Eigenstate deformations as a sensitive probe of quantum chaos.

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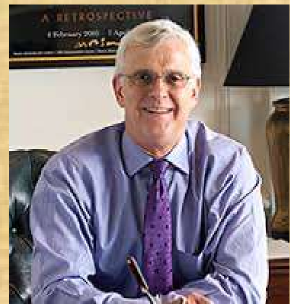


David

Marcos

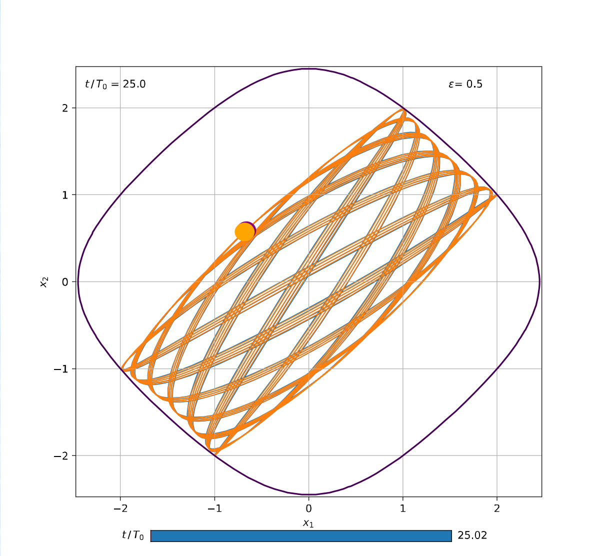
Quantum Matter Meets Math, Lisbon, **03/16/21**

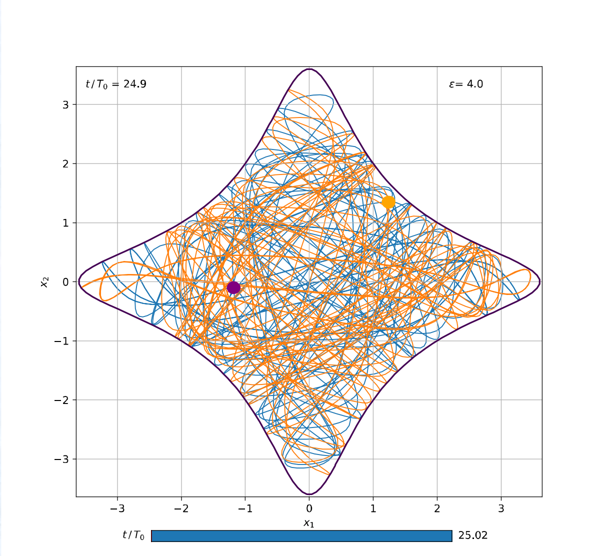
Classical systems

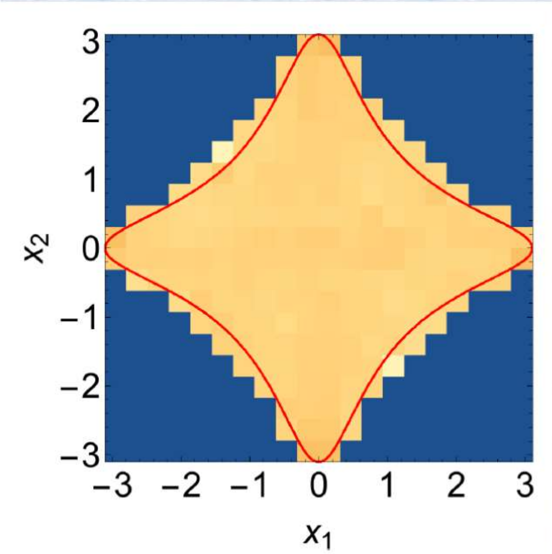
Chaos - sensitivity of trajectories to small perturbations

Ergodicity (loosely): time average = ensemble average

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{\epsilon}{4} x_1^2 x_2^2$$







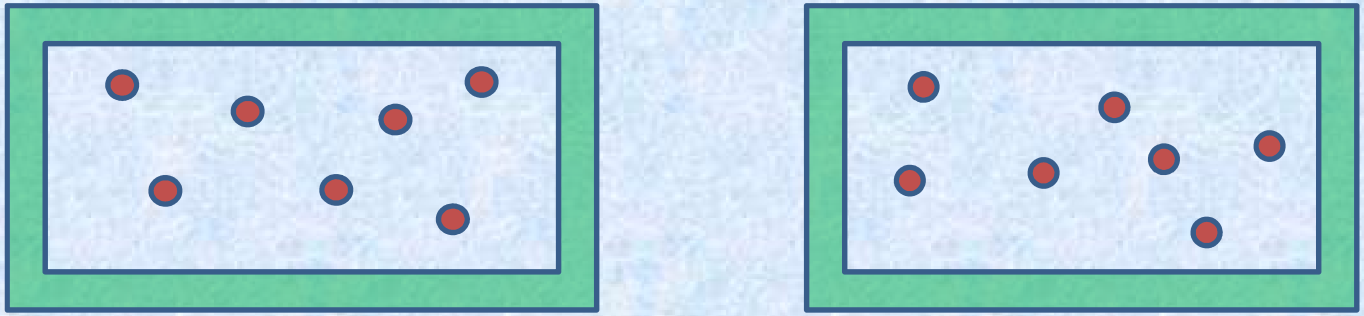
Non-chaotic regime

Chaotic regime

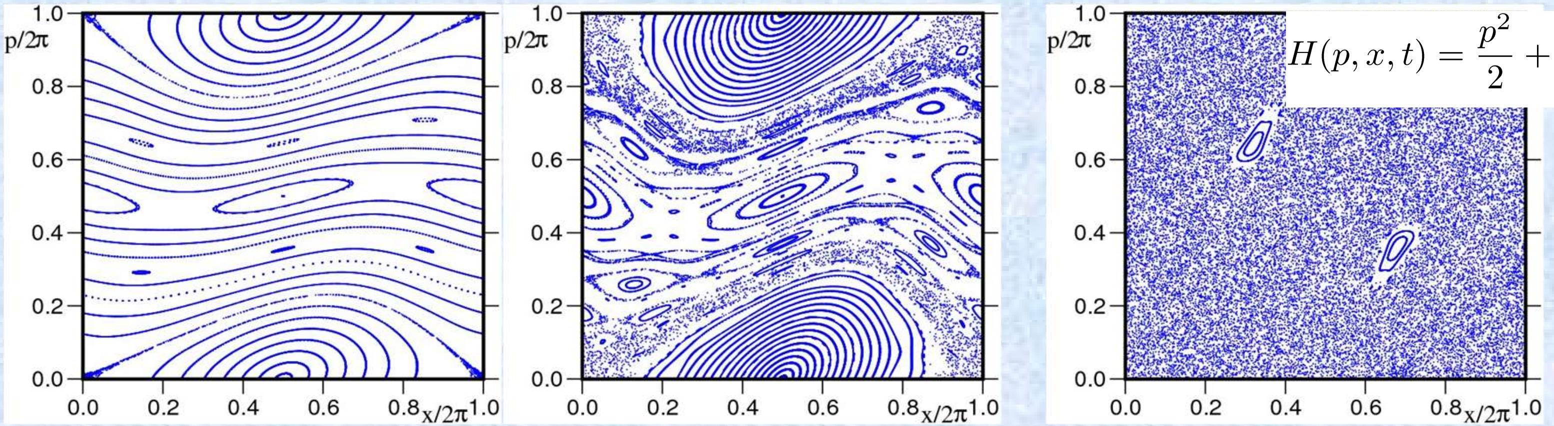
2D $$P_{\rm mc}(\vec{x}) = C \int d\vec{p}\, \delta[E - H(\vec{x},\vec{p})] = \frac{1}{\mathcal{S}}\, \theta[E - V(\vec{x})],$$

Non-ergodic systems

1. Trivial: extra symmetries, conservations laws.



2. Non-trivial (emergent approximate conservation laws)



$$H(p,x,t) = \frac{p^2}{2} + K\cos(x)\sum_n \delta(t-n)$$

Chaotic non-ergodic

Chaotic ergodic

Common assumption: no nontrivial exceptions in TD limit, i.e. chaos is equivalent to ergodicity.

Menu.

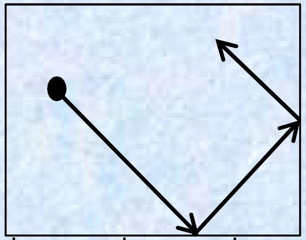
Quantum chaos.

