On Hamiltonian systems with critical Sobolev exponents

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Lisbon WADE

Sumário

Brezis and Nirenberg in [3]

$$-\Delta u = \lambda u^t + u^{2^*-1} \ \text{ in } \ \Omega, \quad u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \ (1.1)$$

• $\Omega \subset \mathbb{R}^N$, $N \geq 3$, bounded smooth domain,
• $1 \leq t < 2^* - 1$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$.

They found an interesting difference between the cases $N \geq 4$ and $N = 3$, the latter named as critical dimension.

For the particular case with $t = 1$

$$-\Delta u = \lambda u + u^{2^*-1} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (1.2)$$

$N \geq 4$ – Existence of solution for every $0 < \lambda < \lambda_1(\Omega)$, the optimal interval for existence.

$N = 3$ – There exists $0 < \lambda^* < \lambda_1(\Omega)$ such that no solution exists for $0 < \lambda < \lambda^*$.

Here $\lambda_1 = \lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

Critical hyperbola — Notion of critical growth for Hamiltonian systems were introduced by Mitidieri [13] and van der Vorst [15], and also appeared in Clément et al. [4].

In 1998 Hulshof et al. [8] considered the version of (1.2) in the framework of Hamiltonian systems

$$\begin{cases} -\Delta u = \lambda v + |v|^{p-1} v \text{ em } \Omega, \\ -\Delta v = \mu u + |u|^{q-1} u \text{ em } \Omega, \\ u, v = 0 \text{ sobre } \partial\Omega, \end{cases}$$

with $N \geq 4$ and (p, q) on the critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}. \tag{1.3}$$

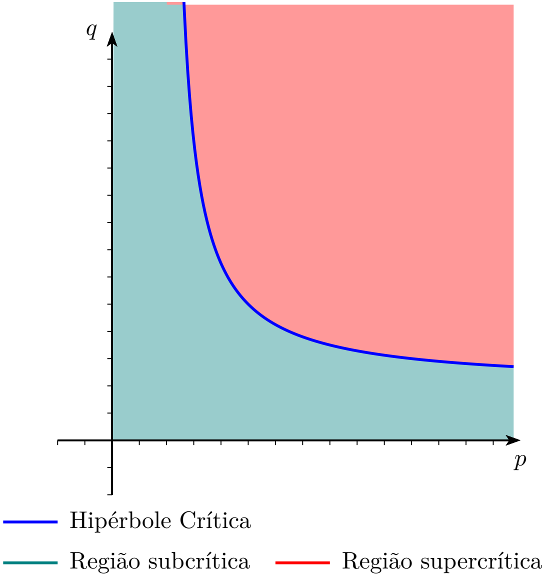

Figura: Critical Hyperbola

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a) What does happen in dimension $N = 3$?
b) What is the meaning of critical dimension for Hamiltonian systems ?
c) The investigation of the general 1-homogenous perturbation of the critical Lane-Emden system, namely (HS) ahead with $rs = 1$, which includes $r = s = 1$ as a particular case.

Regarding c):

The most accurate 1-homogenous perturbation to Hamiltonian systems, given below in (HS), is induced by the hyperbola of points (r, s) such that $rs = 1$ [12].

[12] Schiera-Nornberg-Tavares-MS, [7] Guimarães-MS

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I will be mainly focus on a work in coll. with Angelo Guimarães [7] where we address the three open problems mentioned before.

[12] Schiera-Nornberg-Tavares-MS, [7] Guimarães-MS

Consider the Hamiltonian system

$$\begin{cases} -\Delta u = \lambda |v|^{r-1} v + |v|^{p-1} v \text{ em } \Omega, \\ -\Delta v = \mu |u|^{s-1} u + |u|^{q-1} u \text{ em } \Omega, \\ u, v = 0 \text{ sobre } \partial\Omega, \end{cases} \tag{HS}$$

- $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded regular domain;
- $\lambda > 0$ e $\mu > 0$;
- (p, q) is on the critical hyperbola (1.3);
- (r, s) are such that

$$0 < r < p, \quad 0 < s < q, \quad rs \geq 1. \tag{1.4}$$

System (HS) is a lower-order perturbation of the critical Lane-Emden system

$$-\Delta u = |v|^{p-1}v, \quad -\Delta v = |u|^{q-1}u \quad \text{in} \quad \Omega, \quad u = v = 0 \text{ on } \partial\Omega,$$

as well as (1.1) is for the critical Lane-Emden equation

$$-\Delta u = u^{2^*-1}, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Moreover, condition (1.4) on (r, s) for (HS) corresponds to $1 \leq t < 2^* - 1$ for (1.1).

