The free-fermion eight-vertex model via dimers

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Lisbon IST QM3

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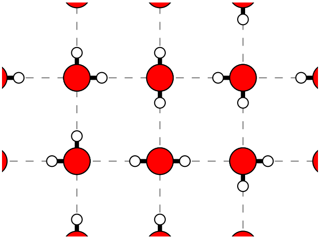
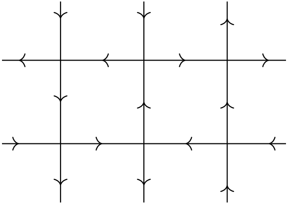
1. The eight-vertex model
2. The free-fermion regime
3. Non-bipartite dimers to bipartite
4. Isoradial Z-invariant case

Six-vertex configurations



$\mapsto$



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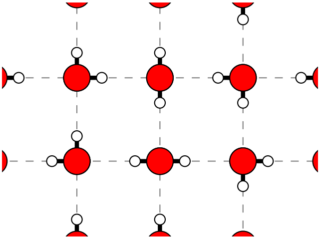

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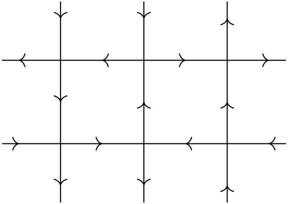
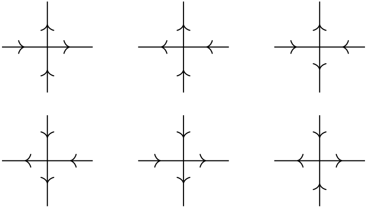
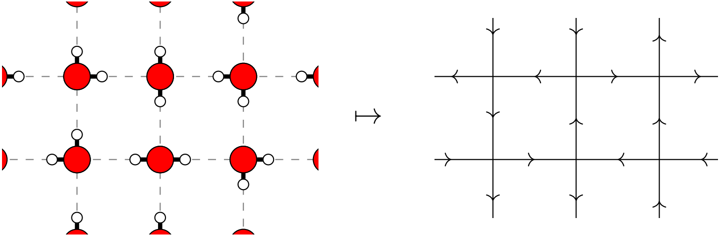


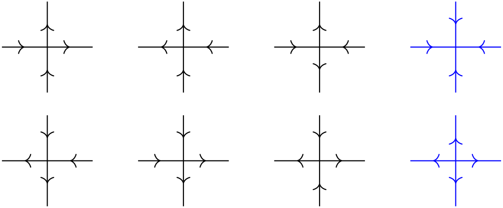
6 local configurations:



Eight-vertex configurations



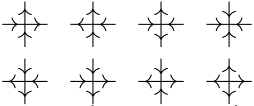
8 local configurations:



...with the introduction of defects.

The (symmetric, classical) eight-vertex model

Fix the local weights $a, b, c, d > 0$:



$a \quad b \quad c \quad d$

On a finite $G \subset \mathbb{Z}^2$, an orientation τ satisfying these rules has *weight*

$$w(\tau) = a^{N_a} b^{N_b} c^{N_c} d^{N_d}.$$

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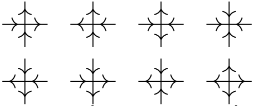

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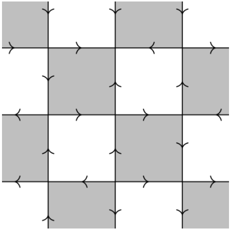

$$w(\tau) = ab^3c^4d.$$

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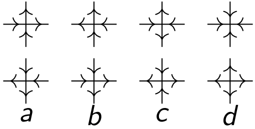

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$$\mathbb{P}(\tau) = \frac{w(\tau)}{Z(G; a, b, c, d)},$$

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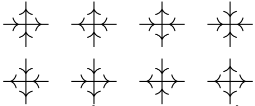

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Related to the XYZ spin chain, Ashkin-Teller model,...

Thermodynamic limit

- Free energy:

$$\lim_{G\to\mathbb{Z}^2} -\frac{1}{|G|}\log\left[Z_{8V}(G;a,b,c,d)\right]\ ?$$

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Computed in the 1970s (Baxter; Johnson, Krinsky, McCoy; Takhtadzhan, Faddeev,...) through Bethe Ansatz methods.

Thermodynamic limit

- Correlations: For edges e, e' faraway in G, is there a **correlation length** ξ s.t.

$$\mathrm{Cov}\left(1_{e\uparrow \text{ dans } \tau}, 1_{e'\uparrow \text{ dans } \tau}\right) \sim \exp\left(-\frac{|e-e'|}{\xi}\right)?$$

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Prediction:

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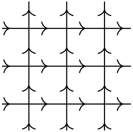
where ν **depends on** a, b, c, d.
However, in the six-vertex case ($d = 0$), ξ is always "infinite" (power law decay of correlations)

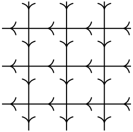
Phase diagram

Predicted in the 1970s-1980s (Baxter, Fan, Lieb, Sutherland, Wu,...):

- If $a \geq b + c + d$, the measure on $\mathbb{Z}^2$ concentrates on



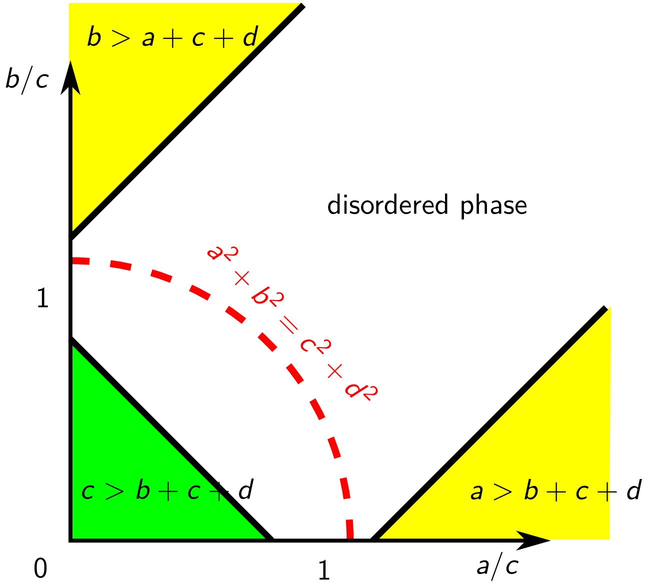
and



. This is a *ferroelectric* phase.

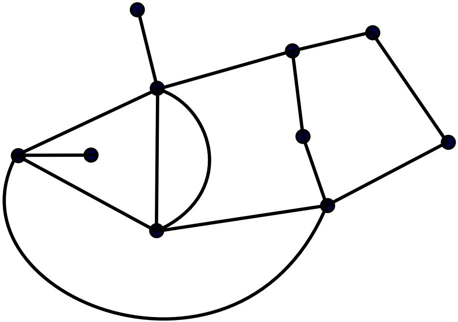
- Similar for $b \geq a + c + d$ (*ferroelectric*).
- Analogous to a *localised* state for $c \geq a + b + d$ or $d \geq a + b + c$ (*anti-ferroelectric*).
- Otherwise, analogous to a *delocalised state*; all configurations appear with positive density (*disordered*).

Phase diagram ($d < c$)



A generalization

G a finite graph on the sphere or the torus.