- Level repulsion. Wigner-Dyson statistics
- Quantum Lyapunov exponents, OTOC, scrambling
- Entanglement of eigenstates
- Canonical typicality. ETH
- Universal operator growth
- Quantum echo (reversibility)
- Diagonal entropy. Thermodynamic relations.
- Today's special: adiabatic transformations

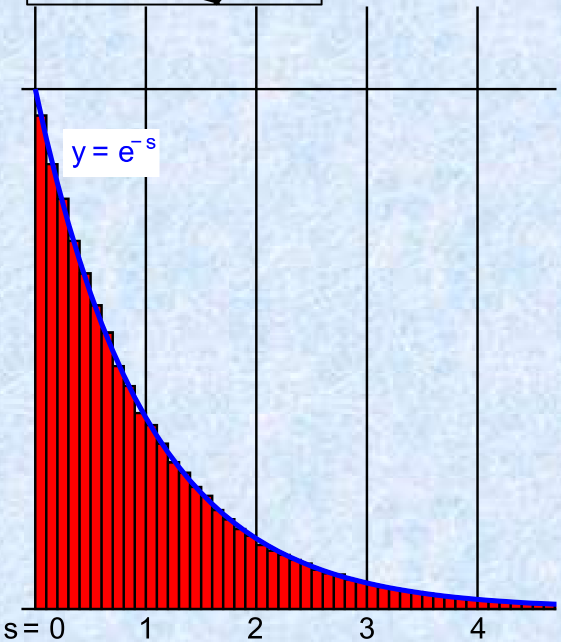
Quantum chaos: two powerful conjectures about energy levels

Berry-Tabor conjecture, 1977: Non-chaotic “generic systems”: expect Poisson statistics. Bohigas, Giannoni, Schmit (BGS) conjecture 1984: random matrix statistics in chaotic generic systems

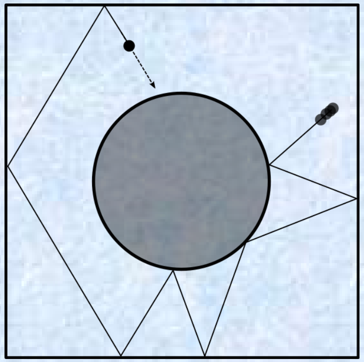
Examples:

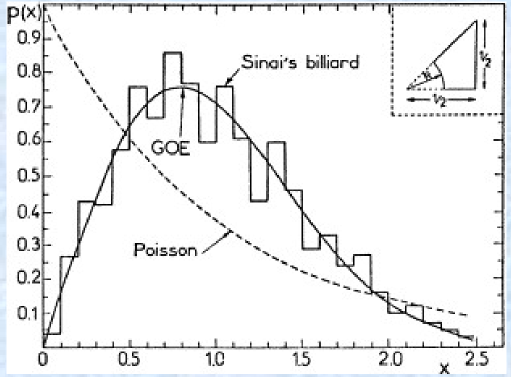


Incommensurate rectangular box

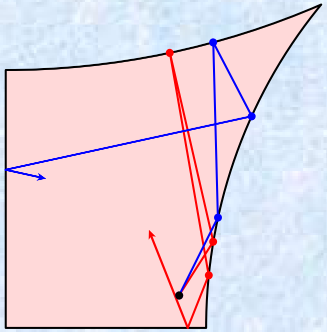


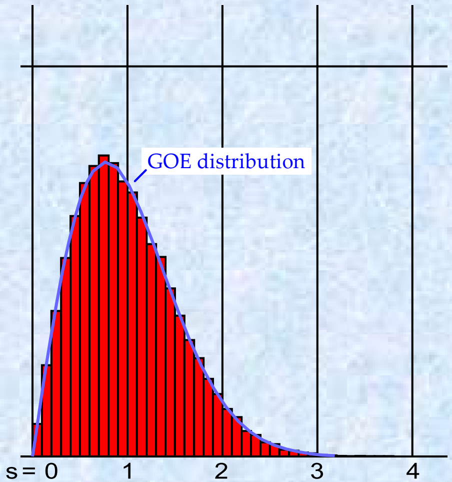
Z. Rudnik, 2008





Sinai Billiard. O. Bohigas et. al. 1984





Another Sinai billiard: Z. Rudnik, 2008

Side remark: RMT level statistics is not special to QM and can be applied to classical systems (P. Claeys and A.P. 2020).

Take a classical (say Gibbs) distribution

$$P(\vec{x},\vec{p}) = \frac{1}{Z}\exp\left[-\beta\left(\frac{\vec{p}^2}{2m} - V(\vec{x})\right)\right]$$

Take the Fourier transform w.r.t. the momentum, define

$$\mathcal{W}(\vec{x}_1,\vec{x}_2) = \int d\vec{p}\, P\left(\frac{\vec{x}_1+\vec{x}_2}{2},\vec{p}\right) \mathrm{e}^{-i\vec{p}(\vec{x}_1-\vec{x}_2)/\epsilon} \propto \exp\left[-\frac{m(\vec{x}_1-\vec{x}_2)^2}{2\beta\epsilon^2} - \beta V\left(\frac{\vec{x}_1+\vec{x}_2}{2}\right)\right]$$

This is a symmetric (generally Hermitian) matrix. Can diagonalize it

$$\mathcal{W}(\vec{x}_1,\vec{x}_2) = \sum_n w_n \psi_n^*(x_1)\psi_n(x_2)$$

Many close parallels with quantum mechanics: Discrete spectrum, representation of observables through Hermitian operators, …:

$$\overline{p} \equiv \int dx\, dp\, p P(x,p) = \sum_n w_n \int dx\, \psi_n^*(x)\, \hat{p}\, \psi_n(x), \quad \hat{p} = -i\epsilon\partial_x$$

Integral equation for Gibbs Eigenstates

$$w_n \psi_n(x) = \int dx' \mathcal{W}(x, x') \psi_n(x') = \int d\xi \mathcal{W}(x, x-\xi) \psi_n(x-\xi)$$

$$\longleftrightarrow$$

$$w_n \psi_n(x) \propto \int d\xi \mathrm{e}^{-\frac{m\xi^2}{2\beta\epsilon^2} - \beta V(x-\xi/2)} \mathrm{e}^{-\xi \frac{d}{dx}} \psi_n(x) = \frac{1}{Z} \mathrm{e}^{-\beta \hat{H}_{\mathrm{Gibbs}}} \psi_n(x)$$

Small β – *s*addle point approximation + leading corrections

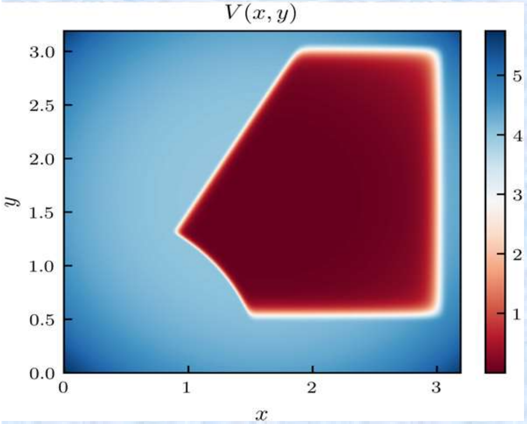
$$\begin{aligned} \hat{H}_{\mathrm{Gibbs}} &= \frac{\hat{p}^2}{2m} + V(x) - \frac{\beta\epsilon^2}{8m} V''(x) \\ &+ \frac{\beta^2\epsilon^2}{24m} \left(\frac{1}{4m} \left[\hat{p}^2 V''(x) + 2\hat{p} V''(x) \hat{p} + V''(x) \hat{p}^2 \right] + V'(x)^2 \right) + O(\beta^3) \end{aligned}$$

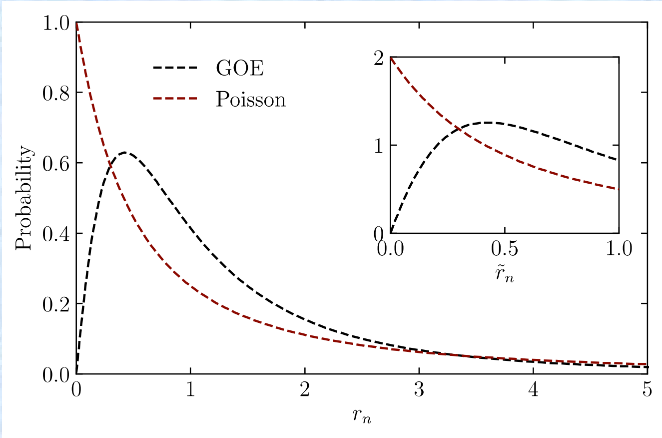
All eigenstates (ground and excited) $\psi_n(x)$ satisfy the Schrödinger, equation. Recover tunneling states, Berry phases, band structures, fermions, bosons, …. There is no semiclassical limit here!

