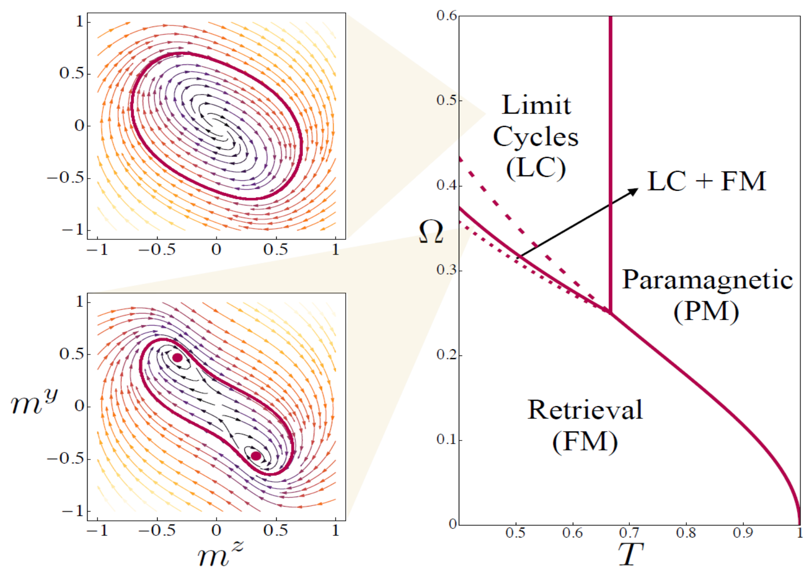
Neural network dynamics in open quantum many-body systems



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24/05/21

Quantum Matter Meets Math QM^3

Journal of Physics A **51**, 115301 (2018)
Physical Review A **99**, 032126 (2019)
Physical Review Research **2**, 013198 (2020)
Physical Review Letters **125**, 070604 (2020)
arXiv:2009.13932, Physical Review Letters *in print*

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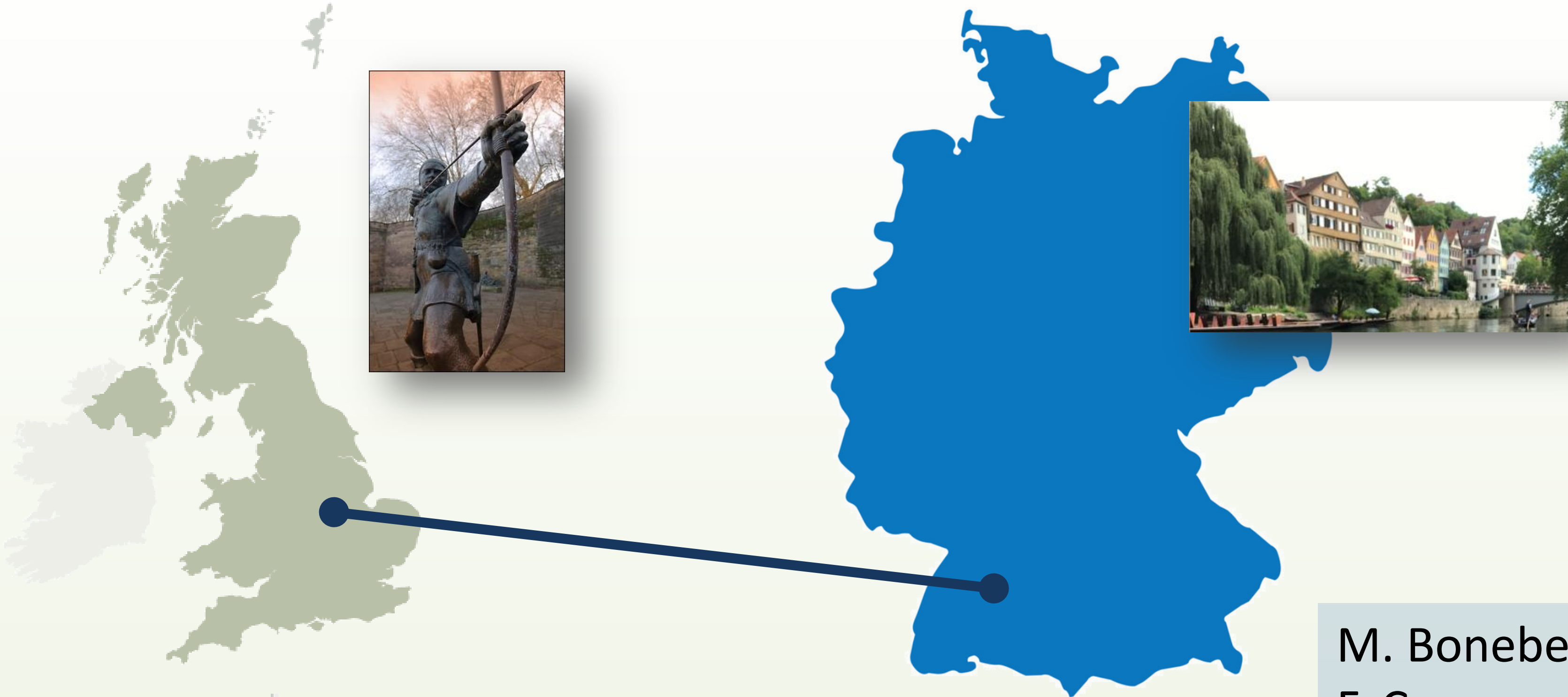

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Rationale of research idea

Neural network
=
System of many interacting spins

→

Retrieval of stored information (patterns)
=
Non-equilibrium dynamics

→

Stationary state of dynamics can be analysed using statistical mechanics methods → phases and phase transitions

→

Dynamics can be augmented with quantum effects

→

New phases/ classes of patterns?