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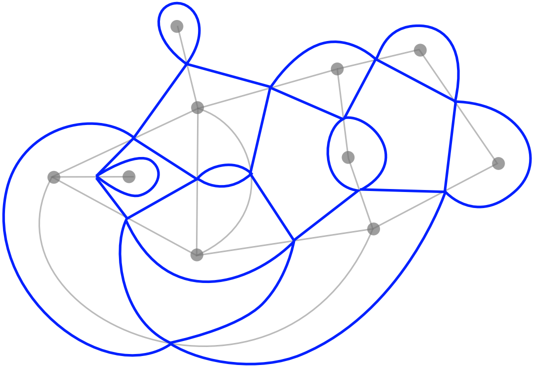

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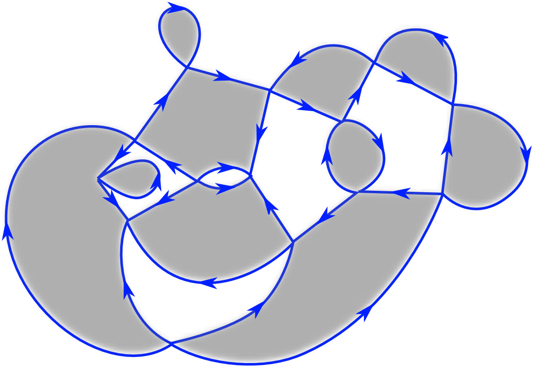

An eight-vertex configuration τ on G_m.

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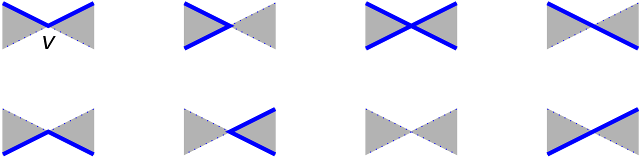
A generalization

Local configurations at a **medial** vertex $v \in V_m$:



$w_v(\tau) = \quad a_v \quad b_v \quad c_v \quad d_v$

$$w(\tau) = \prod_{v \in V_m} w_v(\tau).$$

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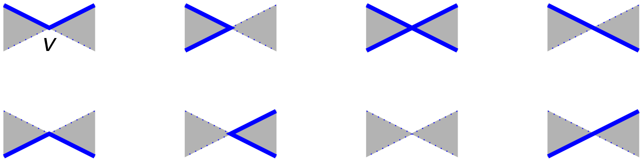

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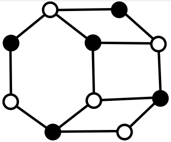
$$Z_{8V}(a, b, c, d) = \sum_{\tau} w(\tau),$$

where a, b, c, d are **functions** of the medial vertices.

The free-fermion regime

The dimer model

Let $G = (V, E)$ be a planar graph, with positive weights $(\mu_e)_{e \in E}$.



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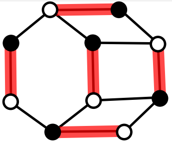

Let $G = (V, E)$ be a planar graph, with positive weights $(\mu_e)_{e \in E}$.

Dimer configuration: subset of edges $m \subset E$ incident to every vertex once. Weight of a configuration:

$$w_{\text{dim}}(m) = \prod_{e \in m} \mu_e.$$

Boltzmann probability:

$$\mathbb{P}(m) = \frac{w_{\text{dim}}(m)}{Z_{dim}(G; \mu)},$$

$$Z_{dim}(G; \mu) = \sum_m \prod_{e \in m} \mu_e.$$

Kasteleyn's theorem

Suppose that G is **bipartite**; $V = W \sqcup B$. After orienting the edges of G, we can define a weighted and oriented matrix $K = (K_{w,b})_{w \in W, b \in B}$:

$$K_{w,b} = \begin{cases} \mu(e) & \text{if } \ {}^{w}\circ\!\xrightarrow{\ e\ }\!\bullet^{b}, \\ -\mu(e) & \text{if } \ {}^{w}\circ\!\xleftarrow{\ e\ }\!\bullet^{b}, \\ 0 & \text{otherwise.} \end{cases}$$

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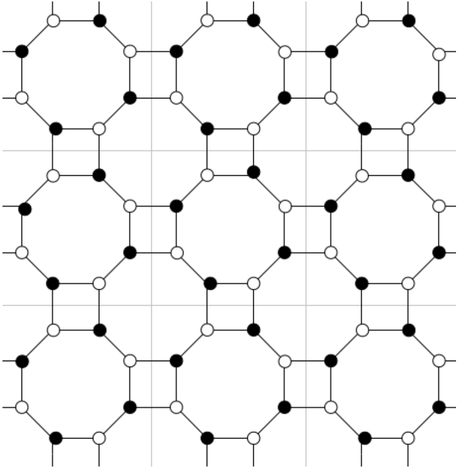
Theorem *[Kasteleyn, Temperley-Fisher; 1961]*

There exists an orientation such that

$$Z_{dim}(G;\mu) = \det K.$$

Free energy for planar bi-periodic bipartite graphs

Now G is planar, bipartite, and $\mathbb{Z}^2$-periodic.



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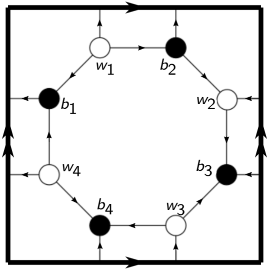
$G_1 = G/\mathbb{Z}^2$, and K_1 its Kasteleyn matrix.



$$K_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

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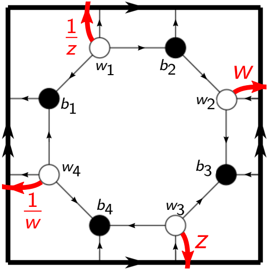
For any $z, w \in \mathbb{C}$, let $K_1(z, w)$ be a modification depending on the crossings of the torus.



$$K_1(z,w) = \begin{pmatrix} 1 & 1 & 0 & 1/z \\ -1 \times w & -1 & 1 & 0 \\ 0 & -1 \times z & 1 & 1 \\ 1 & 0 & 1/w & 1 \end{pmatrix}$$

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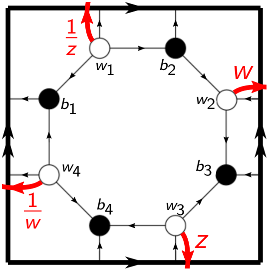
For any $z, w \in \mathbb{C}$, let $K_1(z, w)$ be a modification depending on the crossings of the torus.
$P(z, w) = \det K_1(z, w)$ is called *characteristic polynomial*.



$$P(z, w) = 5 - z - \frac{1}{z} - w - \frac{1}{w}.$$

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Theorem *[Cohn-Kenyon-Propp 2001, Kenyon-Okounkov-Sheffield 2006]*

Let $G_n = G/\left(n\mathbb{Z}\right)^2$.

$$\lim_{n\to\infty} -\frac{1}{n^2}\log\left(Z_{\mathrm{dim}}(G_n,\mu)\right) = \frac{1}{(2i\pi)^2}\int_{\mathbb{T}^2}\log|P(z,w)|\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}.$$

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G is planar, **bipartite**, $\mathbb{Z}^2$-periodic.
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Theorem *[Cohn-Kenyon-Propp 2001, Kenyon-Okounkov-Sheffield 2006]*

Let $e_1 = \{w_1 b_1\}, \ldots, e_k = \{w_k, b_k\}$ be edges of G. La probability that they are all present is

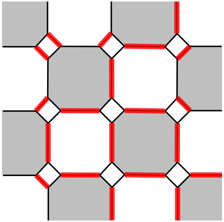
$$\mathbb{P}(e_1, \ldots, e_k \in m) = \left(\prod_{i=1}^{k} K_{w_i, b_i} \right) \det \left(K^{-1}_{b_i, w_j} \right)_{1 \leq i,j \leq k}$$

where

$$K^{-1}_{b, w+(n,m)} = \frac{1}{(2i\pi)^2} \int_{\mathbb{T}^2} \frac{\left[{}^t\mathrm{Com} K_1(z,w)\right]_{b,w}}{P(z,w)} z^n w^m \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w}.$$

Decorated graphs

Dimer configuration $m \mapsto$ **six**-vertex configuration τ.



