Categories

Global 2-group symmetry makes sense for any objects $\mathbf{C}$ in a bicategory $\mathcal{B}$, as a (strict) 2-functor:

$$\Phi : \mathcal{G} \to End(\mathbf{C})$$

If $\mathbf{C}$ is a category (so $\mathcal{B} = \mathbf{Cat}$), this amounts to a functor $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ satisfying:

$$\begin{array}{ccc}
\mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow{\otimes \times Id_{\mathbf{C}}} & \mathcal{G} \times \mathbf{C} \\
{\scriptstyle Id_{\mathcal{G}} \times \hat{\Phi}} \downarrow & & \downarrow {\scriptstyle \hat{\Phi}} \\
\mathcal{G} \times \mathbf{C} & \xrightarrow[\hat{\Phi}]{} & \mathbf{C}
\end{array}$$

Given an action of $\mathcal{G}$ on $\mathbf{C}$, the transformation 2-groupoid $\mathbf{C}/\!\!/\mathcal{G}$ is the groupoid in **Cat** with:

- **Category of objects**: $(\mathbf{C}/\!\!/\mathcal{G})^{(0)} = \mathbf{C}$.
- **Category of morphisms**: $(\mathbf{C}/\!\!/\mathcal{G})^{(1)} = \mathcal{G} \times \mathbf{C}$

with structure maps that amount to is a *double category*:

	$C^{(0)}$	$C^{(1)}$
Objects	x	$x \xrightarrow{f} y$
Morphisms	$x \xrightarrow{g} z$ (vertical)	$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \swarrow F & \downarrow \\ z & \longrightarrow & w \end{array}$

If the 2-group acts on a groupoid, this is a *double groupoid*.

$\mathbf{C}/\!\!/\mathcal{G}$ is a category internal in **Cat**, whose data are seen in this square:

$$\begin{array}{ccc} x & \xrightarrow{\quad f \quad} & y \\ {\scriptstyle(\gamma,x)}\Big\downarrow & {\scriptstyle((\gamma,\eta),f)}\Downarrow & \Big\downarrow{\scriptstyle((\partial\eta)\gamma,y)} \\ \gamma \blacktriangleright x & \xrightarrow[(\gamma,\eta)\blacktriangleright f]{} & (\partial\eta)\gamma \blacktriangleright y \end{array}$$

(The morphism on the bottom is the diagonal of the naturality square associated with (γ, η) and f.)
In particular, we're interested in generalizing our original construction in gauge theory, which means we want the transformation double groupoid:

$$\mathbf{Conn}(\Sigma)/\!\!/\mathbf{Gauge}(\Sigma)$$

Double Groupoid of Connections

Generalizing to flat connections on gerbes (analogous to principal bundles, but based on a 2-group $\mathcal{G}$), we again expect a moduli space based on the 2-groupoid of **transport functors**:

$$2Fun(\Pi_2(M), \mathcal{G}) \tag{1}$$

where $\Pi_2(M)$ is the **fundamental 2-groupoid** of M consisting of

- **Objects**: $x \in M$
- **Morphisms**: Paths $I \to M$
- **2-Morphisms**: Homotopies $I^2 \to M$ fixing endpoints (up to homotopy)

Then $2\mathit{Fun}(\Pi_2(M), \mathcal{G})$ has:

- **Objects**: 2-functors from $\Pi_2(M)$ to $\mathcal{G}$
- **Morphisms**: Pseudonatural transformations between 2-functors
- **2-Morphisms**: Modifications

But this is a 2-groupoid, not a double groupoid, which suggests we need to give up our correspondence:

$$\mathit{Fun}(\Pi_1(M), G) \simeq \mathit{Conn}(M) /\!\!/ \mathit{Gauge}(M)$$

But not necessarily!