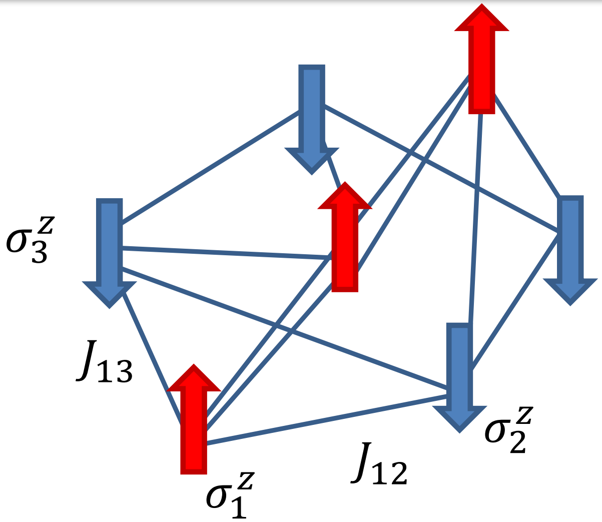
→

Are there physical realisation/manifestation of this physics in quantum many-body systems

Outline

1) Introduction

- Hopfield neural network
- pattern storage and retrieval
- statistical physics perspective

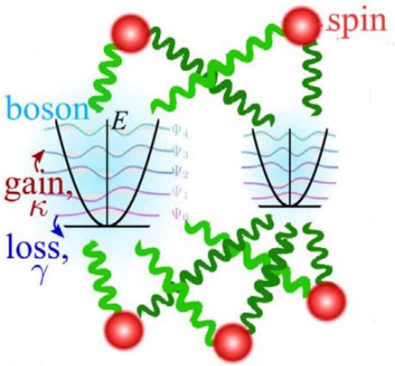


2) Quantum effects

- bringing classical and quantum dynamics together
- thermal vs. quantum fluctuations
- limit cycle phase = „novel quantum pattern“?

3) Manifestation in physical systems

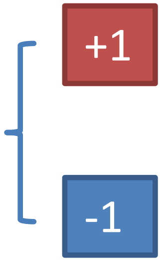
- realisation of Hopfield neural network dynamics with atoms coupled to light



Hopfield network

- simple model of associative memory
  [J.J. Hopfield, PNAS **79**, 2554 (1989)]

- if input is "similar enough to stored pattern" it will be retrieved
- "neurons" are represented by binary variables: σ_k^z



Input

-1	-1	-1	-1	+1
-1	+1	+1	-1	-1
-1	+1	-1	+1	-1
-1	+1	+1	+1	-1
-1	+1	-1	+1	-1
-1	-1	-1	+1	-1

Time evolution →

Output pattern

-1	-1	-1	-1	-1
-1	+1	+1	+1	-1
-1	+1	-1	+1	-1
-1	+1	+1	+1	-1
-1	+1	-1	+1	-1
-1	+1	-1	+1	-1

Hopfield network

- patterns are represented by N-dimensional vectors $\boldsymbol{\xi}^{\mu}$

-1	-1	-1	-1	-1
-1	+1	+1	+1	-1
-1	+1	-1	+1	-1
-1	+1	+1	+1	-1
-1	+1	-1	+1	-1
-1	+1	-1	+1	-1

first row   second row

$$\boldsymbol{\xi}^1 = \{\underbrace{-1,-1,-1,-1,-1}_{\text{first row}},\underbrace{-1,+1,+1,+1,-1}_{\text{second row}},\ldots,-1\,\}$$

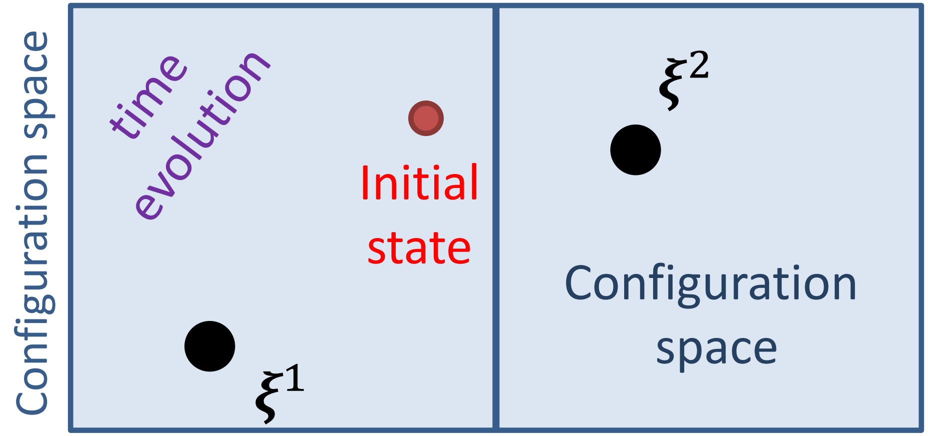
Discrete dynamics with coupling constants J_{ij} is constructed such that patterns are fixed points

$$\sigma_j^z(t+1) = \text{sign}\left[\sum_i J_{ij}\sigma_i^z(t)\right] \quad \text{with } \sigma_j^z = \pm 1$$

Order parameter = overlap of spin configuration with pattern

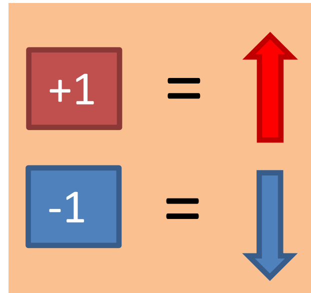
$$m_\mu^z = \frac{1}{N}\sum_k \xi_k^\mu \sigma_k^z$$

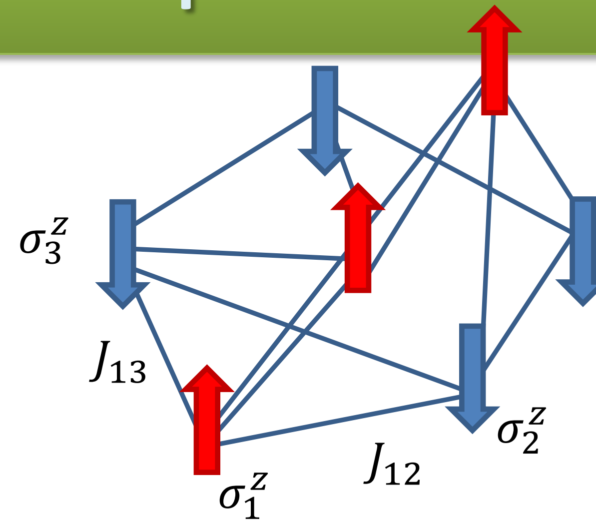
basin of attraction of pattern 1   basin of attraction of pattern 2



Statistical mechanics perspective

- Hopfield network can be interpreted as spin glass (network of “randomly” coupled spins)