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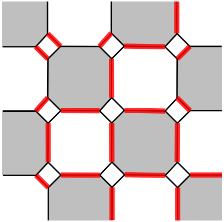

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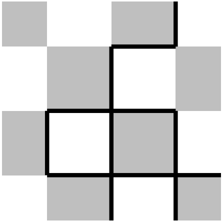

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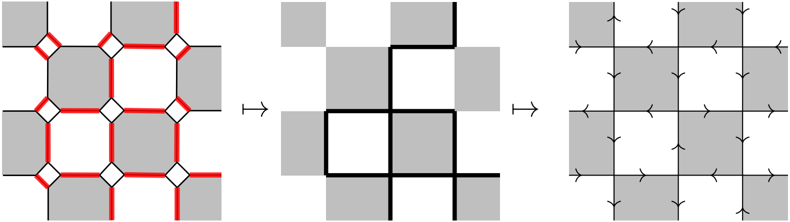

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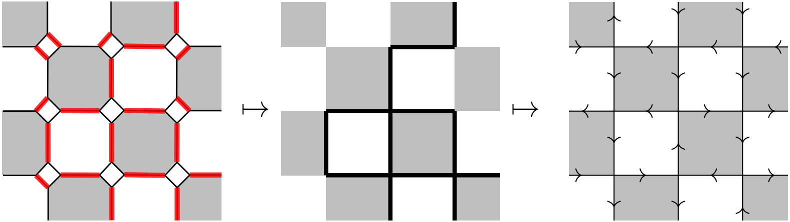

Proposition *[Fan, Lin, Wu 1970s]*

If the **six**-vertex model ($d = 0$) satisfies $a^2 + b^2 = c^2$ (at every vertex), then for some dimer weights,

$$\sum_{m \mapsto \tau} w_{\text{dim}}(m) = w(\tau).$$

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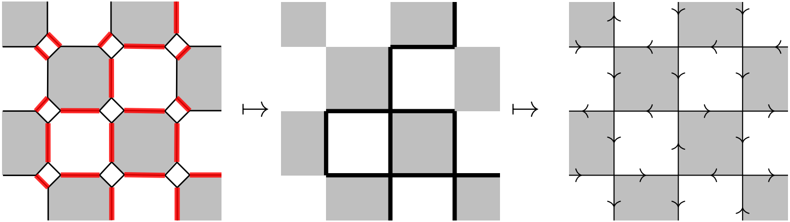

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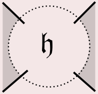
Free energy, correlations, Gibbs measure,...

Existence of decorations

Question

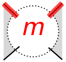
For the **eight**-vertex model, are there decorations $\mathfrak{h}$ s.t.

$$\sum_{m \mapsto \tau} w_{\dim}(m) = w(\tau)?$$



For instance, at a single vertex $v \in V_m$,

$$\sum_{m \text{ s.t. } \boxed{m}} w_{\dim}(m) = a_v.$$



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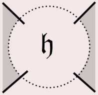


Lemma

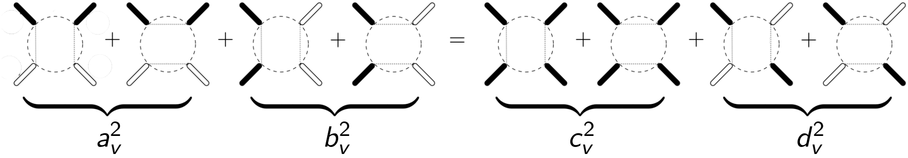
If such a **planar** $\mathfrak{h}$ exists, then

$$\forall v \in V_m, \quad a_v^2 + b_v^2 = c_v^2 + d_v^2 \qquad \text{(“free-fermion” regime.)}$$

Claim:

If such a **planar** $\mathfrak{h}$ exists, then $\forall v \in V_m, \quad a_v^2 + b_v^2 = c_v^2 + d_v^2.$

Proof:



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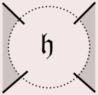

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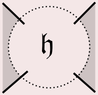

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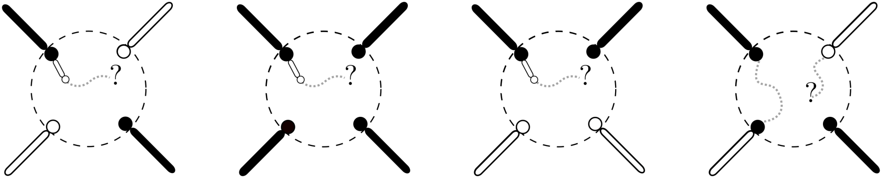
Lemma

If such a **planar bipartite** $\mathfrak{h}$ exists, then

$$\forall v \in V_m, \quad a_v^2 + b_v^2 = c_v^2 + d_v^2 \ \ \textbf{and} \ a_v b_v c_v d_v = 0.$$

Claim:
If such a **planar bipartite** $\mathfrak{h}$ exists, then one of a_v, b_v, c_v, d_v is zero.

Proof:

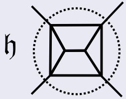


Existence of decorations

Proposition *[Hsue, Lin, Wu 1970s]*

If the eight-vertex model satisfies

$$a^2 + b^2 = c^2 + d^2,$$



then for some dimer weights on the $\mathfrak{h}$-decorated graph ($\mathfrak{h}$ non-bipartite),

$$\sum_{m \mapsto \tau} w_{\text{dim}}(m) = w(\tau).$$

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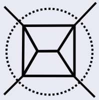
$\mathfrak{h}$ 

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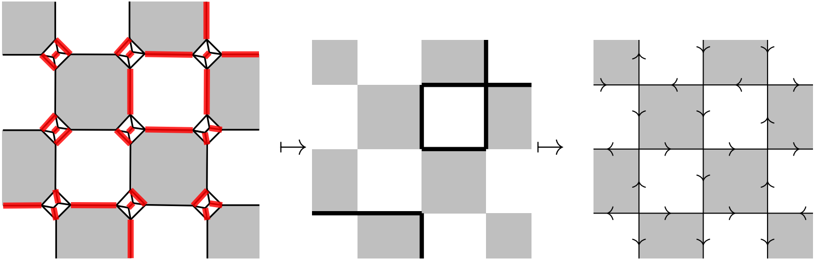
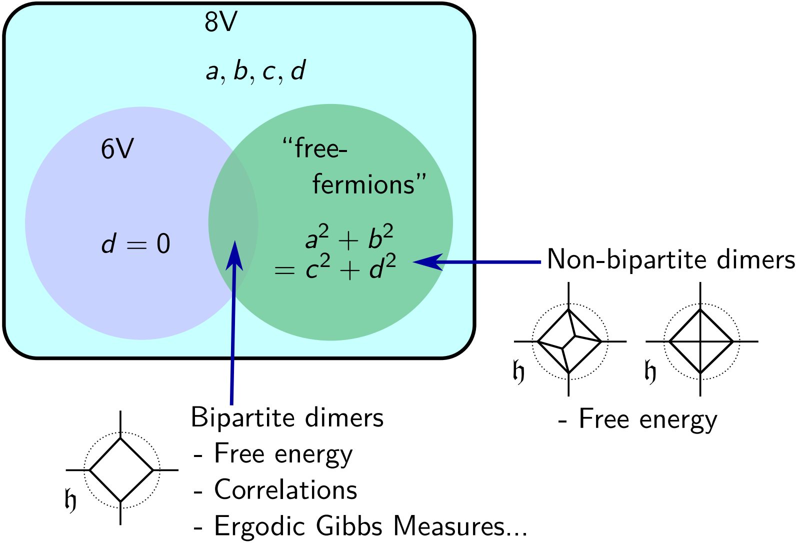