Strict and Costrict Transformations

We can use the fact that there are "strict" and "costrict" pseudonatural transformations (see Lack).
For 2-functors

$$F, G : \mathcal{C} \to \mathcal{D}$$

a *strict* (pseudonatural) transformation $s : F \Rightarrow G$ is just a natural transformation: for each object x it assigns a morphism $s_x : F(x) \to G(x)$, satisfying, for all $f : x \to y$

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle s_x}\downarrow & & \downarrow{\scriptstyle s_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

A *costrict* (pseudonatural) transformation, $c : F \Rightarrow G$ can only exist if for all $x \in \mathbf{A}$, we have $F(x) = G(x)$. Then it assigns, to every $f : x \to y$, a 2-cell c_f filling this square:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \| & \swArrow c_f & \| \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

That is, strict transformations relate objects in a way that "coheres" with morphisms; costrict ones relate morphisms in a way that "coheres" with objects.

Any pseudonatural transformation p is uniquely a composite of a strict and a costrict transformation:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
{\scriptstyle s_x}\downarrow & & \downarrow{\scriptstyle s_y} \\
G'(x) & \xrightarrow{G'(f)} & G'(y) \\
\| & \swarrow c_f & \| \\
G(x) & \xrightarrow[G(f)]{} & G(y)
\end{array}
\quad = \quad
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
{\scriptstyle n(x)}\downarrow & \swarrow c_f \circ 1_{s_x} & \downarrow{\scriptstyle n(y)} \\
G(x) & \xrightarrow[G(f)]{} & G(y)
\end{array}
\tag{2}
$$

So that $n_x = s_x$ and $n_f = c_f \circ 1_{s_x}$.
(Similarly, it is also uniquely a composition of a costrict and strict, in the other order.)

If **A** and **B** are bicategories, there is a double category $Hom_{\square}(\mathbf{A}, \mathbf{B})$ with:

- **Objects**: 2-functors from **A** to **B**
- **Vertical Morphisms**: Strict natural transformations between 2-functors
- **Horizontal Morphisms**: Costrict Pseudonatural transformations between 2-functors
- **Squares**: Modifications $M : s_2 \circ c_F \Rightarrow c_G \circ s_1$:

$$\begin{array}{ccc} F_1 & \xrightarrow{c_1} & G_1 \\ {\scriptstyle s_F}\big\downarrow & \Swarrow M & \big\downarrow{\scriptstyle s_G} \\ F_2 & \xrightarrow[c_2]{} & G_2 \end{array} \tag{3}$$

Its squares are in 1-1 correspondence with the bigons of the ordinary 2-category $Hom(\mathbf{A}, \mathbf{B})$.

So finally we recover a generalization of the correspondence for $A(\Sigma)$:

$$\mathbf{Conn}/\!\!/\mathbf{Gauge} \simeq Hom_{\Box}(\Pi_2, \mathcal{G})$$

To make this work **Conn** is a category of connections:

- objects are $\mathcal{G}$-connections, which assigning G-valued holonomies to paths, and H-valued holonomies to surfaces
- morphisms are *costrict* gauge transformations, which assign H-valued holonomies to paths