Spin glass energy function

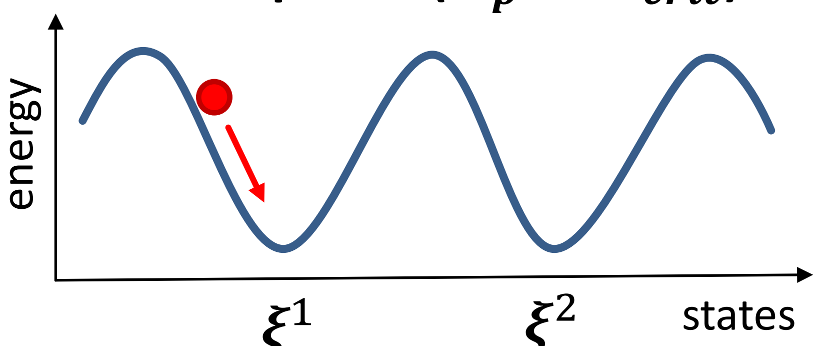
$$E_{HF} = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i^z \sigma_j^z$$

with coupling constants (Hebbian rule)

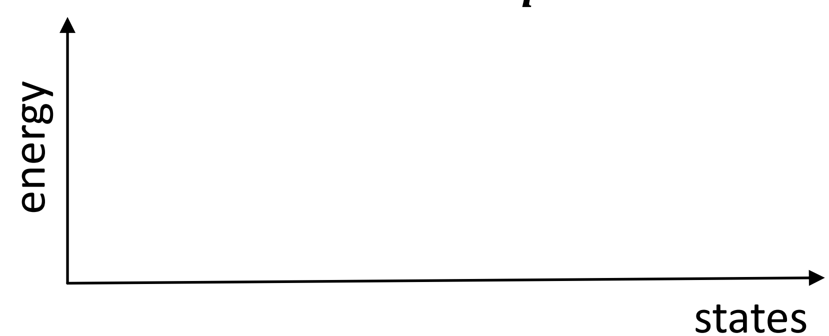
$$J_{ij} = \frac{1}{N_p}\sum_{\mu=1}^{N_p} \xi_i^\mu \xi_j^\mu$$

Capacity: number of patterns N_p that can be stored $\approx 0.138 \times$ N

retrieval phase ($N_p < N_{crit}$)

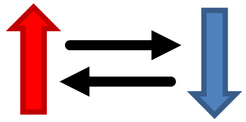


spin glass phase ($N_p > N_{crit}$)

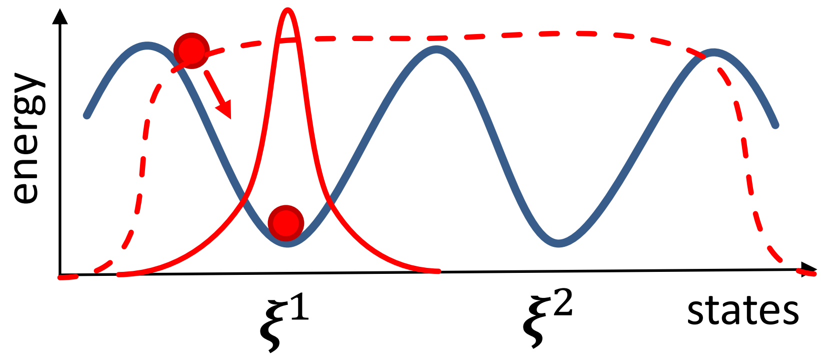


Statistical mechanics perspective

- Statistical mechanics approach allows to introduce temperature = thermal fluctuations



Thermal equilibrium state: $\rho_{\mathrm{thermal}} \propto \exp(-\beta E_{HF})$



Inverse temperature

$$\beta = \frac{1}{k_B T}$$

Retrieval phase
- energy minima correspond to patterns
- which pattern is selected (basin of attraction) depends on initial conditions
- Thermal fluctuations are small

Paramagnetic phase
- thermal fluctuations are so large that more than a single pattern is populated

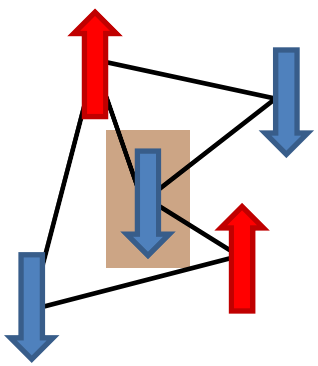
increasing temperature →

Amit et al., PRA **32**, 1007 (1985)

Formulation of dynamics

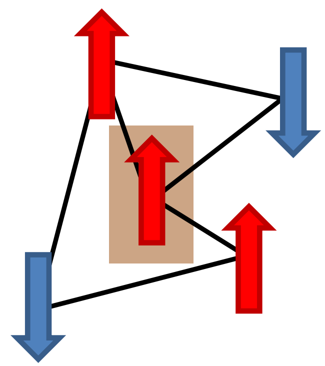
Glauber dynamics

- dynamics performs flips of individual spins
- stationary state is thermal equilibrium state



$$\Gamma_{k+} = \frac{\exp(\beta\ \Delta E_k)}{2\cosh(\beta\ \Delta E_k)}$$

$$\Gamma_{k-} = \frac{\exp(-\beta\ \Delta E_k)}{2\cosh(\beta\ \Delta E_k)}$$



$$\Delta E_k = \sum_{i\neq k} J_{ik}\sigma_j^z$$

is change of energy under flipping of k-th spin

- Probability of having a certain spin configuration is encoded in classical probability vector $\boldsymbol{p}$

$$\dot{\boldsymbol{p}} = \left[\sum_k \Gamma_{k+}(\sigma_k^+ - [1 - n_k]) + \Gamma_{k-}(\sigma_k^- - n_k)\right]\boldsymbol{p}$$

flip up — flip down

Two spins

$$\boldsymbol{p} = \begin{pmatrix} p_{\uparrow\uparrow} \\ p_{\uparrow\downarrow} \\ p_{\downarrow\uparrow} \\ p_{\downarrow\downarrow} \end{pmatrix}$$