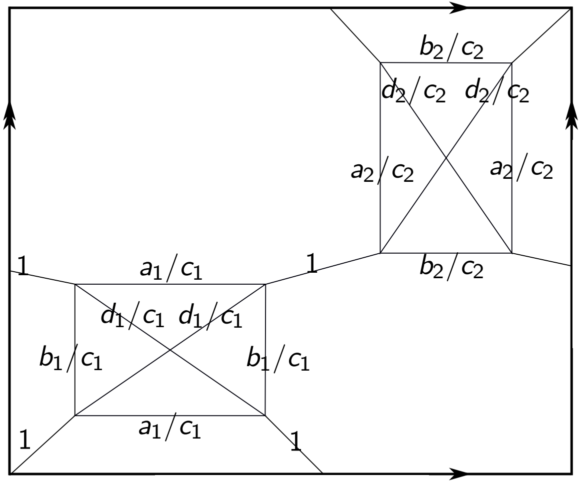
Recap



8V
a, b, c, d
6V
$d = 0$
"free-fermions"
$a^2 + b^2 = c^2 + d^2$
Non-bipartite dimers
𝔥
𝔥
- Free energy
Bipartite dimers
- Free energy
- Correlations
- Ergodic Gibbs Measures...
𝔥

Characteristic polynomial: examples

The free-fermion 8V model for $G = \mathbb{Z}^2$ gives dimers:



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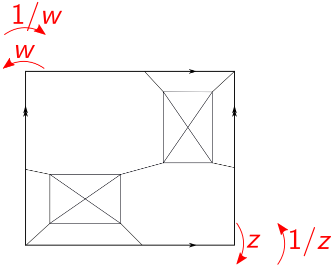

For $z, w \in \mathbb{C}$, Kasteleyn matrix $K(z, w)$.

$$P(z, w) := \det K(z, w)$$

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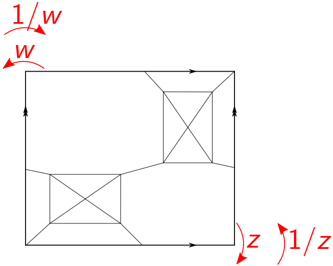

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Free energy *[Cohn-Kenyon-Propp 2001], [Kenyon-Okounkov-Sheffield 2006]*:

$$f = \frac{1}{2\pi^2} \int_{\mathbb{T}^2} \log |P(z, w)| \frac{dz}{z} \frac{dw}{w}$$

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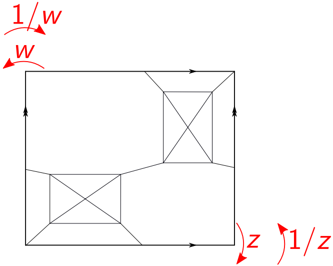

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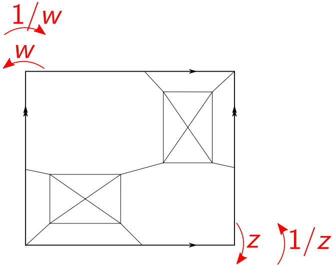

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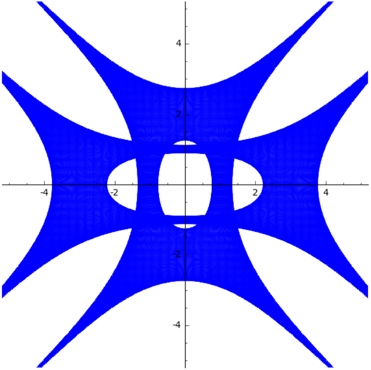
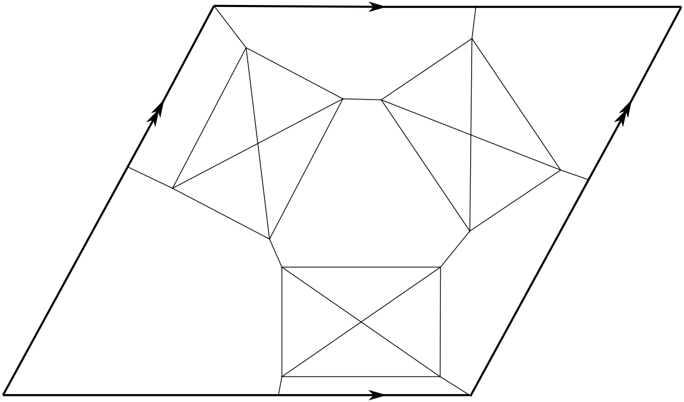


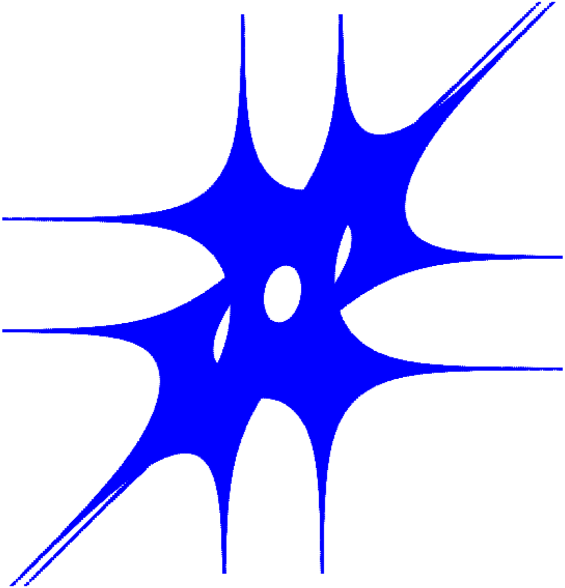
Amoeba: image of the zero locus of P by $(z,w) \mapsto (\log|z|, \log|w|)$.

Characteristic polynomial: examples

The free-fermion 8V model for $G =$ triangular lattice gives dimers:



Characteristic polynomial: examples



Non-bipartite to bipartite

Non-bipartite to bipartite (1)

Theorem *[M. 2020]*

For a finite graph on the torus equipped with a free-fermion **8V** model (a, b, c, d), let $P(z, w)$ be the characteristic polynomial of the corresponding (non-bipartite) dimers.
Then there exists two free-fermion **6V** models, (a_1, b_1, c_1) et (a_2, b_2, c_2), with characteristic polynomials P_1, P_2 s.t.

$$P(z, w) = P_1(z, w)P_2(z, w).$$

Consequence: $f = f_1 + f_2$.