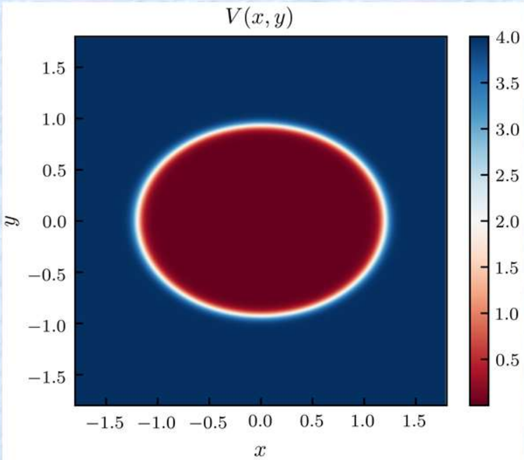
Chaotic systems. BGS and Berry-Tabor conjectures

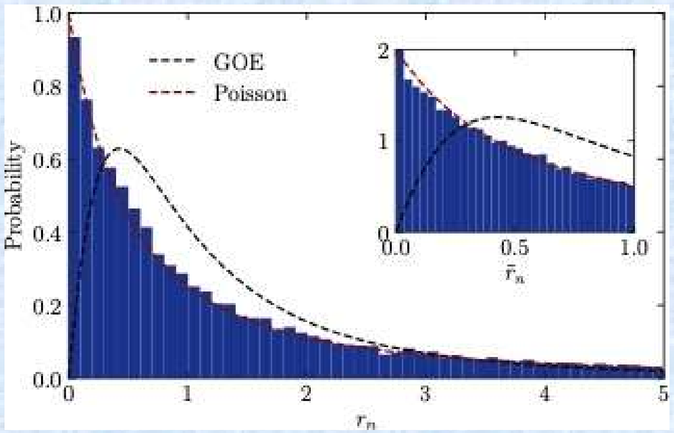
(thanks to M. Berry for the suggestion to check)

$$r_n = \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}} = \frac{\log(w_n/w_{n+1})}{\log(w_{n+1}/w_{n+2})}$$









The classical Gibbs ensemble knows whether the system is chaotic or integrable.

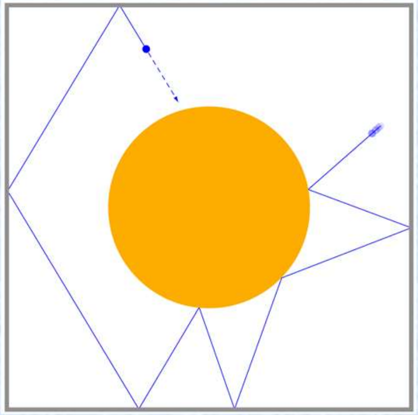
Experimentally can detect chaos from a series of static images: no dynamics.

Connection to Lyapunov exponents? Both definitions of chaos are entirely within the classical framework.

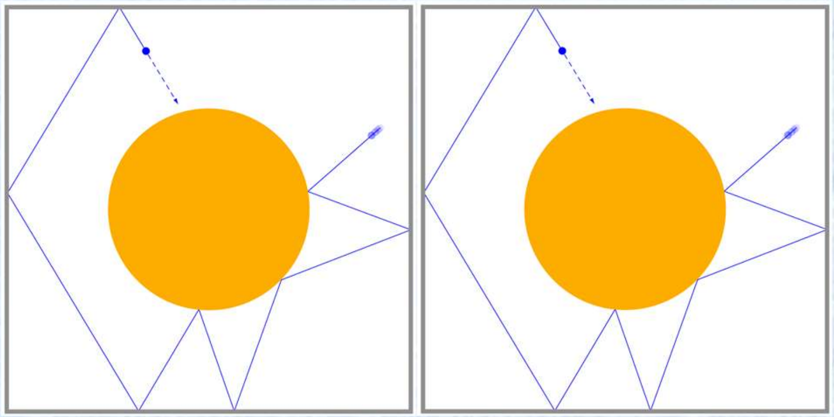
Experiments?

Level statistics is a measure of ergodicity, not chaos.

RMT, ETH imply stationary states (long time average) are thermal (J. Deutsch 1992; M. Srednicki 1994; M. Rigol, V. Dunjko, M. Olshanii 2008).



Chaotic ergodic.
GOE level statistics



Chaotic, non-ergodic.
Mixed level statistics

TD limit: usually chaos implies ergodicity, so the measure is fine

OTOC (= quantum echo) $F(t) \sim \langle |[O(t), O(0)]|^2 \rangle \propto \exp[2\lambda t]$

Only works for quantum systems near a classical limit (commutators can be replaced with Poisson brackets leading to the same result). Does not apply to e.g. quantum systems with local interactions (B. Fine et. al. 2013, I. Kukuljan, S. Grozdanov, T. Prosen 2017)

Key idea: use eigenstate sensitivity to probe quantum chaos

Family of Hamiltonians $H(\lambda)$. Transformations of eigenstates:

$$|n(\lambda)\rangle = U(\lambda)|n_0\rangle, \quad U^\dagger(\lambda)H(\lambda)U(\lambda) = \mathrm{diag}(E_j(\lambda))$$

$$i\,\partial_\lambda|n\rangle = \mathcal{A}_\lambda|n\rangle, \qquad \mathcal{A}_\lambda = i(\partial_\lambda U)U^\dagger$$

$\mathcal{A}_\lambda$ – adiabatic gauge potential (AGP) – generator of adiabatic transformations.

It defines a natural distance metric (a.k.a. geometric tensor, fidelity susceptibility) between the eigenstates (Provost Valee, 1980)

$$g_{\lambda\lambda} = \langle\partial_\lambda n|\partial_\lambda n\rangle_c = \langle n|\mathcal{A}_\lambda^2|n\rangle_c = \sum_{m\neq n}|\langle n|\mathcal{A}_\lambda|m\rangle|^2$$

Intuitively: expect a very large distance in chaotic systems because of their high sensitivity to perturbations.

This talk: explore this idea in detail

Hellmann-Feynman theorem (first order perturbation theory)

$$\langle n|\mathcal{A}_\lambda|m\rangle = i\langle n|\partial_\lambda|m\rangle = i\frac{\langle n|\partial_\lambda H|m\rangle}{E_m - E_n}$$

$$g_{\lambda\lambda} = \sum_{m\neq n} |\langle n|\mathcal{A}_\lambda|m\rangle|^2 = \sum_{m\neq n} \frac{|\langle n|\partial_\lambda H|m\rangle|^2}{(E_n - E_m)^2}$$

Chaotic/ergodic systems satisfying – ETH (RMT)

$$|\langle n|\partial_\lambda H|m\rangle| \sim \exp[-S/2],$$
$$\min|E_m - E_n| \sim \exp[-S]$$

$$||\mathcal{A}_\lambda||^2 \propto \exp[S]$$

Relation to the spectral (autocorrelation) function:

$$||\mathcal{A}_\lambda||^2 = \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(\mu^2+\omega^2)^2} \overline{|f_\lambda(\omega)|^2} \sim \frac{\overline{|f(\mu)|^2}}{\mu}, \quad \mu \to 0$$

$$\overline{|f_\lambda(\omega)|^2} = \frac{1}{4\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\overline{\langle n|\{\partial_\lambda H(t), \partial_\lambda H(0)\}|n\rangle_c}$$

Choose $\mu \propto L\, 2^{-L}$ for smoothening. Physically: exponentially long cutoff time but less than the Heisenberg time.

Alternatively analyze a typical $\chi = \exp[\overline{\log g_{\lambda\lambda}}]$.

Adiabatic transformations and conservation laws

$$\langle n|\mathcal{A}_\lambda|m\rangle = i\frac{\langle n|\partial_\lambda H|m\rangle}{E_m - E_n} \quad \Leftrightarrow \quad [\underbrace{\partial_\lambda H + i[\mathcal{A}_\lambda, H]}, H] = 0$$

G_λ - conserved operator

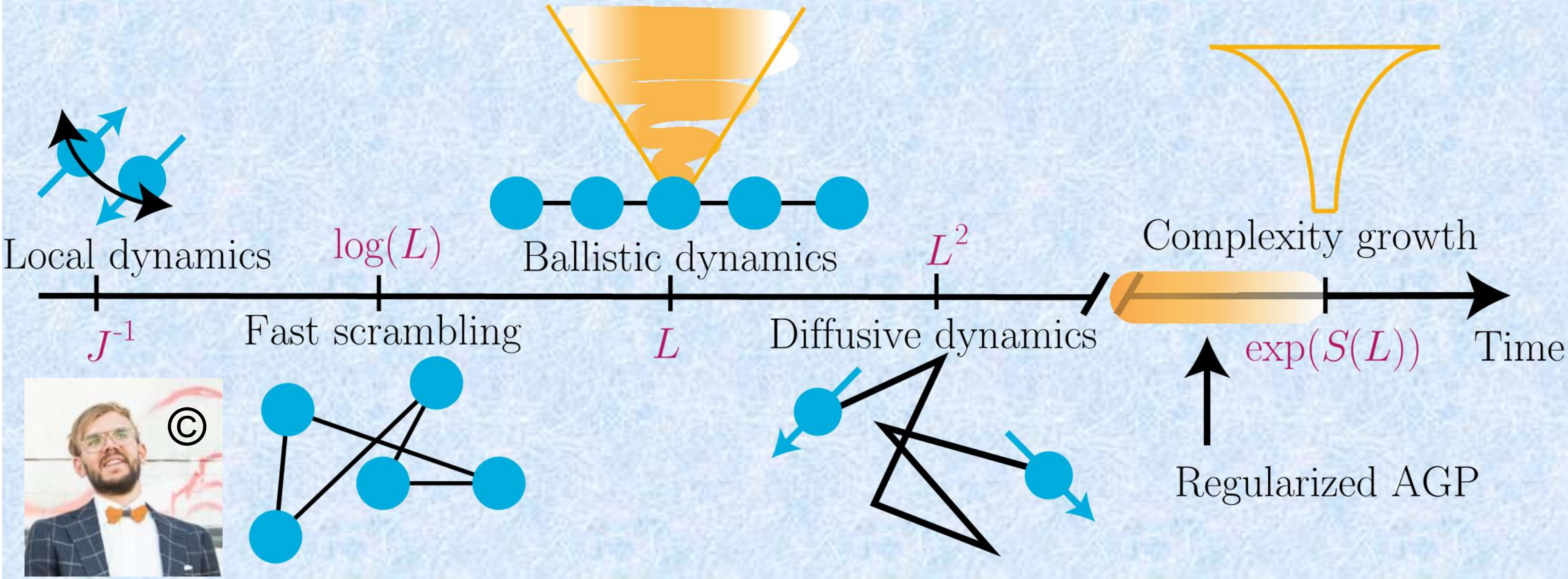
Straightforward to check:

$$\mathcal{A}_\lambda = -\frac{1}{2}\int_{-\infty}^{\infty} dt\, \mathrm{sgn}(t)(\partial_\lambda H)(t)\mathrm{e}^{-\mu|t|}, \quad (\partial_\lambda H)(t) \equiv \mathrm{e}^{iHt}\partial_\lambda H \mathrm{e}^{-iHt}$$

$$G_\lambda \equiv \partial_\lambda H + i[\mathcal{A}_\lambda, H] = \frac{\mu}{2}\int_{-\infty}^{\infty} dt\, (\partial_\lambda H)(t)\mathrm{e}^{-\mu|t|}$$

G_λ are used to find approximate conservation laws in perturbed integrable models (M. Mierzejewski, T. Prosen, and P. Prelovek, PRB 2015). Local adiabatic transformations imply local conservation laws.

General hierarchy of time scales in chaotic systems



General hope – exponentially long times are exponentially sensitive to small perturbations. Shorter times – hopeless to detect weak chaos!

Mini Summary of Expectations

$$||\mathcal{A}_\lambda||^2 = \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(\mu^2+\omega^2)^2} |f_\lambda(\omega)|^2 \sim \frac{|f(\mu)|^2}{\mu}, \quad \mu \to 0$$

Ergodic/ETH

$$||\mathcal{A}_\lambda||^2 \propto \mathrm{e}^{S(L)}$$

Free like TFI – AGP
is a local extensive operator

$$||\mathcal{A}_\lambda||^2 \propto L$$

Generic integrable – expect
something in between

$$||\mathcal{A}_\lambda||^2 \propto L^\beta, \beta > 1,$$

$$||\mathcal{A}_\lambda||^2 \propto \mathrm{e}^{\alpha S(L)}, \; \alpha < 1$$

Models

1. Interacting integrable: XXZ chain

$$H_{\rm XXZ} = \sum_{i=1}^{L-1} (\sigma_{i+1}^x \sigma_i^x + \sigma_{i+1}^y \sigma_i^y) + \Delta \sum_{i=1}^{L-1} \sigma_{i+1}^z \sigma_i^z$$

2. ETH/ergodic: Ising chain

$$H_{\rm Ising} = \sum_{i=1}^{L-1} \sigma_{i+1}^z \sigma_i^z + h_z \sum_{i=1}^{L} \sigma_i^z + h_x \sum_{i=1}^{L} \sigma_i^x, \quad h_x = (\sqrt{5}+5)/8,\ h_z = (\sqrt{5}+1)/4$$

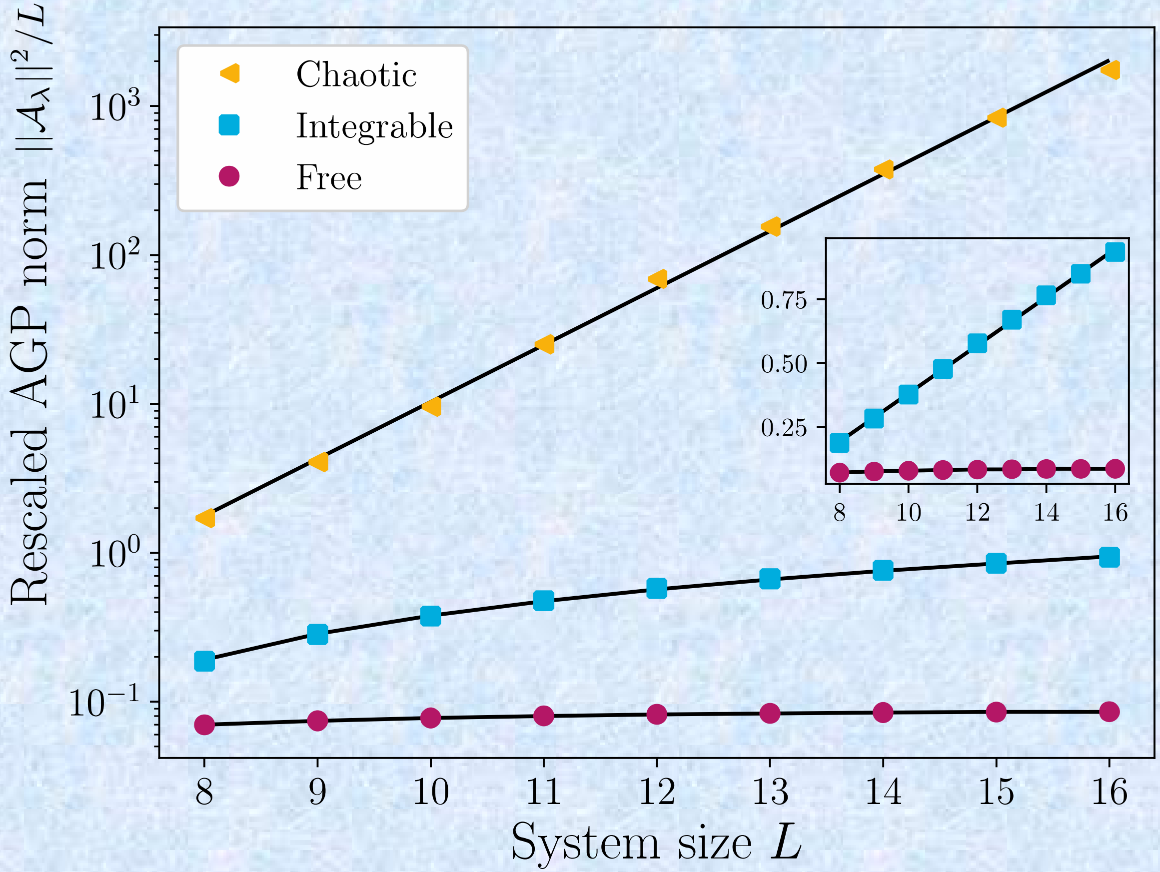
3. Free, also Ising chain with $h_z = 0, h_x = 0.83$.

4. Break integrability

$$H \rightarrow H + \epsilon V \qquad V = \sigma_{[L+1]/2}^z, \quad V = \sum_j \sigma_j^z \sigma_{j+2}^z$$

Numerical results

$$||\mathcal{A}_\lambda||^2 = \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(\mu^2+\omega^2)^2} |f_\lambda(\omega)|^2$$