Quantum effects

- goal: lets make this "quantum"
- starting point: formulate classical stochastic dynamics in terms of quantum master equation propagating the density matrix ρ

classical

$$\dot{\boldsymbol{p}} = \left[\sum_k \Gamma_{k+}(\sigma_k^+ - [1-n_k]) + \Gamma_{k-}(\sigma_k^- - n_k)\right]\boldsymbol{p}$$

probability vector

quantum

$$\dot{\rho} = \sum_{k,\alpha=\pm} L_{k\alpha}\rho L_{k\alpha}^+ - \frac{1}{2}\{L_{k\alpha}^+ L_{k\alpha}, \rho\}$$

density matrix

with jump operators $L_{k\pm} = \sqrt{\Gamma_{k\pm}}\sigma_k^\pm$

"square root" of Hopfield Glauber rates

- master equation is direct translation of classical stochastic dynamics
- both lead to **identical** time evolution and stationary state

Quantum effects

- Formulation in terms of quantum master equation
  permits inclusion of quantum effects

$$\dot{\rho} = \sum_{k,\alpha=\pm} L_{k\alpha} \rho L_{k\alpha}^{+} - \frac{1}{2}\{L_{k\alpha}^{+} L_{k\alpha}, \rho\}$$

classical Glauber dynamics
(controlled by temperature,
i.e. thermal fluctuations)

- next step: explore competition between classical noise and quantum fluctuations → new patterns/phases?

- our choice: quantum process that flips spins coherently (transverse magnetic field): $H = \Omega \sum_k \sigma_k^x$

Quantum evolution $\exp(-i\,H\,t)\,|{+1}\rangle = \cos(\Omega t)\,|{+1}\rangle - i\sin(\Omega t)\,|{-1}\rangle$

Order parameter and dynamics

- To analyse dynamics we use the pattern-overlap as order parameter (overlap of the state of the system with a pattern $\boldsymbol{\xi}^{\mu}$)

order parameter

$$m_{\mu}^{z} = \frac{1}{N}\sum_{k}\langle \xi_k^{\mu}\sigma_k^{z}\rangle$$

$$m_{\mu}^{y} = \frac{1}{N}\sum_{k}\langle \xi_k^{\mu}\sigma_k^{y}\rangle$$

component does not exist
classical Hopfield dynamics

equations of motion

$$\dot{\boldsymbol{m}}^{z} = 2\Omega\,\boldsymbol{m}^{y} + \frac{1}{N}\sum_{i}\boldsymbol{\xi}_i \tanh(\beta\boldsymbol{\xi}_i\cdot\boldsymbol{m}^{z}) - \boldsymbol{m}^{z}$$

$$\dot{\boldsymbol{m}}^{y} = -2\Omega\,\boldsymbol{m}^{z} - \frac{1}{2}\boldsymbol{m}^{y}$$

pattern $\boldsymbol{\xi}^{\mu}$

solutions in retrieval phase: $\boldsymbol{m}^{z} = \{0, \dots, 0,1,0, \dots 0\}$

Overlap with one specific pattern

solution in paramagnetic phase: $\boldsymbol{m}^{z} = \{0,0, \dots 0\}$

No overlap with any pattern

Stationary state

stationarity condition: $\dot{\boldsymbol{m}}^z = \dot{\boldsymbol{m}}^y = 0$

$$0 = 2\Omega\, \boldsymbol{m}^y - \boldsymbol{m}^z + \frac{1}{N}\sum_i \boldsymbol{\xi}_i \tanh(\beta \boldsymbol{\xi}_i \cdot \boldsymbol{m}^z)$$

y-component is trivially related to z-component at stationarity

$$0 = -2\Omega\, \boldsymbol{m}^z - \frac{1}{2}\boldsymbol{m}^y$$

$$\frac{1+8\Omega^2}{\beta}\beta \boldsymbol{m}^z = \overline{\boldsymbol{\xi} \tanh(\boldsymbol{\xi} \cdot \beta \boldsymbol{m}^z)}$$

- this equation of state is identical to classical Hopfield model
- new effective temperature $T \rightarrow T(1 + 8\Omega^2)$
- stationary points of dynamics are unchanged but transition temperature between retrieval phase and paramagnetic phase is shifted

Is this really everything?

Phase diagram

Analysis of dynamics:

(i) numerically solve dynamical equations for small number of patterns, assuming that **patterns are random sequences of -1 and +1** with probability distribution

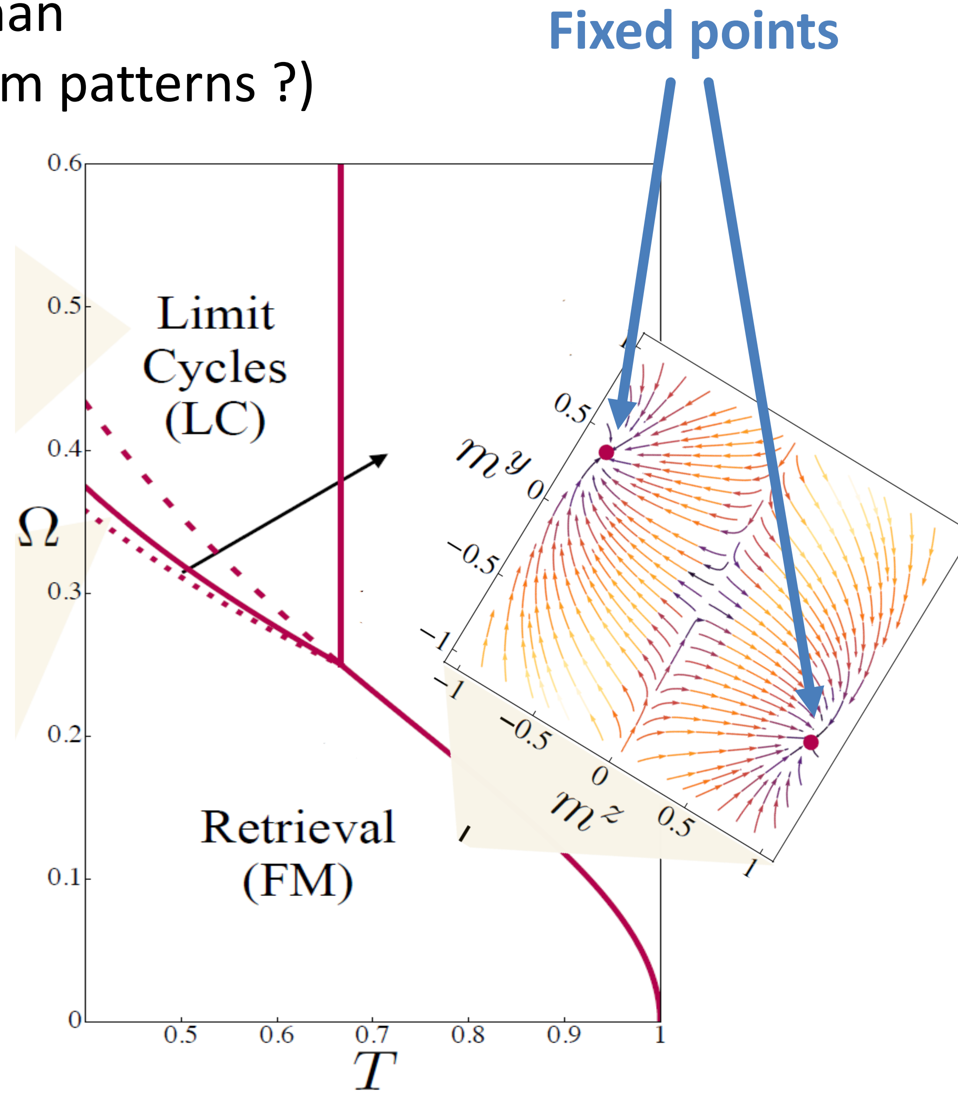
$$p(\xi_i^\mu) = \frac{1}{2}\left[\delta(\xi_i^\mu - 1) + \delta(\xi_i^\mu + 1)\right]$$