The transformation

$$\left[a : b : c : d\right] = \left[\sin\left(\tfrac{\alpha+\beta}{2}\right) : \cos\left(\tfrac{\alpha+\beta}{2}\right) : \cos\left(\tfrac{\beta-\alpha}{2}\right) : \sin\left(\tfrac{\beta-\alpha}{2}\right)\right]$$

$$\mapsto \left[a_1 : b_1 : c_1\right] = \left[\sin\alpha : \cos\alpha : 1\right],$$

$$\left[a_2 : b_2 : c_2\right] = \left[\sin\beta : \cos\beta : 1\right].$$

In these variables, the free-fermion 8V model becomes a 6V one when $\alpha = \beta$.

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In these variables, the free-fermion 8V model becomes a 6V one when $\alpha = \beta$.

Remark: The previous result can be generalized into

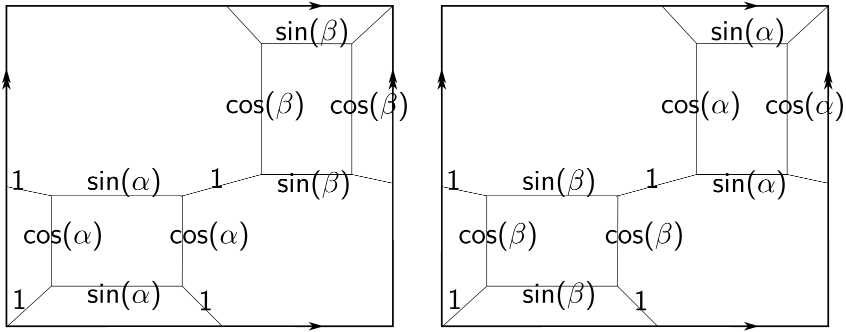
$$P_{\alpha,\beta}(z,w)P_{\alpha',\beta'}(z,w) = P_{\alpha,\beta''}(z,w)P_{\alpha',\beta}(z,w).$$

Example

For the initial "classical" 8V model on $\mathbb{Z}^2$, with

$$\left[a : b : c : d\right] = \left[\sin\left(\frac{\alpha+\beta}{2}\right) : \cos\left(\frac{\alpha+\beta}{2}\right) : \cos\left(\frac{\beta-\alpha}{2}\right) : \sin\left(\frac{\beta-\alpha}{2}\right)\right],$$

the two 6V models correspond to dimers



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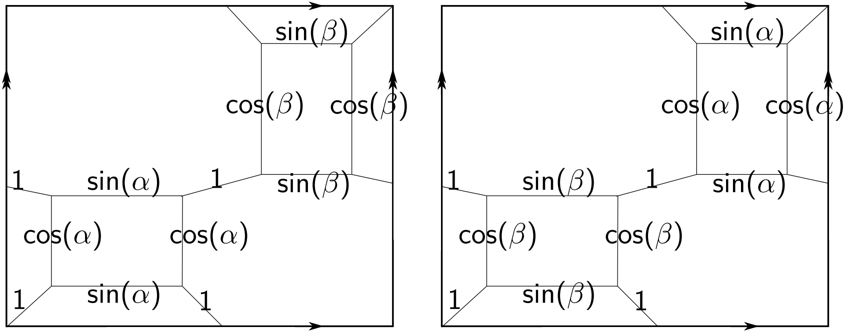

"Staggered" 6V models, in a gas phase for $\alpha \neq \beta$.

Idea of proof

Find a relation on Kasteleyn matrices.

Proposition *[M. 2020]*

Consider a graph on the sphere, torus, or the whole plane, equipped with a free-fermion 8V-model. Let

- K be the Kasteleyn matrix of the (non-bipartite) dimers from (a, b, c, d),
- K_1, K_2 those of (bipartite) dimers from (a_1, b_1, c_1) and (a_2, b_2, c_2),

then

$$K^{-1} = \frac{1}{2}\left((I + T)K_1^{-1} + (I - T)K_2^{-1}\right).$$

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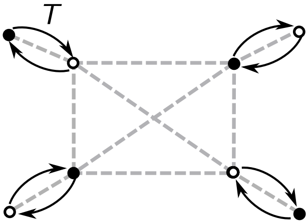
T is a simple switching operator (with some weight).



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Check relation by hand, plus some work to deduce determinants from this linear relation...

Consequence: Correlations of the 8V-model can actually be expressed in terms of those of 6V-models.

Remark: The relation can be generalized into

$$K_{\alpha,\beta}^{-1} = \frac{1}{2}\left((I+T)K_{\alpha,\beta'}^{-1} + (I-T)K_{\alpha',\beta}^{-1}\right).$$

Non-bipartite to bipartite (2)

Theorem *[M. 2020]*

For a finite planar graph, equipped with a free-fermion 8V-model a, b, c, d,

$$Z_{8V}(a,b,c,d)^2 = Z_{6V}(a_1,b_1,c_1)\ Z_{6V}(a_2,b_2,c_2).$$

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Moreover, let τ, τ' be random 8V-configuration sampled from the Boltzmann measure of (a, b, c, d), and τ_1, τ_2 from those of $(a_1, b_1, c_1), (a_2, b_2, c_2)$, all independent. Then

$$\tau \triangle \tau' \overset{d}{=} \tau_1 \triangle \tau_2.$$

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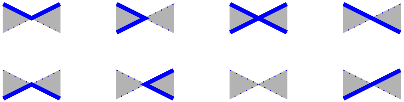
Remark: $\tau_{\alpha,\beta} \bigtriangleup \tau_{\alpha',\beta'} \overset{d}{=} \tau_{\alpha,\beta'} \bigtriangleup \tau_{\alpha',\beta}$

1. Duality

[Kramers-Wannier 1941], [Wu 1969], [Kadanoff-Ceva 1971], [Dubédat 2011]...

Other encoding of configurations:



$(0,1)$ $(1,0)$ $(0,0)$ $(1,1)$

Sketch of proof

1. Duality

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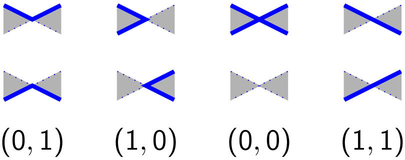

Hence the set of 8V-configurations can be seen as $H \subset \left(\mathbb{Z}_2^2\right)^E$.

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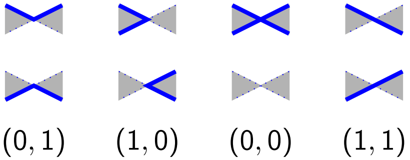

Hence the set of 8V-configurations can be seen as $H \subset \left(\mathbb{Z}_2^2\right)^E$.
Compatibility: $H = \ker(\Psi)$ where

$$\Psi : \left(\mathbb{Z}_2^2\right)^E \to \mathbb{Z}_2^{V \cup F}$$

$$(x_e, y_e)_{e \in E} \mapsto \left(\sum_{e \sim v} y_e, \sum_{e \sim f} x_e \right)_{v \in V, f \in F}$$

Sketch of proof

1. Duality

$$\left(\mathbb{Z}_2^2\right)^E \xrightarrow{\Psi} (\mathbb{Z}_2)^{V \cup F}$$

$H = \ker(\Psi)$

Sketch of proof

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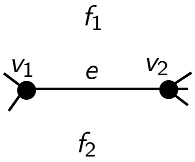
$$(\mathbb{Z}_2)^{V\cup F} \xrightarrow{\Phi} \left(\mathbb{Z}_2^2\right)^E \xrightarrow{\Psi} (\mathbb{Z}_2)^{V\cup F}$$

$H = \ker(\Psi) = \mathrm{Im}(\Phi)$
where

$$\Phi\left(\sigma_V, \sigma_F\right)_e = \left(\sigma_{v_1} + \sigma_{v_2}, \sigma_{f_1} + \sigma_{f_2}\right)$$



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By equipping $(\mathbb{Z}_2)^{V\cup F}$ with the standard scalar product $(\cdot,\cdot)$ and $\left(\mathbb{Z}_2^2\right)^E$ with the symplectic one $\langle\cdot|\cdot\rangle$:

$$\langle (x_e, y_e) | (x'_e, y'_e) \rangle = \sum_{e\in E} x_e y'_e + x'_e y_e,$$

one has $\Phi = \Psi^*$. Hence

$$H = H^\perp.$$

1. Duality

We have a space $(\mathbb{Z}_2^2)^E$ equipped with a (symplectic) form $\langle\cdot|\cdot\rangle$, so there is a Fourier transform:

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Applying this to the set of 8V-configurations $H = H^\perp$ and with g being the weight function, we get Wu's abelian duality.