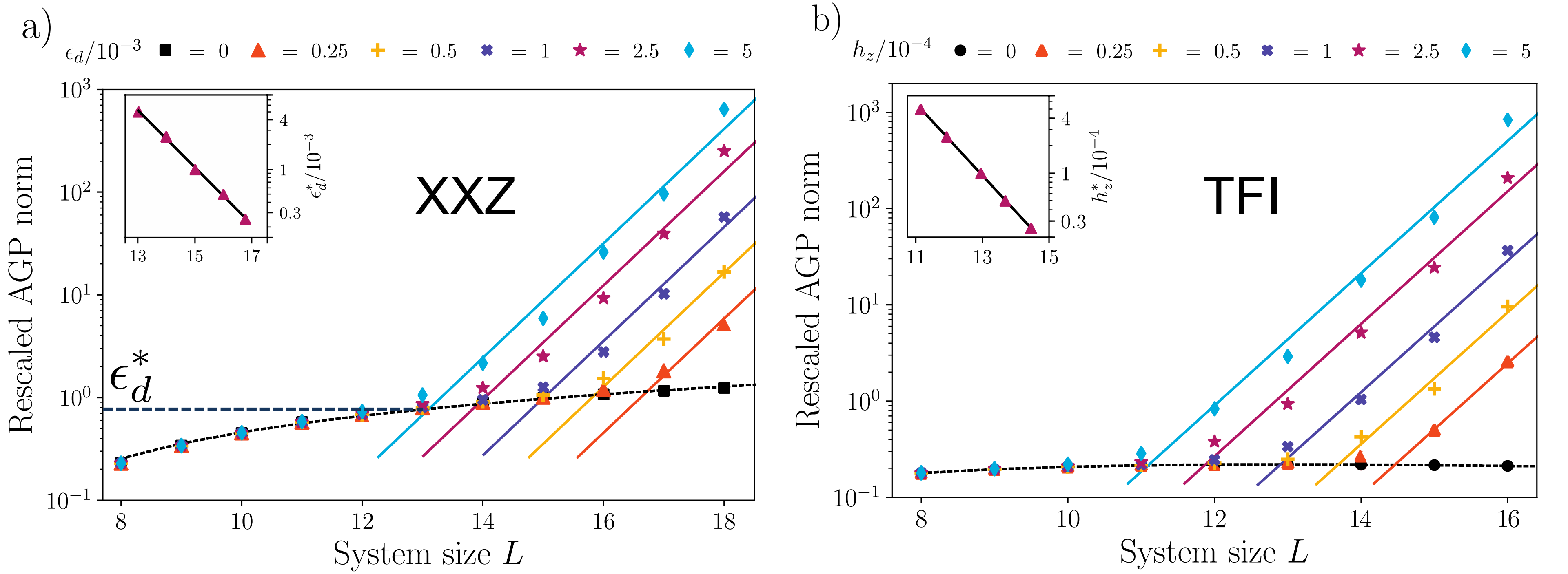


Nonuniversal Δ-dependent power law for the interacting integrable model

$\mathcal{A}_\lambda$ is a quasi-long range operator for generic integrable models.

$|f_\lambda(\omega)|^2 \to 0, \omega \to 0$ (similar conclusions T. LeBlond, M. Rigol et. al.). Long time oscillatory dynamics of $\partial_\lambda H(t)$ after Thouless time, diffusion equation is incomplete. Seems to be generic for interacting integrable models.

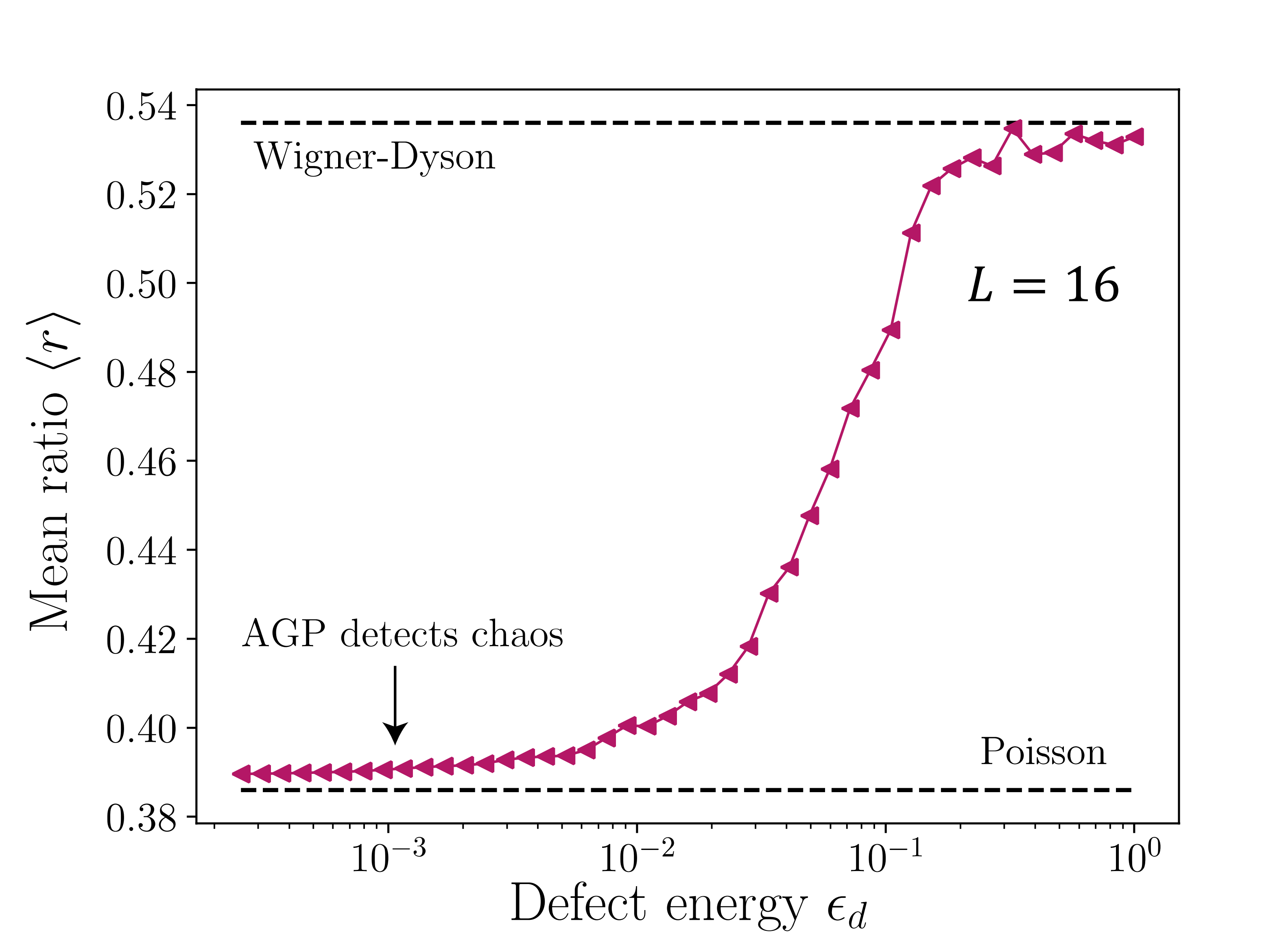
Break integrability



Fit: $||\mathcal{A}_\lambda||^2 \sim \epsilon_d^\alpha \mathrm{e}^{\beta L},\ \epsilon_d^* \sim \mathrm{e}^{-\frac{\beta}{\alpha}L},\qquad \beta \approx 2\log(2),\ \alpha \approx 1.7$

Exponentially small in L threshold for the onset of chaos

Comparison with other methods



AGP is orders of magnitude more sensitive than level statistics and the spectral form factor – standard measures of chaos.

$$||\mathcal{A}_\lambda||^2 \sim \frac{|f_\lambda(\mu)|^2}{\mu} \quad \rightarrow \quad |f_\lambda(\mu)|^2 \sim \mu||\mathcal{A}_\lambda||^2$$

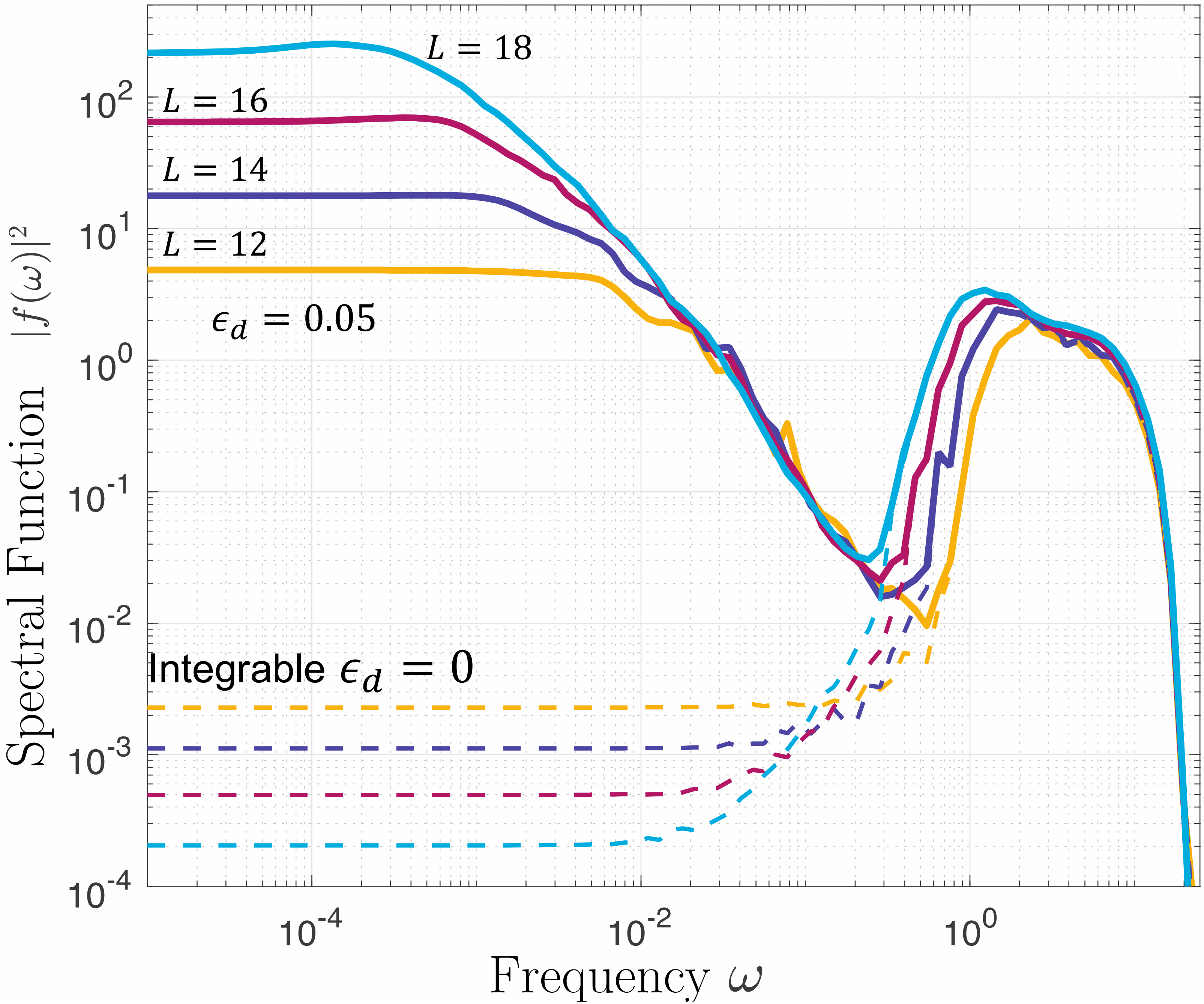
$$|f_\lambda(\omega)|^2 = \frac{1}{4\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle n|\{\partial_\lambda H(t), \partial_\lambda H(0)\}|n\rangle_c \;\Rightarrow\; |f_\lambda(\omega \rightarrow 0)|^2 \propto \tau$$