+1 -1

(ii) perform stability analysis around stationary points

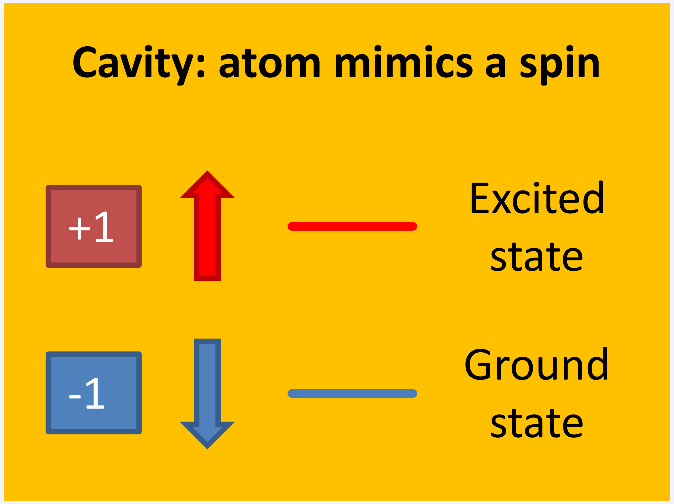
Phase diagram

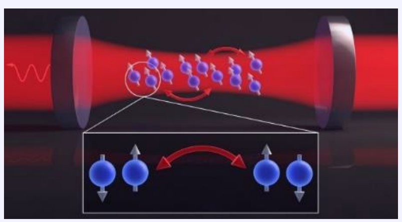
- quantum fluctuations do more than just rescale temperature (quantum patterns ?)



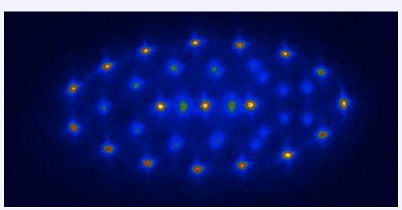
Journal of Physics A **51**, 115301 (2018)

Physical realisation

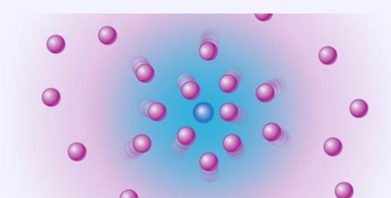




Cavity QED



(Rydberg) ions



Impurities in BEC/ mixtures

- dynamics of density matrix ρ is governed by **quantum master equation**

$$\dot{\rho} = L\,\rho = -i[H,\rho] + \sum_{n=l,g} L_n \rho L_n^\dagger - \frac{1}{2}\{L_n^\dagger L_n, \rho\}$$

Coherent dynamics

$$H = \sum_{\mu=1}^{M} \omega_\mu a_\mu^\dagger a_\mu + \sum_{\mu=1}^{M}\sum_{i=1}^{N} g_{i\mu}\sigma_i^z (a_\mu^\dagger + a_\mu) + \Omega \sum_{i=1}^{N} \sigma_i^x$$

bosons — coupling between spins and bosons — spins

- Patterns encoded in spin-boson coupling constants

$$\xi_i^\mu \leftrightarrow g_{i\mu}$$

Dissipative processes

Boson loss

$$L_l = \sqrt{\gamma}a$$

Boson gain

$$L_g = \sqrt{\kappa}a^\dagger$$

Dynamics (classical limit)

Strong dissipation: dynamics becomes effectively classical

Classical master equation:
(non-thermal stationary state)

$$\dot{p}_{\vec{\sigma}} = \sum_{\vec{\sigma}'} (W_{\vec{\sigma}' \to \vec{\sigma}} p_{\vec{\sigma}'} - W_{\vec{\sigma} \to \vec{\sigma}'} p_{\vec{\sigma}})$$

Probability of spin configuration $\vec{\sigma}$

$$W_{\vec{\sigma} \to \vec{\sigma}'} = \frac{2\Omega^2}{\omega} \int_0^\infty d\tau e^{-\frac{2g_i^2 \nu}{\omega^2}(f(\tau)+\tau)} \cos\left[16\frac{\Delta E_i \tau - g_i^2 s(\tau)}{\omega^2(\eta^2+4)}\right]$$

$$f(\tau) = \frac{8 - 2\eta^2}{\eta(\eta^2+4)}[1 - e^{-\frac{\eta}{2}\tau}\cos(\tau)] - \frac{8e^{-\frac{\eta}{2}\tau}}{\eta^2+4}\sin(\tau)$$

$$s(\tau) = \frac{4\eta[e^{-\frac{\eta}{2}\tau}\cos(\tau) - 1] + [\eta^2 - 4]e^{-\frac{\eta}{2}\tau}\sin(\tau)}{\eta^2 + 4}$$