Sketch of proof

1. Duality

$$Z_{8V}(a,b,c,d) = Z_{8V}(\hat{a},\hat{b},\hat{c},\hat{d})$$

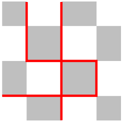
where

$$\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

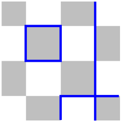
More generally, allows for tracking *order-disorder* variables.

2. Spin switching

τ



τ'



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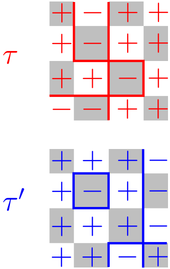

Sketch of proof

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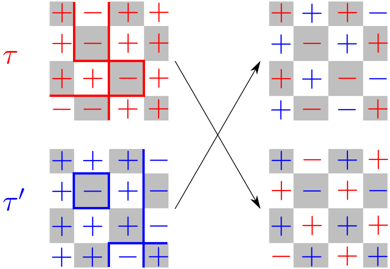

Sketch of proof

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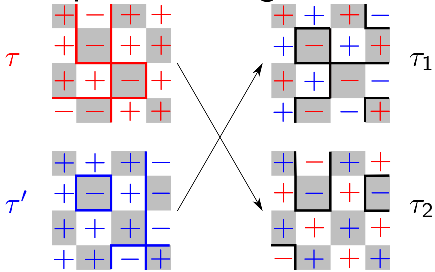

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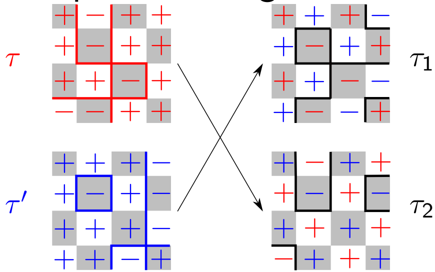
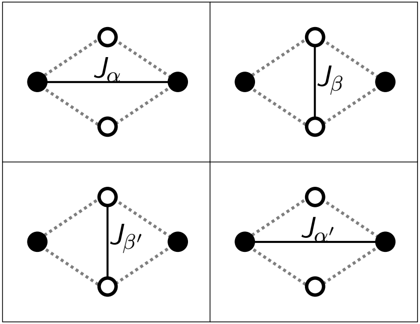


If $ab = cd$ and $a'b' = c'd'$,

$$\begin{aligned} &Z_{8V}(a, b, c, d) \\ &\times Z_{8V}(a', b', c', d') \\ = &Z_{8V}(a_1, b_1, c_1, d_1) \\ &\times Z_{8V}(a_2, b_2, c_2, d_2) \end{aligned}$$

Sketch of proof

3. Wrapping up



$\longrightarrow 8V^{\Gamma}_{\alpha,\beta} \longleftrightarrow 8V_{\alpha,\beta}$ *(f-f)*

$\longrightarrow 8V^{\Gamma}_{\alpha',\beta'} \longleftrightarrow 8V_{\alpha',\beta'}$