$$||\mathcal{A}_\lambda{}^2|| \sim 2^{\beta L}, \;\; \beta > 1 \quad \Rightarrow \quad \tau \sim 2^{(\beta-1)L}$$

At the chaos onset the system develops exponentially long (in the system size) relaxation times.

Physically: the Drude weight in the integrable limit is transferred to frequencies of the order of the level spacing.

Extracted spectral function



Full transition/crossover from integrability to ergodicity
(T. LeBlond, D. Sels, A. P., M. Rigol, 2020)

$$\hat{H}_{\rm cln} = \sum_{i=1}^{L} \left[\frac{J}{2} \left(\hat{S}_i^+ \hat{S}_{i+1}^- + \text{H.c.} \right) + \Delta \hat{S}_i^z \hat{S}_{i+1}^z + \Delta' \hat{S}_i^z \hat{S}_{i+2}^z \right]$$

$$J = \sqrt{2}, \qquad \Delta = (\sqrt{5}+1)/4, \qquad \Delta' \in [10^{-4}, 10^{1}]$$

Analyze typical fidelity susceptibility (also suitable for disordered systems, no need for cutoff).

$$\chi = L \exp \left[\overline{ \log \left(\sum_{m \neq n} \frac{|\langle n|\hat{O}|m\rangle|^2}{(E_n - E_m)^2} \right) } \right]$$

$$\hat{O} = \hat{K}_{\rm nn} = \frac{1}{L} \sum_{i=1}^{L} \left(\hat{S}_i^+ \hat{S}_{i+2}^- + \text{H.c.} \right),$$

$$\hat{O} = \hat{U}_{\rm n} = \frac{1}{L} \sum_{i=1}^{L} \hat{S}_i^z \hat{S}_{i+1}^z,$$

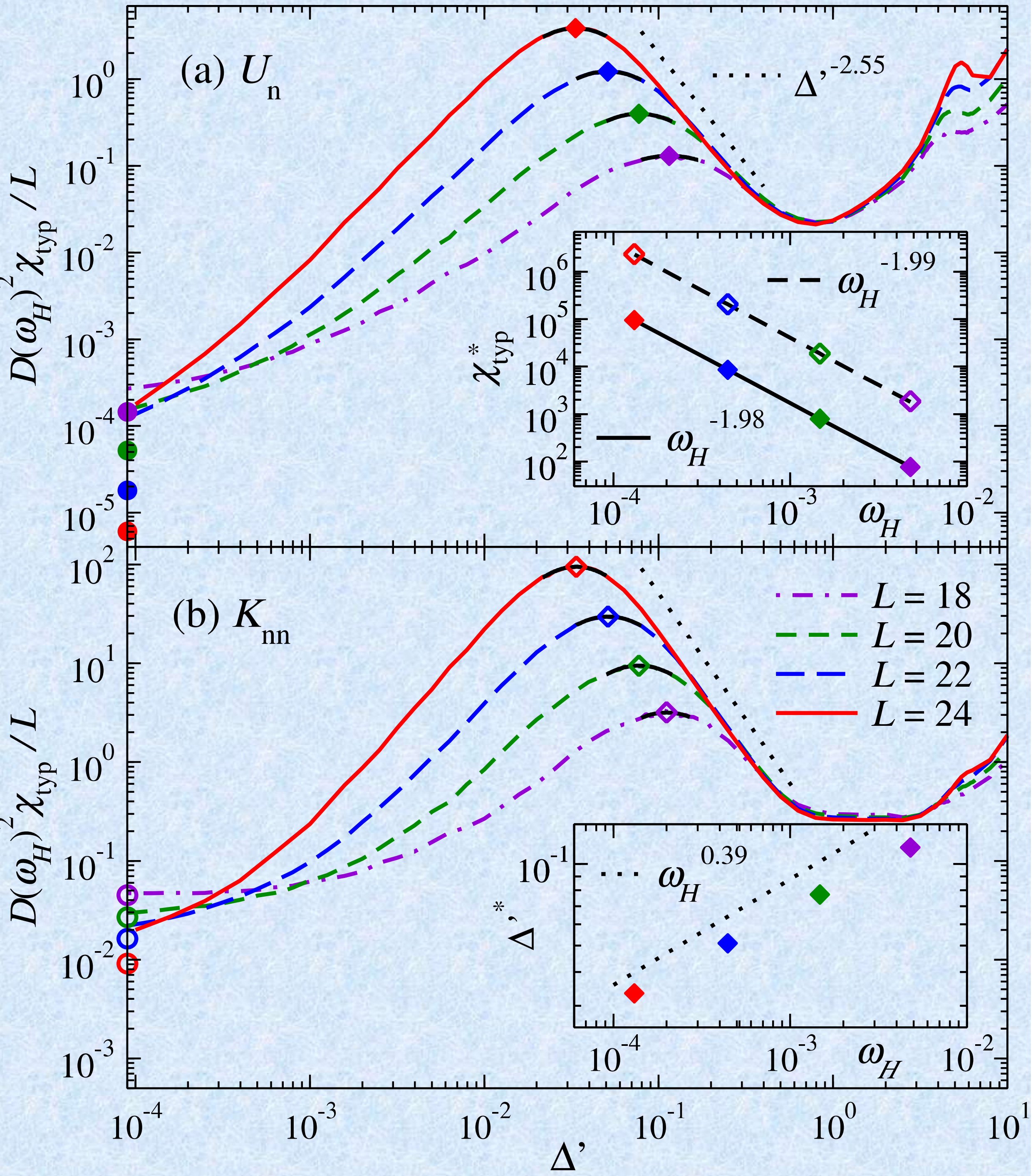
Results

- Universal maximum chaos regime separating integrable and ergodic phases.