Transition rates depend on energy change of single spin flip

$$\Delta E_i = E(\sigma_i^z = 1) - E(\sigma_i^z = -1)$$

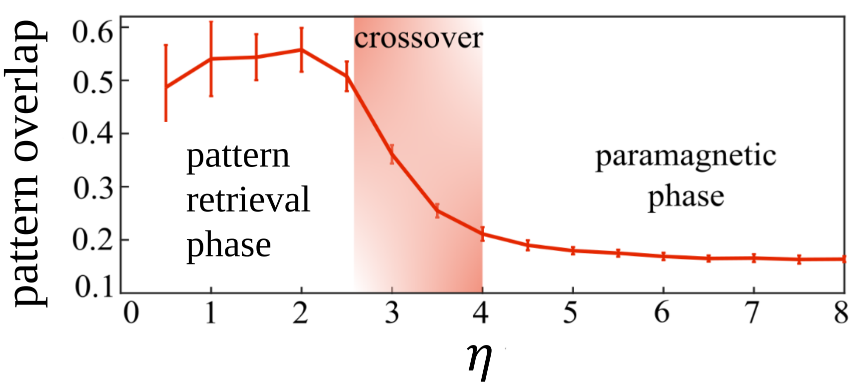
„Energy function“ is determined by spin-photon coupling constants

$$E = -\frac{1}{4}\sum_{ij}\sum_{\mu} g_{i\mu} g_{j\mu} \sigma_i^z \sigma_j^z$$

Hebbian rule

Parameter controling the strength of fluctuations

$$\eta = \frac{\gamma - \kappa}{\omega} = \frac{\text{photon loss} - \text{photon gain}}{\text{photon frequency}}$$



Beyond the classical limit

- Problem can be solved (to some extent) considering quantum effects
- Equations of motion become that of **multi-mode Dicke model**

<u>Meanfield equations of motion</u>

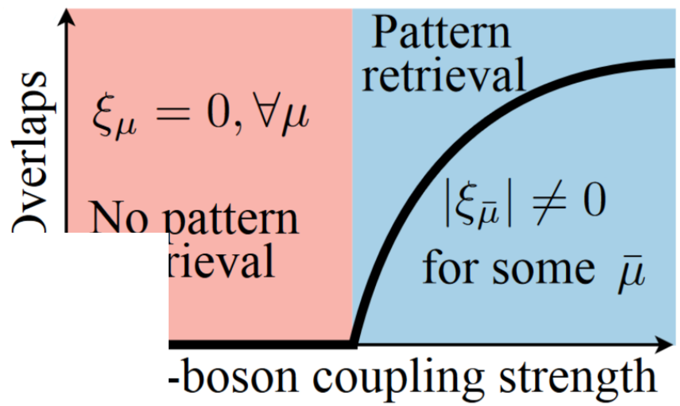
Dynamics of pattern overlap $m_{a,k}$ $(a = x, y, z)$

$\dot{m}_{\mathrm{a,k}}$

coupling constants

Bosonic mode amplitude

$\dot{\alpha}_\mu$



- mean field equations are exact in thermodynamic limit
  proof in arXiv:2009.13932 (2020)
- steady-state solution shows pattern retrieval phase transition
- Pattern retrieval signalled by boson mode occupation

Summary and outlook

- Hopfield neural networks can be studied within a **statistical mechanics framework**
- Ability to retrieve patterns is connected to a phase transitions within an all-to-all connected spin system

- **Quantum effects** can change the nature of the observed phases
    - Patterns can feature quantum coherence (oscillatory motion)
    - Mixed quantum-classical phase
        - Is this useful?

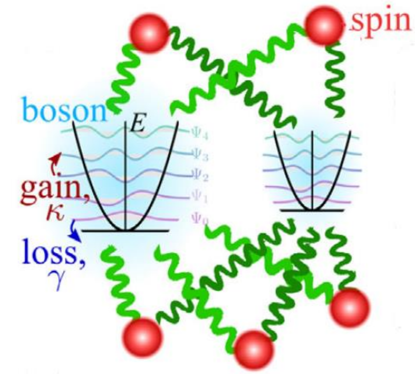
- Hopfield neural network can be physically realised with **atom-cavity system**
    - Reminiscent of multi-mode Dicke model
    - Patterns are encoded in atom-light coupling constants
    - Features phase transition between paramagnetic and retrieval phase

P. Rotondo et al., Journal of Physics A **51**, 115301 (2018)
E. Fiorelli et al., Physical Review A **99**, 032126 (2019)
E. Fiorelli et al., Physical Review Research **2**, 013198 (2020)
E. Fiorelli et al., Physical Review Letters **125**, 070604 (2020)
F. Carollo and IL, arXiv:2009.13932 (2020)

Summary and outlook

Open questions

- Can quantum effects enhance the capabilities of (Hopfield) neural networks, e.g. pattern retrieval speed?
- Can study be generalised beyond Hopfield model? [ongoing work with M. Müller (Aachen)]
- Is link to modern machine learning problems possible? (Would dynamical spin-boson coupling parameter allow to implement learning?)



Other recent works

Many-body quantum engines
PRL **125**, 240602 (2020)
PRL **124**, 170602 (2020)
Time crystals
PRE **100**, 060105(R) (2019)
PRL **122**, 015701 (2019)

Kinetically constrained systems
PRL **125**, 033602 (2020)
PRL **126**, 103002 (2021)
Sub- and superradiance
PRL **124**, 093601 (2020)
PRA **102**, 043711 (2020)

Many-body interactions
PRL **125**, 133602 (2020)
ML open system dynamics
PRR **3**, 023084 (2021)
Quantum Mpemba effect
arXiv:2103.05020