$\downarrow 8V^{\Gamma}_{\alpha,\beta'} \updownarrow 8V_{\alpha,\beta'}$

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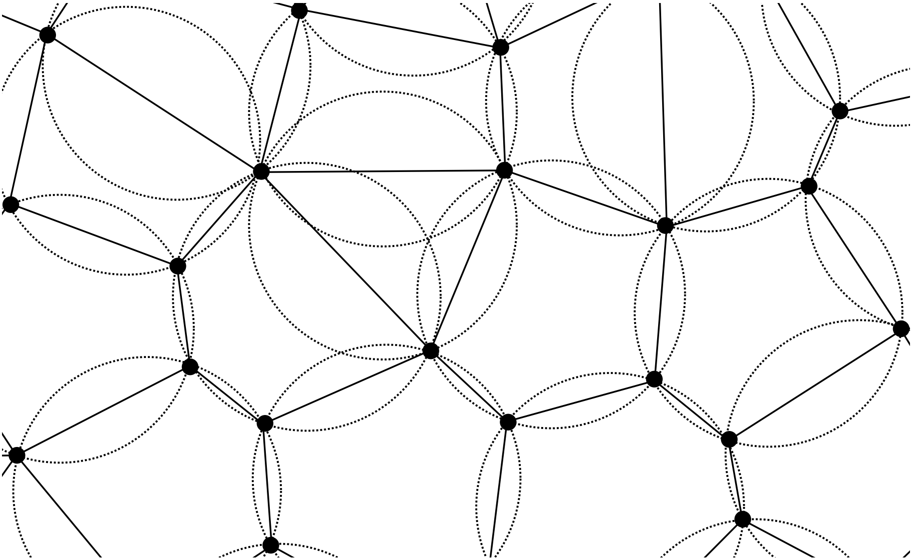
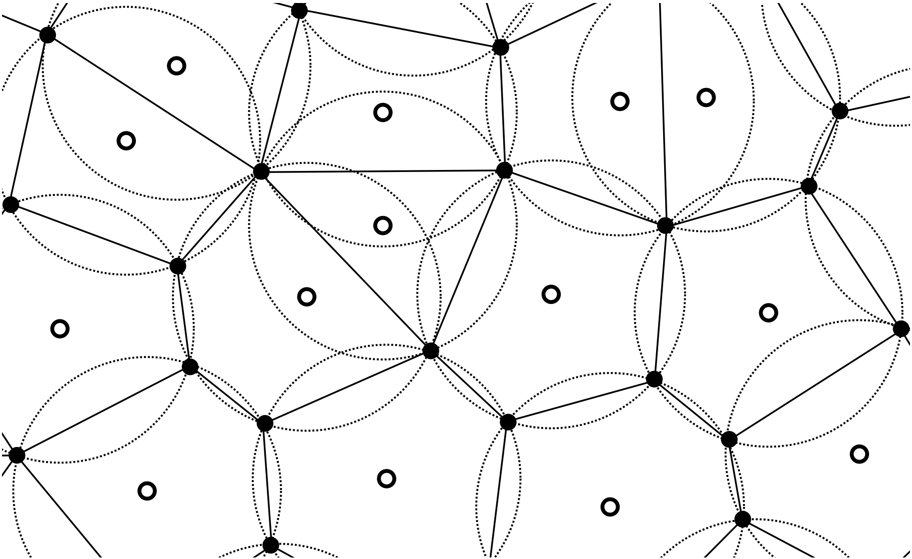
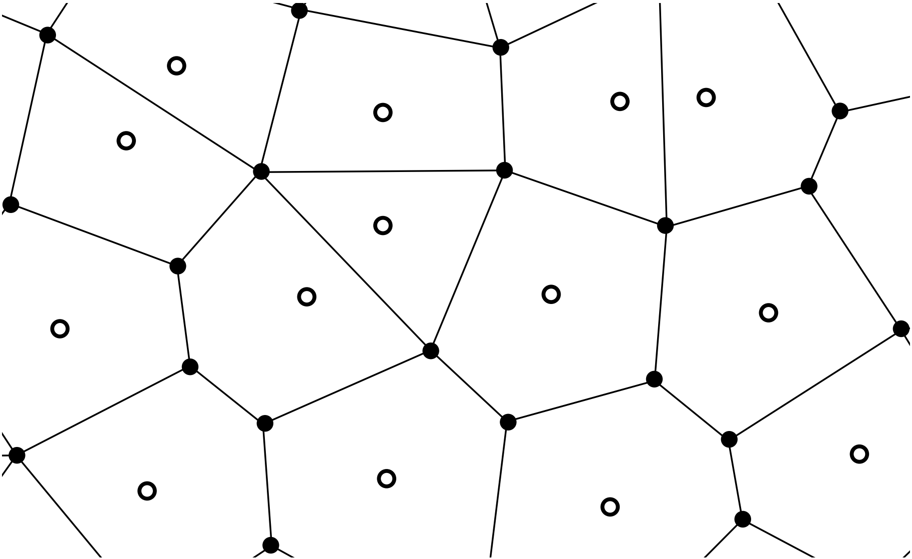
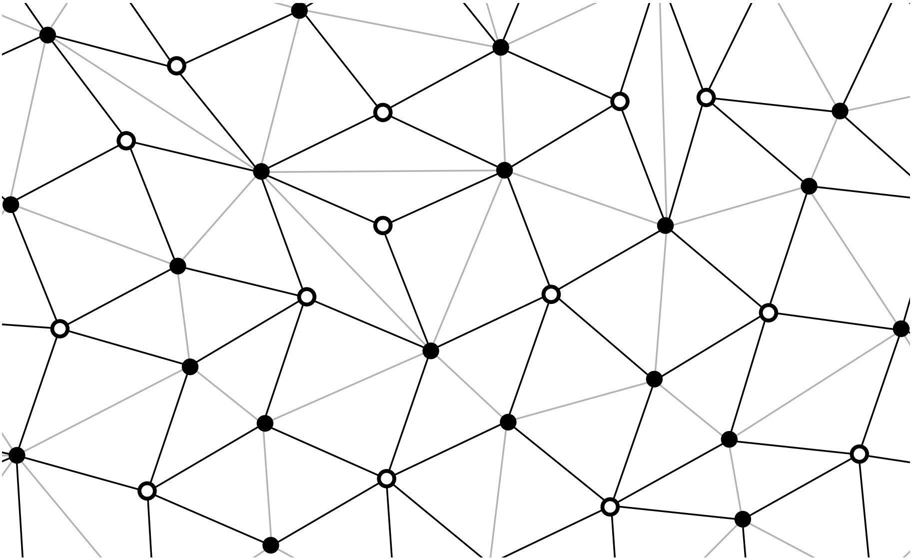
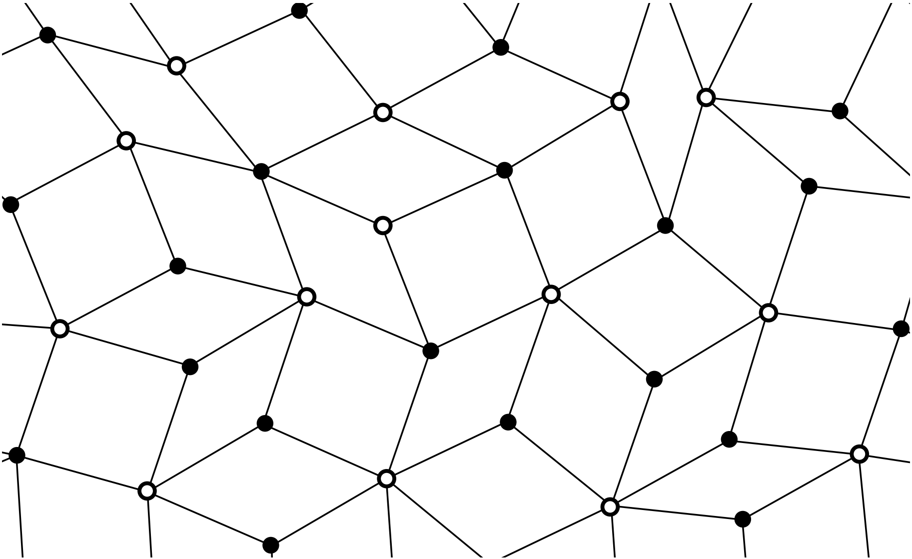
Application to the isoradial Z-invariant setting