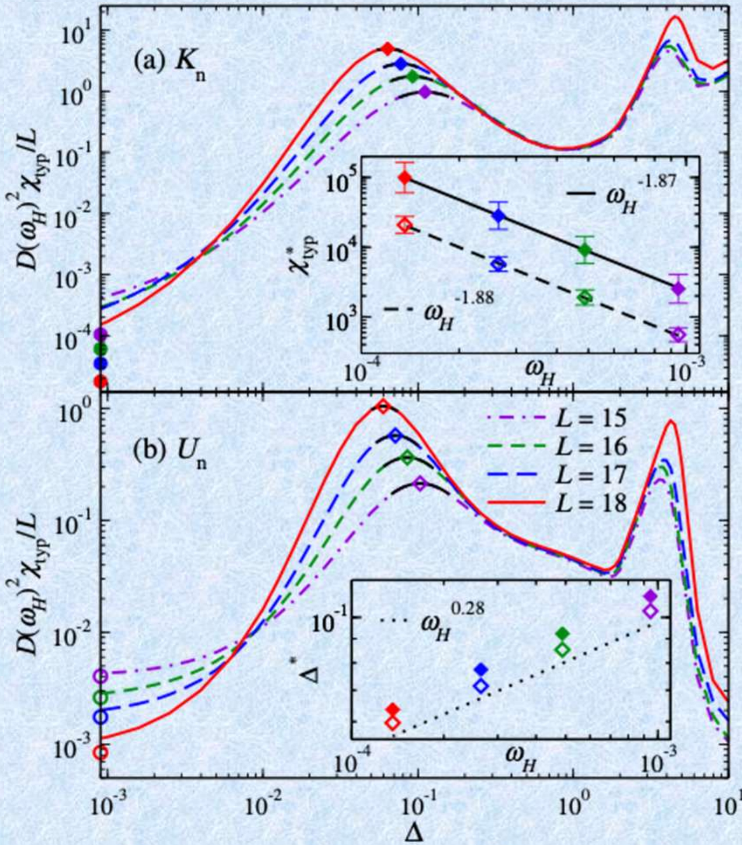
- Need exponentially small perturbation in L to induce ETH.

$$\chi^* \sim \frac{1}{\omega_H^2} \sim 4^L,$$

$$\Delta'^* \sim \omega_H^{0.39} \sim 2^{-0.39L}$$



Disordered Anderson model with interactions



$$\hat{H}_{\rm dsr} = \sum_{i=1}^{L} \left[\frac{J}{2} \left(\hat{S}_i^+ \hat{S}_{i-1}^- + H.c. \right) + h_i \hat{S}_i^z + \Delta \hat{S}_i^z \hat{S}_{i+1}^z \right],$$

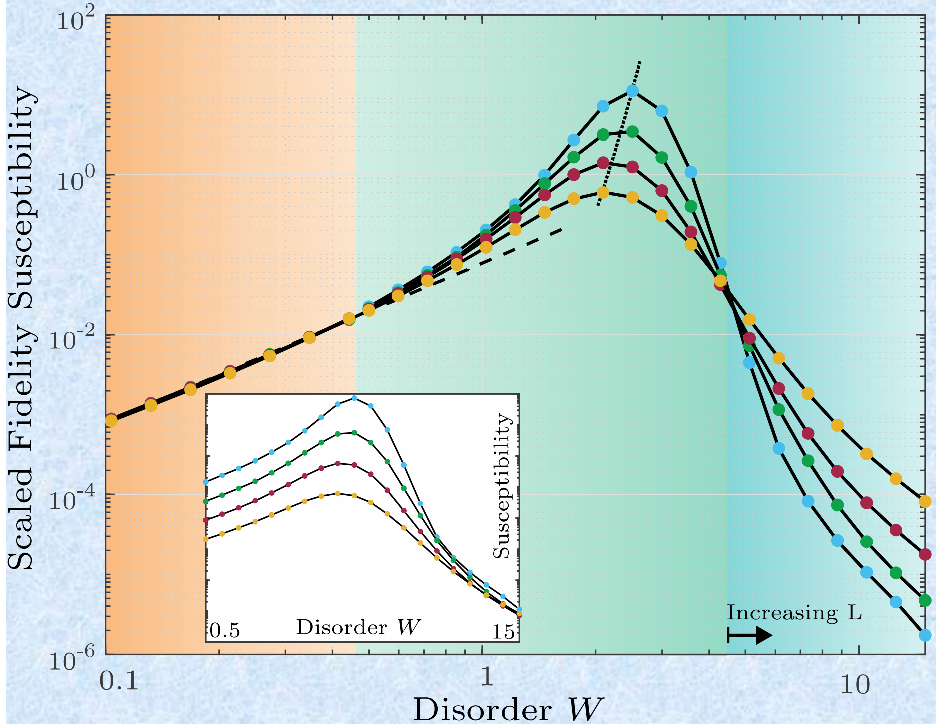
$$J = \sqrt{2},\ h_i \in [-0.81, 0.81],\ \Delta \in [10^{-3}, 10]$$

Very similar behavior of fidelity in clean and disordered models.

Strong indication of exponential (or large degree polynomial) scaling of the critical integrability breaking with the system size.

Strong disorder (D. Sels and A.P. 2020)

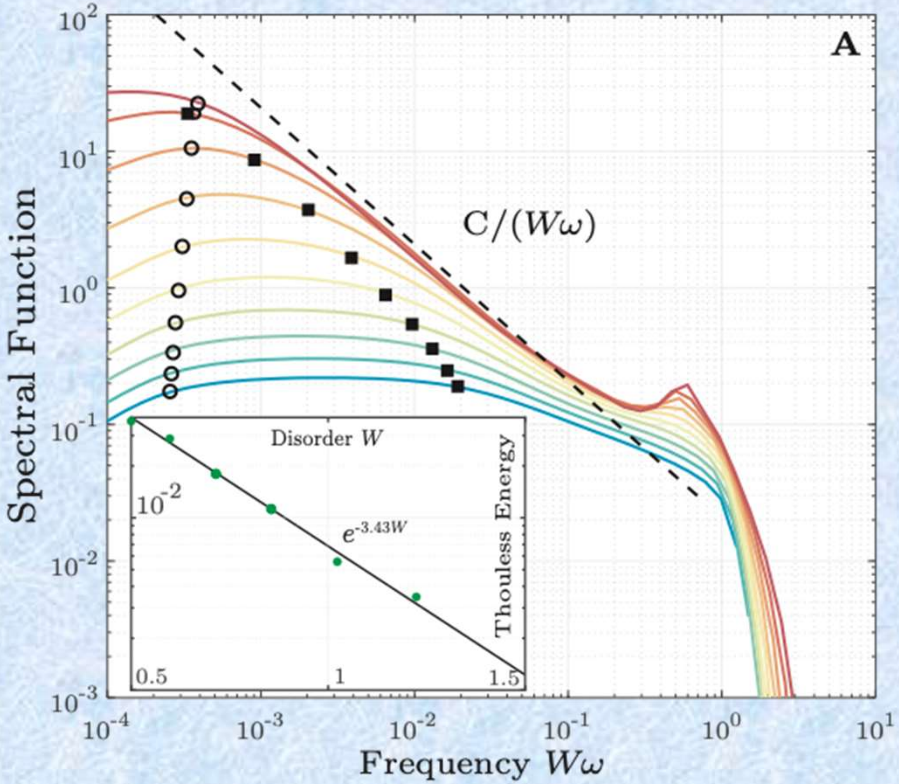
$$\hat{H} = \frac{1}{W}\sum_j (\hat{S}^x_j \hat{S}^x_{j+1} + \hat{S}^y_j \hat{S}^y_{j+1} + \Delta \hat{S}^z_j \hat{S}^z_{j+1}) + \sum_j h_j \hat{S}^z_j, \quad \Delta = 1.1,\ h_j \in [-1, 1]$$

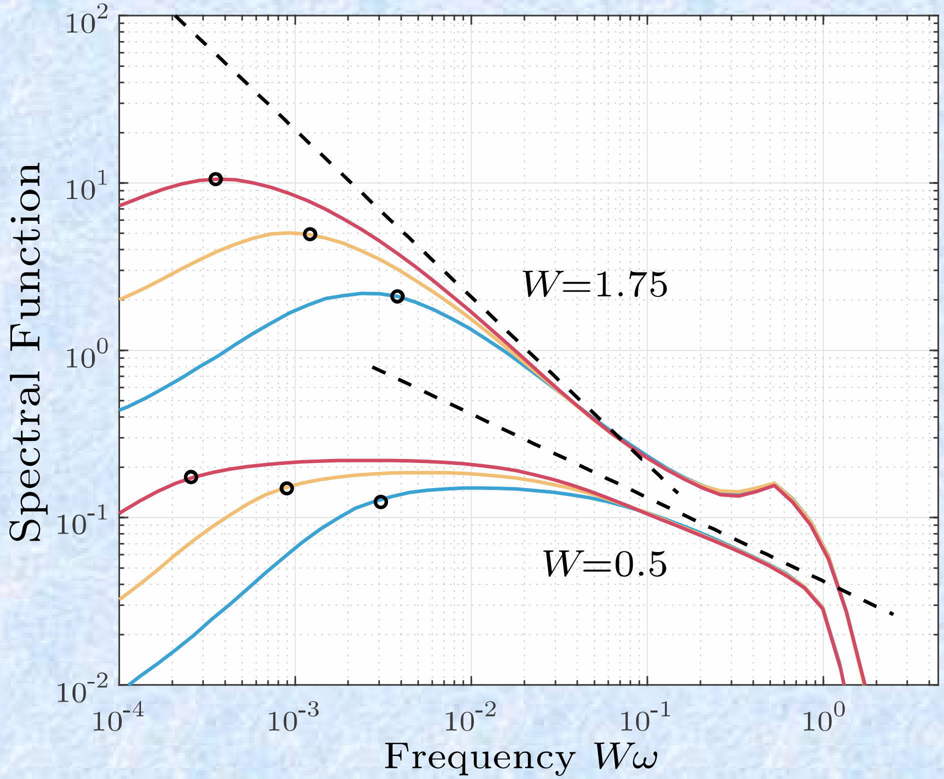


Linear drift of the maximum of fidelity with the system size.