Physical realisation

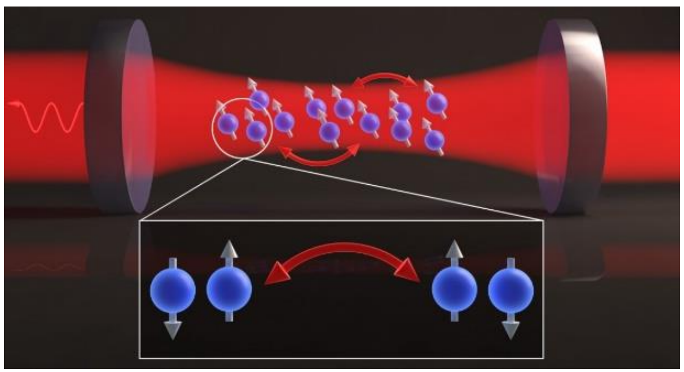
Many-body cavity electrodynamics

Featured in Physics | Editors' Suggestion

Sign-Changing Photon-Mediated Atom Interactions in Multimode Cavity Quantum Electrodynamics

Yudan Guo, Ronen M. Kroeze, Varun D. Vaidya, Jonathan Keeling, and Benjamin L. Lev
Phys. Rev. Lett. **122**, 193601 – Published 14 May 2019

Physics See Synopsis: A Step Toward Simulating Spin Glasses



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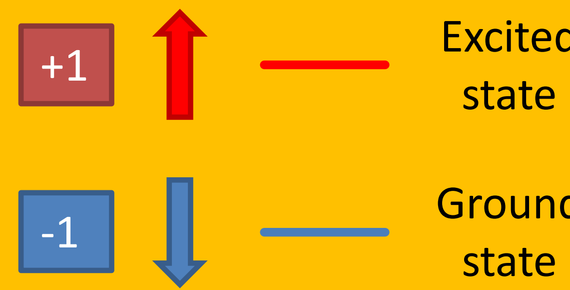


Quantum dynamics governed by

$$H = \sum_{\mu=1}^{M} \omega_\mu a_\mu^\dagger a_\mu + \sum_{\mu=1}^{M}\sum_{i=1}^{N} g_{i\mu}\sigma_i^z (a_\mu^\dagger + a_\mu) + \Omega \sum_{i=1}^{N} \sigma_i^x$$

$$\dot\rho = -i[H,\rho] + \sum_{n=l,g} L_n \rho L_n^\dagger - \frac{1}{2}\{L_n^\dagger L_n, \rho\}$$

Photon loss: $L_l = \sqrt{\gamma} a$

Photon gain: $L_g = \sqrt{\kappa} a^\dagger$

- Cavity „traps“ photons
- atoms are confined inside cavity
- Photons interact with atoms
- Photons can be „integrated out“ to yield spin-only model

Dynamics

Strong dissipation: dynamics becomes effectively classical

Classical master equation: $$\dot{p}_{\vec\sigma} = \sum_{\vec\sigma'} \left(W_{\vec\sigma'\to\vec\sigma} p_{\vec\sigma'} - W_{\vec\sigma\to\vec\sigma'} p_{\vec\sigma} \right)$$

Probability of spin configuration $\vec\sigma$

$$W_{\vec\sigma\to\vec\sigma'} = \frac{2\Omega^2}{\omega} \int_0^\infty d\tau\, e^{-\frac{2g_i^2 \nu}{\omega^2}(f(\tau)+\tau)} \cos\left[16 \frac{\Delta E_i \tau - g_i^2 s(\tau)}{\omega^2(\eta^2+4)} \right]$$

$$f(\tau) = \frac{8-2\eta^2}{\eta(\eta^2+4)}\left[1 - e^{-\frac{\eta}{2}\tau}\cos(\tau)\right] - \frac{8e^{-\frac{\eta}{2}\tau}}{\eta^2+4}\sin(\tau)$$

$$s(\tau) = \frac{4\eta[e^{-\frac{\eta}{2}\tau}\cos(\tau) - 1] + [\eta^2 - 4]e^{-\frac{\eta}{2}\tau}\sin(\tau)}{\eta^2+4}$$

Transition rates depend on energy change of single spin flip

$$\Delta E_i = E(\sigma_i^z = 1) - E(\sigma_i^z = -1)$$

„Energy function“ is determined by spin-photon coupling constants

$$E = -\frac{1}{4}\sum_{ij}\sum_{\mu} g_{i\mu} g_{j\mu} \sigma_i^z \sigma_j^z$$

Parameter controling the strength of fluctuations

$$\eta = \frac{\gamma - \kappa}{\omega} = \frac{\text{photon loss} - \text{photon gain}}{\text{photon frequency}}$$

E. Fiorelli et al.,
PR Research **2**, 013198 (2020)

Pattern retrieval

- interaction between spins is described by energy function reminiscent of Hopfield model
- patterns are encoded in coupling constants
- retrieval dynamics is not thermal
- many-body system behaves similar to „neural network“

Energy function

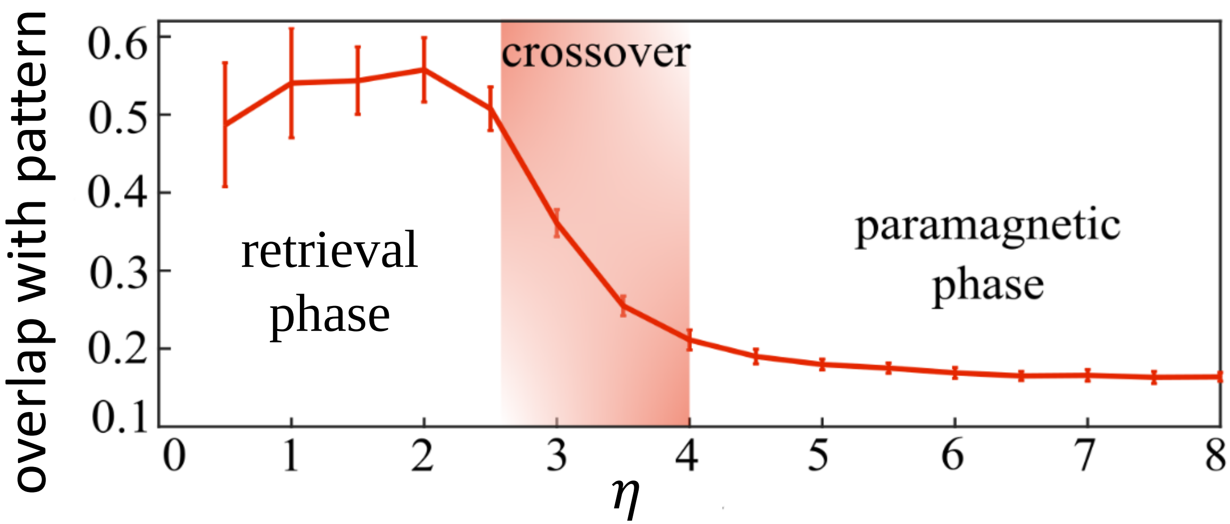
$$E = -\frac{1}{4}\sum_{ij}\underbrace{\sum_{\mu} g_{i\mu} g_{j\mu}}_{J_{ij}}\sigma_i^z \sigma_j^z$$