And similarly, **Gauge** is a 2-group

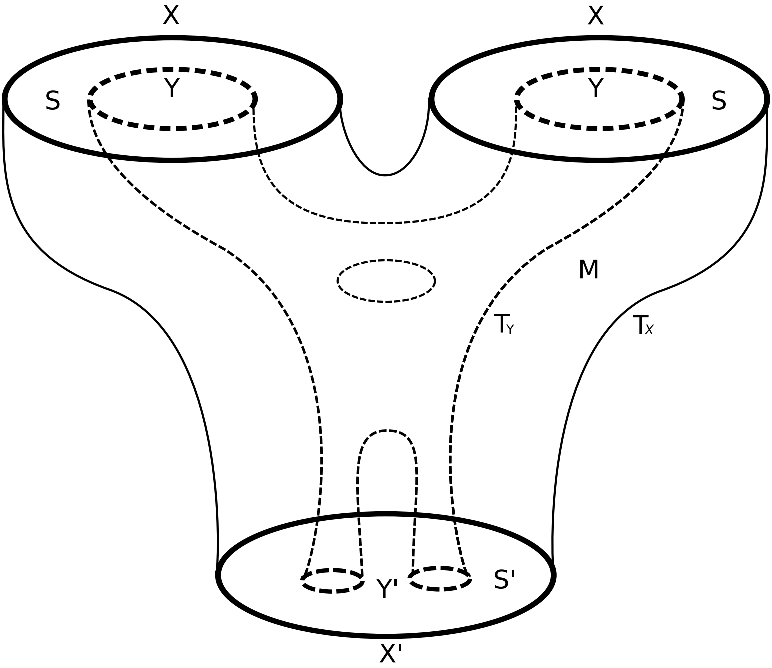
- **Objects**: Strict gauge transformations, which can be seen as G-valued functions
- **Morphisms**: Gauge modifications, which can be seen as H-valued functions

It acts by "conjugation", in some sense.

Double (Bi-)Categories of Cobordisms

Another context where double categories appear in TQFT is $\mathbf{nCob_2}$: a double category of cobordisms with corners. Intuitively, this consists of:

- **Objects**: $(n-2)$-manifolds X (supporting boundary conditions)
- **Horizontal Morphisms**: Cobordisms S (thought of as "spacelike" regions with boundary)
- **Vertical Morphisms**: Cobordisms T (thought of as "timelike" evolutions of boundary manifolds)
- **Squares**: Cobordisms with corners M (thought of as "spacetimes" containing evolving surfaces bounding spacelike regions on which fields evolve)


X
X
S
Y
Y
S
M
T_Y
T_X
Y'
S'
X'

These can be understood as *double cospans*, which naturally assemble into a double category (provided special composition squares - in this case pushout squares - exist):

$$\begin{array}{ccccc} X_1 & \longrightarrow & S & \longleftarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ T_1 & \longrightarrow & M & \longleftarrow & T_2 \\ \uparrow & & \uparrow & & \uparrow \\ X_1' & \longrightarrow & S' & \longleftarrow & X_2' \end{array} \tag{4}$$

Applying a contravariant functor turns this into a span of spans. Our classical field theory uses the functor

$$A(-) = Hom(\Pi_2(-), \mathcal{G})$$

(or the uncategorified version, $Hom(\Pi_1(-), G)$).

Questions

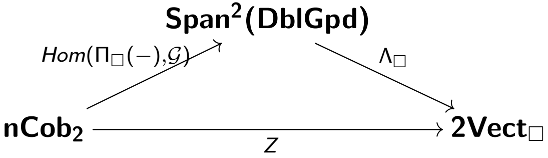
It turns out that our $Hom_{\square}(\Pi_2(\Sigma), \mathcal{G})$ is just the internal hom in **DblCat** between the *vertical double categories* associated to $\Pi_2(\Sigma)$ and $\mathcal{G}$. (That is, with only identity horizontal morphisms.)

Question 1: If this double categorical setting is significant, is there a better generalization which preserves it?
Suggestion: Suppose Σ is a manifold with *causal structure*: tangent vectors at each point can be classified as *timelike* or *spacelike*. Then define

$$\Pi_{\square}(\Sigma)$$

with vertical morphsims the timelike paths, and horizontal morphisms the spacelike paths. Squares are "world-sheets" of spacelike "strings".