Results are consistent with Suntajs et. al. 2019

Spectral function





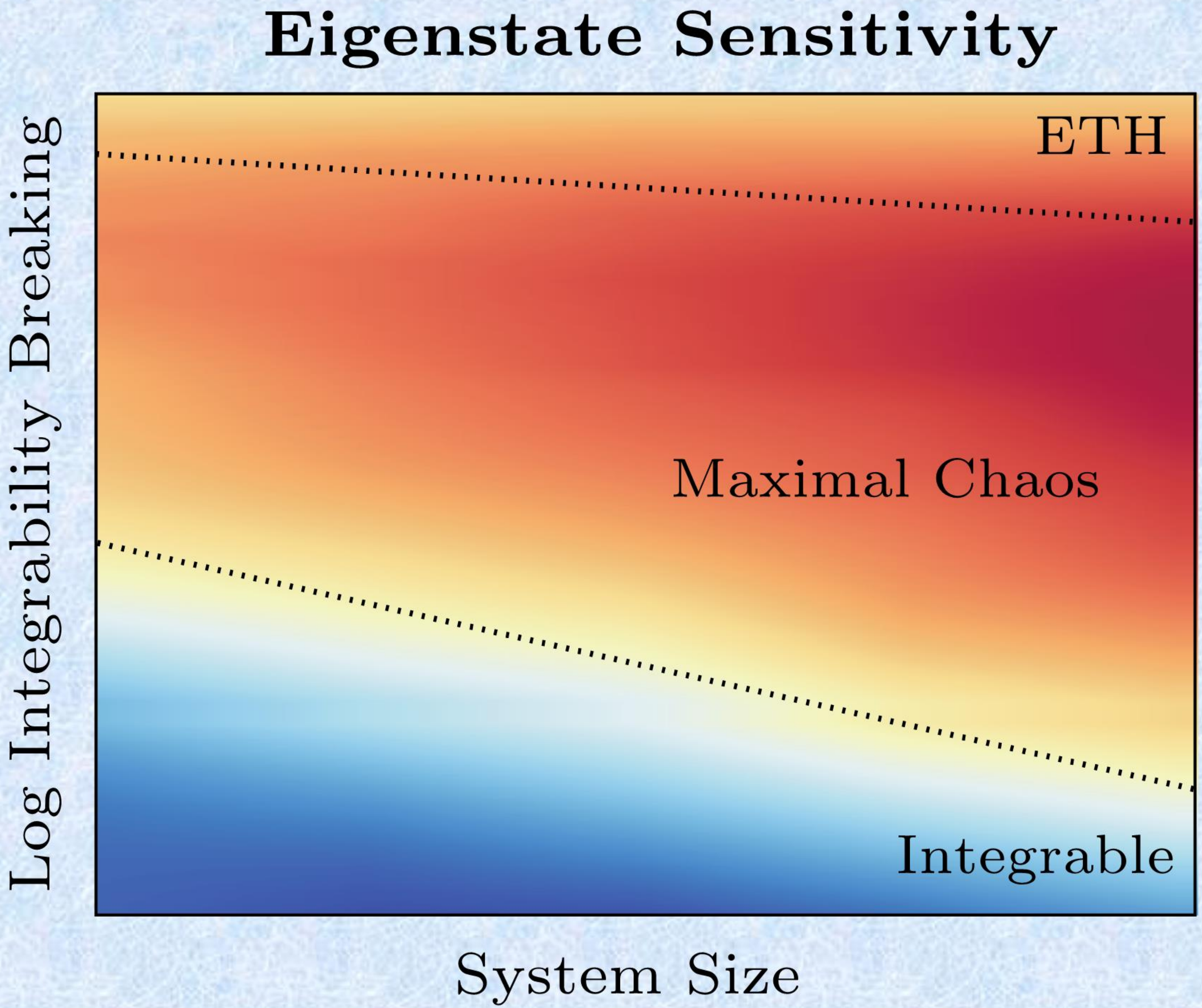
- Indication for ω^{-1}scaling ($1/f$ noise?). No variable subdiffusion exponent.

- Inconsistency of the sum rule with the transition in TD limit. Can be fixed with Log corrections, but
- leads to other inconsistencies.

$$\int_{-\infty}^{\infty} |f^2(\omega)| d\omega = \langle O^2 \rangle_c \sim 1$$

Existence of intermediate maximally chaotic regime is missed in RG treatments invalidating them.

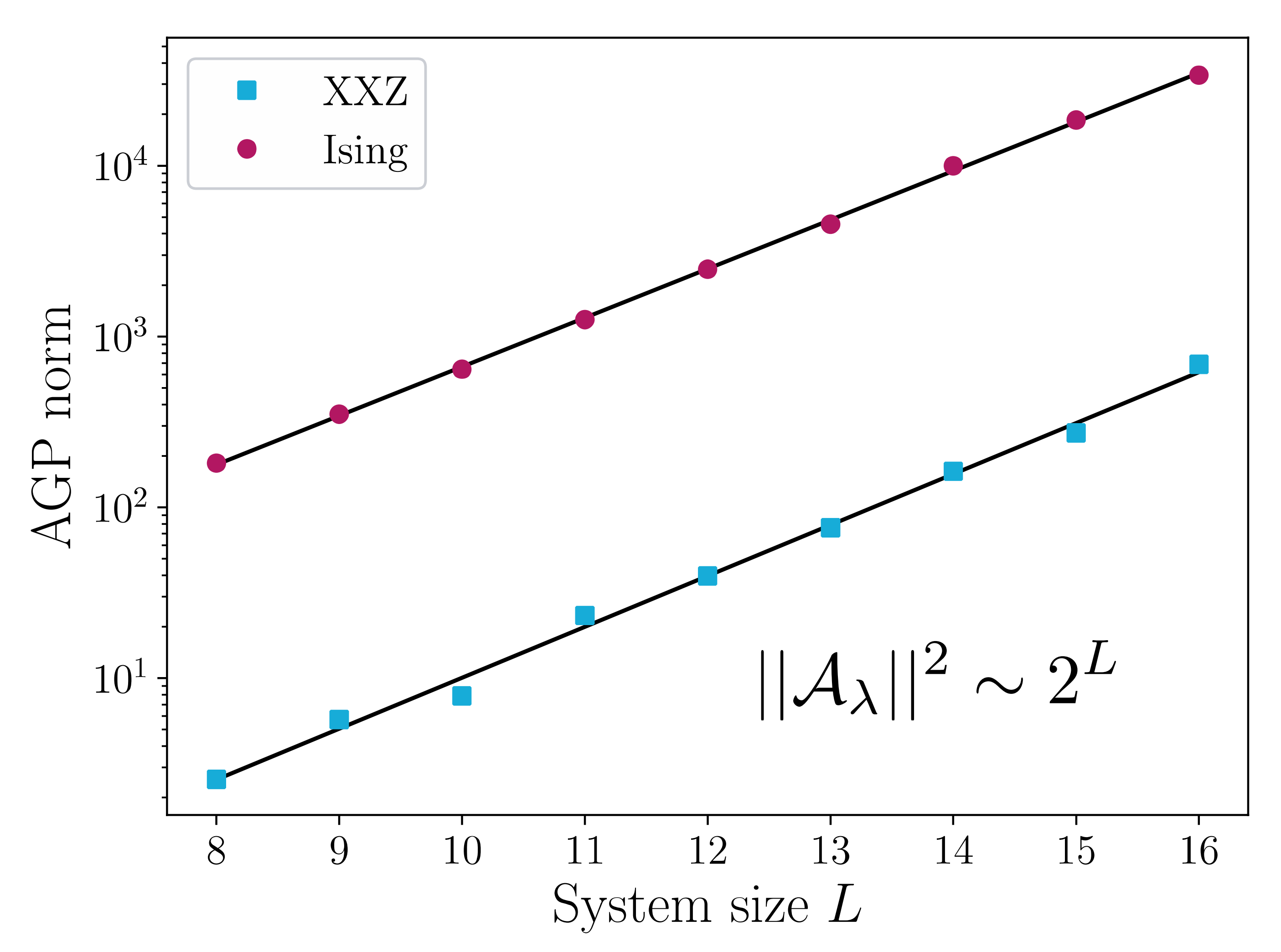
Qualitative chaos phase diagram for clean models



Conclusions

- AGP is a very (exponentially) sensitive probe of chaos. Much more sensitive than other measures. “Classical chaos” – exponential sensitivity of trajectories. “Quantum chaos” – exponential sensitivity of eigenstates.
- Exponentially long (in L) relaxation times for weak integrability perturbations for integrable observables
- Universal crossover from integrability to ergodicity through maximally chaotic (maximally sensitive) regime.
- Similar behavior of disordered and non-disordered systems.

Direction $\lambda = h_z$, parallel to the integrability-breaking perturbation. Look right at the integrable point



Exponential behavior from the onset. No threshold.