Hopfield energy function

$$E_{HF} = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i^z \sigma_j^z$$

$$J_{ij} = \frac{1}{N_p}\sum_{\mu=1}^{N_p} \xi_i^{\mu} \xi_j^{\mu}$$

***Paramagnet – Retrieval Phase* Transition**



PRL **125**, 070604 (2020)

Beyond the classical limit – Dicke model

Problem: full quantum dynamics (including bosons) can only be simulated for few spins

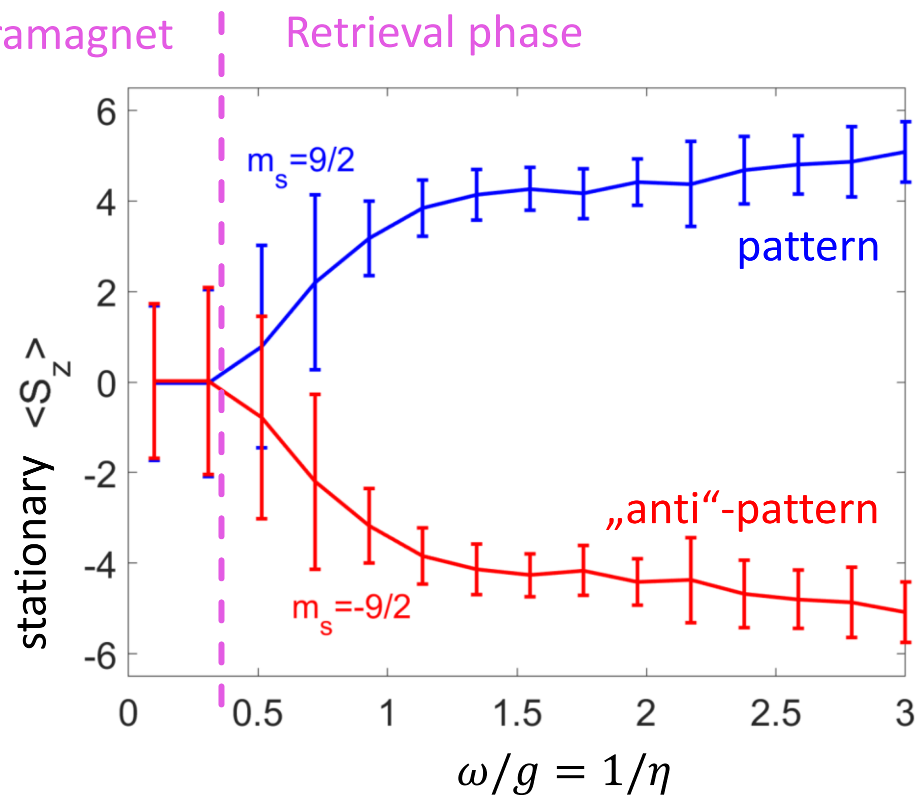
Simpler model

Boson + macroscopic spin

$$H = \omega a^\dagger a + \Omega S^x + g S^z (a + a^\dagger)$$

Boson loss

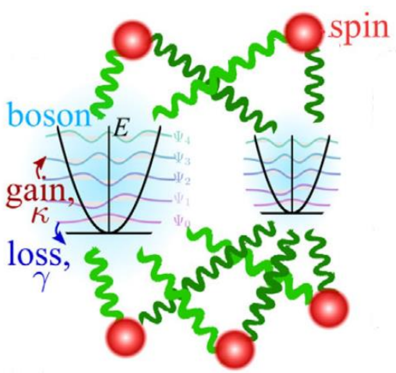
$$L = \sqrt{\gamma} a$$



- For one memory this is exactly the problem we have been dealing with so far
- In fact, this is essentially the dissipative **Dicke model**
- Monte-Carlo simulations show crossover between paramagnet and retrieval phase
- basin of attraction (pattern/anti-pattern) chosen by initial condition

Beyond the classical limit – Dicke model

- multi-memory case (N spins, M bosons) is reminiscent of **multi-mode Dicke model**
- not exactly solvable either (for finite N), but amenable to mean field treatment



<u>Meanfield equations of motion</u>

$$\dot{m}_{\mathrm{a,k}} = -2\Omega \sum_{\mathrm{b}} \epsilon_{\mathrm{xab}} m_{\mathrm{b,k}} - 2g \sum_{\mathrm{b},\mu} \epsilon_{\mathrm{zab}} f^{M}_{\mu,\mathrm{k}} \left(\alpha^{\dagger}_{\mu} + \alpha_{\mu}\right) m_{\mathrm{b,k}}$$

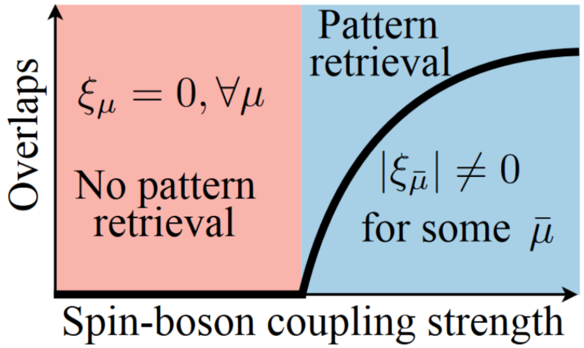
$$\dot{\alpha}_{\mu} = -\left(i\Omega_{\mu} + \frac{\kappa_{\mu}}{2}\right)\alpha_{\mu} - ig \sum_{\mathrm{k}=1}^{2^{M-1}} f^{M}_{\mu,\mathrm{k}} m_{\mathrm{z,k}}$$

bosonic mode amplitude

coupling constants

pattern overlap

- mean field equations can be proven to be exact when N → ∞
- steady-state solution shows transition between retrieval and paramagnetic phase



F. Carollo and IL, arXiv:2009.13932 (2020)