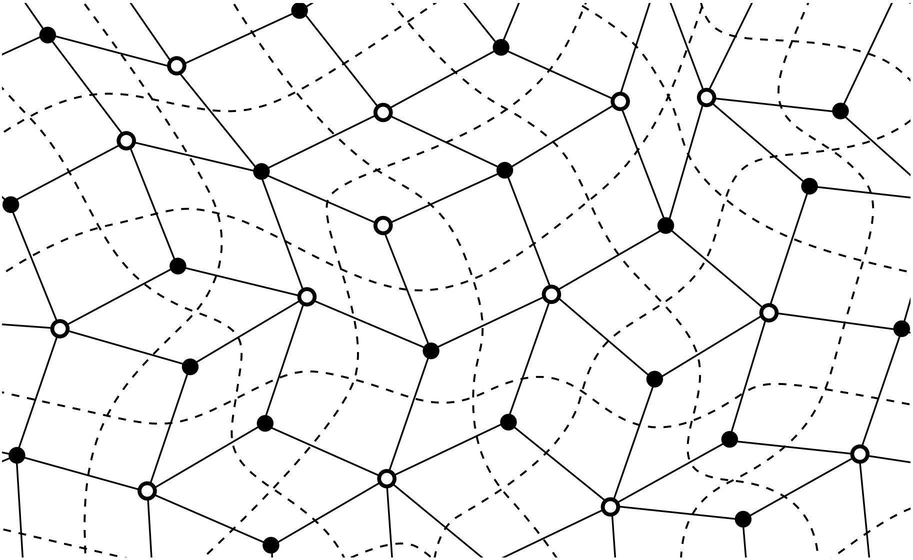


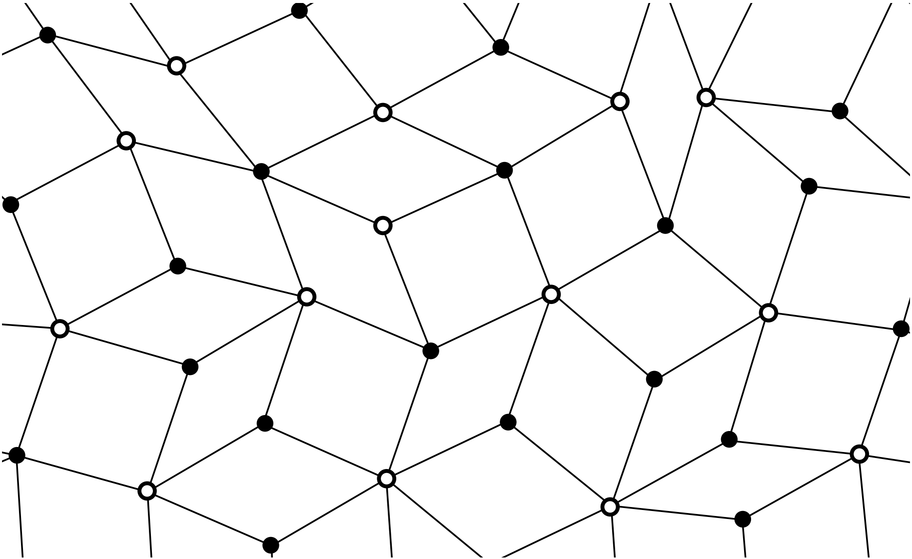


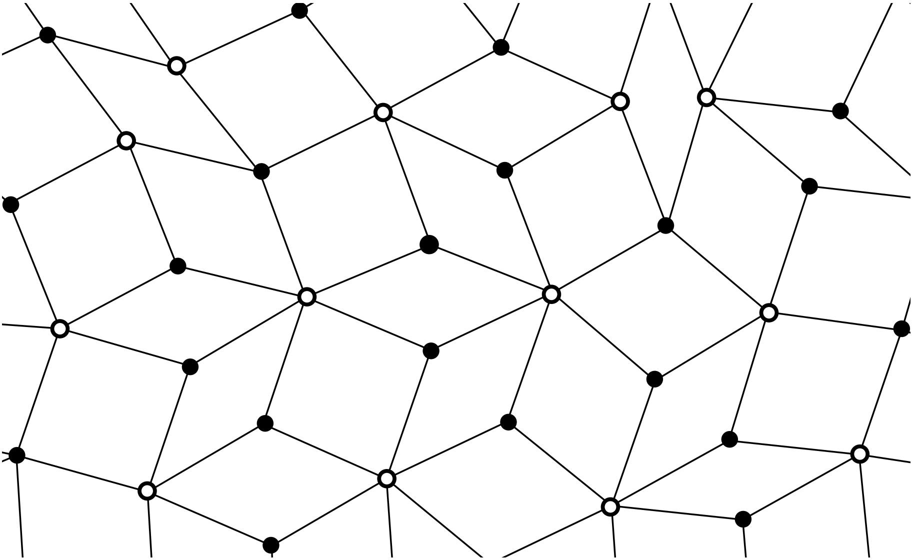




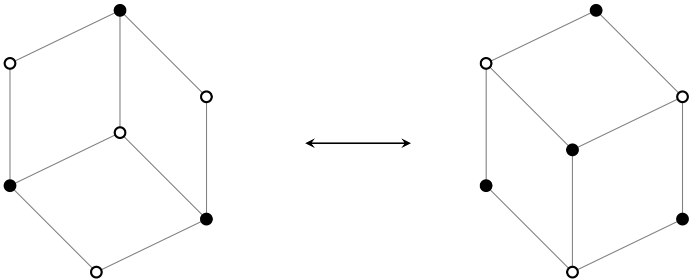








Star-triangle move for lozenges



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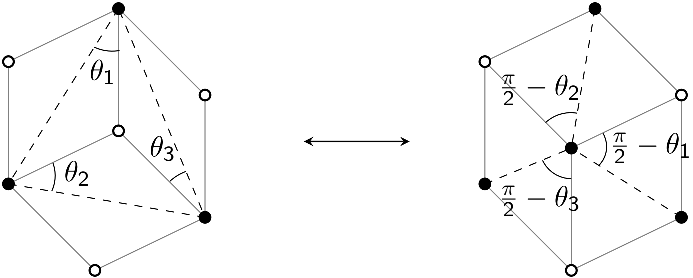
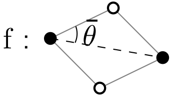


Star-triangle move for lozenges

Free-fermion 8V weights on the face of lozenges given by the geometry $(k, \ell \in [0,1))$:



$$a(\mathrm{f}) = \mathrm{sn}\,(\theta|k) + \mathrm{sn}\,(\theta|\ell)$$
$$b(\mathrm{f}) = \mathrm{cn}\,(\theta|k) + \mathrm{cn}\,(\theta|\ell)$$
$$c(\mathrm{f}) = 1 + \mathrm{sn}\,(\theta|k)\,\mathrm{sn}\,(\theta|\ell) + \mathrm{cn}\,(\theta|k)\,\mathrm{cn}\,(\theta|\ell)$$
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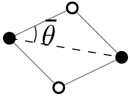
f : 

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Proposition *[M. 2020]*

This 8V model is invariant in distribution under star-triangle transformation of lozenges.

Star-triangle move for lozenges

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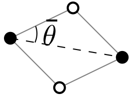
f : 

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Proposition *[M. 2020]*

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Z-**invariant** regime (Baxter).
Corresponding 6V models in *[Boutillier-de Tilière-Raschel 2016]*

Gibbs measure

Fix $k, \ell \in [0, 1)$ and the Z-invariant weights of the 8V model.

Theorem *[M. 2020]*

For any isoradial graph, and any $k, \ell \in [0, 1)$, there exists an ergodic Gibbs measure $\mathcal{P}_{k,\ell}$ on 8V configurations.

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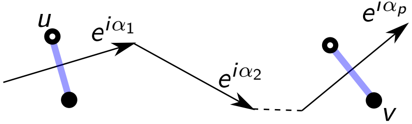
The operator $K_{k,\ell}^{-1}$ is explicit and **local**:



$$K_{k,\ell}^{-1}[u,v] = g_{k,\ell}(\alpha_1, \ldots, \alpha_p).$$

Correlations

Theorem

If $0 < k < \ell < 1$, as $|x - y| \to \infty$,

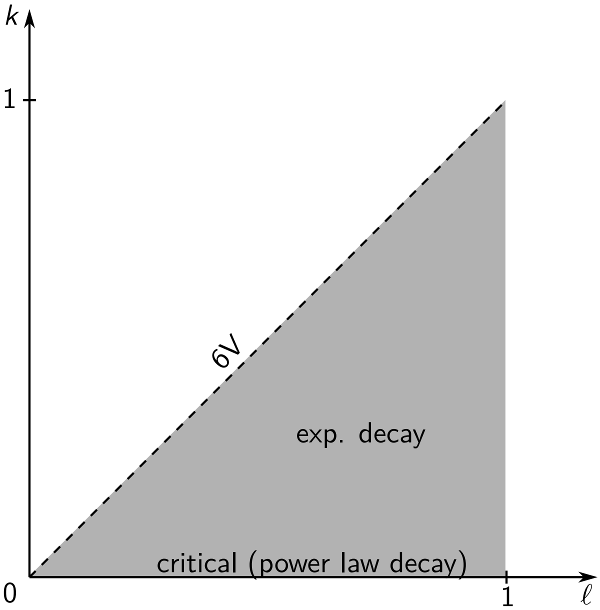
$$K_{k,\ell}^{-1}[x, y] \sim |x - y|^{-\frac{1}{2}} \exp\left(-\frac{|x - y|}{\xi_k}\right).$$

When $k \to 0$,

$$\xi_k = \Theta\left(k^{-2}\right) = \Theta\left((\beta - \beta_c)^{-1}\right).$$

Critical exponent $\nu = 1$ (universality class of the Ising model).

Correlations



Summary

In free-fermion vertex models, there exists a generic relation

$$8V^2 = 6V_1 \times 6V_2.$$

It can be made global, local, algebraic, probabilistic...

Perspectives

- Computation of density of states a, b, c, d.
- Geometric interpretation/embedding? (Regge symmetry)
- Analytic extension?