Question 2:
The preceding examples suggest there should be a double-categorical variation of the Freed-Hopkins-Lurie-Teleman construction



Most of the construction is clear, but:

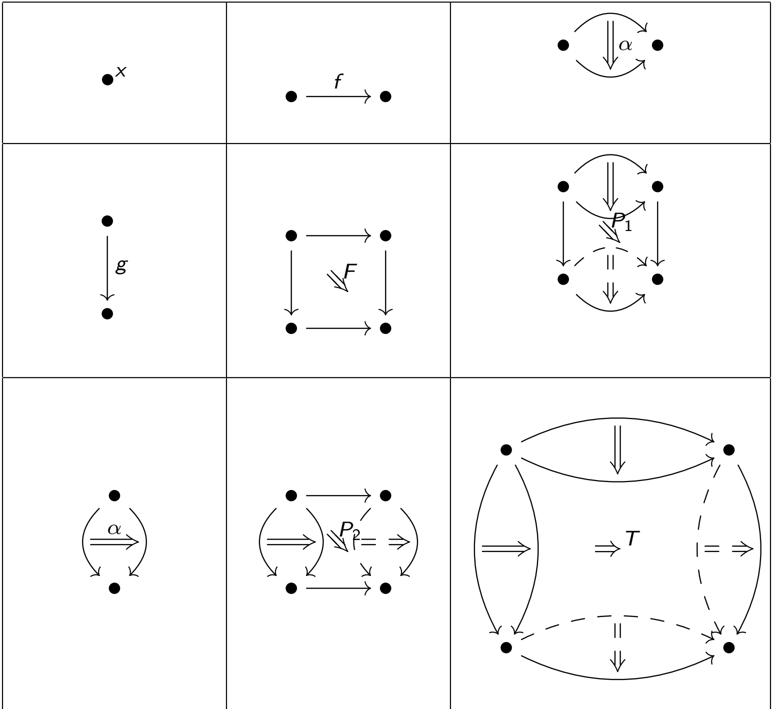
- What is a natural choice of double category to call $\mathbf{2Vect}_\square$?
- What is the double-categorical analog of the quantization functor, $\Lambda_\square$?

Suggestion: If we use the double category **Q(2Vect)** of *quintets* of the 2-category **2Vect** (whose horizontal and vertical morphisms are both morphisms of **2Vect**), the usual Λ should extend naturally. But is this the only choice?

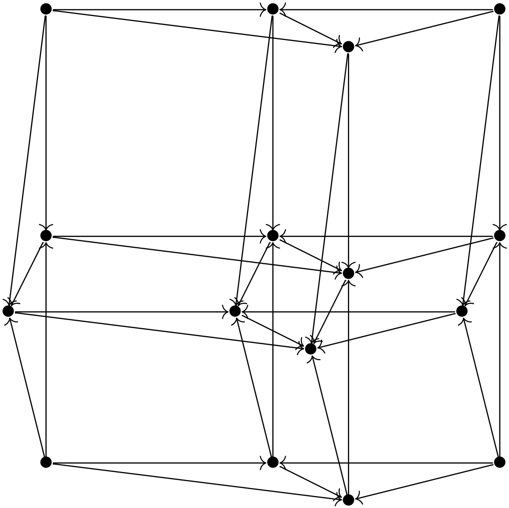
Question 3: Does the preceding generalize to HGT based on even higher *n*-groups?

In particular: the symmetries of a category, acted upon by a 2-group (group object in **Cat**), give a *transformation double category*, and in particular a *double groupoid*, because this is an internal groupoid in **Cat**.
What do we get from the symmetries of a bicategory, acted upon by a 3-group (group-object in **Bicat**)? An internal groupoid in **Bicat**

- Do we still get the correspondence with an internal hom in tricategories?
- Strictification is different in tricategories: what complications are introduced?
- It's possible to extend $\mathbf{nCob_2}$ to a *double bicategory* by allowing gluing (and span-morphisms in the span-of-spans). Is this related?



x
f
α
g
F
P1
α
P2
T



Question 4: What does the cubical picture look like if we attempt to extend beyond codimension 2? Does any of the preceding generalize, or is there something special about this case?

- Repeating internalization to get *n*-fold categories (triple, quadruple, etc.) makes sense, and applies to the cobordism category.
- We can get *n*-fold spans-of-spans of any *k*-groupoids.
- The symmetry construction only naturally extends to 2-fold categories... unless the underlying object being acted upon can be a cubical structure also?