Our main result

Theorem 1.1.

Let $\lambda > 0$, $\mu > 0$, assume (1.4) and in the case of $rs = 1$ also suppose that $\lambda\mu^r$ is suitably small. If $N = 3$ and $p \leq 7/2$ or $p \geq 8$, or if $N \geq 4$, then (HS) has a positive solution.

[11] - Melo-dos Santos, [15] - Van der Vorst

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The precise condition on the size of $\lambda\mu^r$ (for $rs = 1$) is given ahead in (2.1) and (2.2). It corresponds to $\lambda < \lambda_1$ for (1.2) (see [11] and [15]).

Note that in case $N = 3$, $(\frac{7}{2}, 8)$ and $(8, \frac{7}{2})$ are symmetric points on the critical hyperbola (1.3).

[11] - Melo-dos Santos, [15] - Van der Vorst

In case $\lambda = \mu$, $r = s$, $p = q$, any solution of (HS) is such that $u = v$ (see [14]) ((HS) and (1.1) are equivalent in this case).

For $N = 3$, critical dimension for (1.1), we prove existence for (p, q) in some parts of the critical hyperbola (even if $r = s = 1$), which brings new results.

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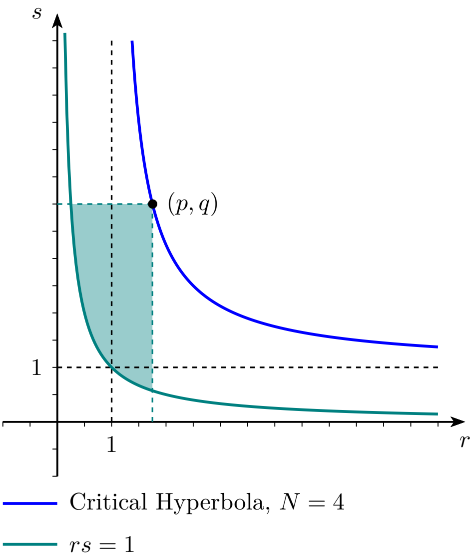
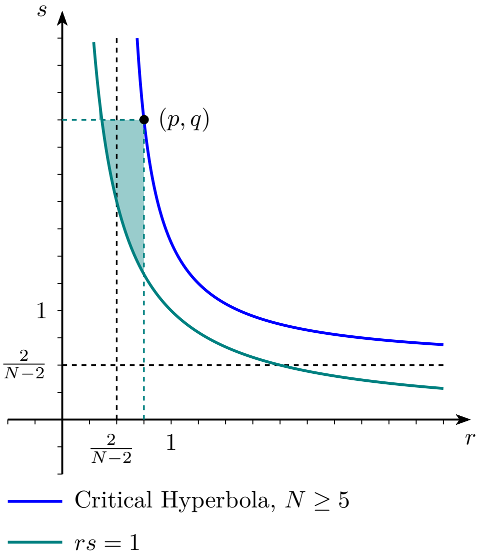
For $N \geq 4$, we cover all the points (p, q) on the critical hyperbola, including cases with $p < 1$ or $q < 1$.

[8] Hushof et al. , [14] Nornberg-Soave-MS


s
r
1
1
(p, q)
Critical Hyperbola, N = 4
rs = 1


s
(p, q)
1
2
N−2
r
Critical Hyperbola, N ≥ 5
rs = 1

For $N = 3$ it is not possible existence of solution to (1.1) in the full interval $1 \leq t < 5$. Indeed, as in [3, Corollary 2.3] and [2], such result only holds $3 < t < 5$. This leads:

Definition 1.2.

For $N = 3$, let (p, q) be a point on the critical hyperbola (1.3), Ω a regular bounded domain, and (r, s) as in (1.4). We say that (p, q) is on a *Critical region* if, for some Ω and some (r, s), (HS) does not have a positive solution for some λ and μ small. On the other hand, (p, q) is on a *Noncritical region* if for all Ω, all (r, s) as in (1.4), and λ, μ suitably small, (HS) has a positive solution.

[3] Brézis-Nirenberg 1983, [2] Brézis-Lieb 1985

Connection with critical dimension for the biharmonic operator

According to [16], $N = 5, 6, 7$ are the critical dimensions for

$$\Delta^2 u = \mu u + u^{\frac{N+4}{N-4}} \text{ em } \Omega, \quad u = \Delta u = 0 \text{ sobre } \partial\Omega, \qquad (1.5)$$

a particular case of (HS) with $\lambda = 0$ and $p = 1$.

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[16] Van der Vorst 1995, [5] Gazzola-Grunau 2001 and [6] Gazzola-Weth 2010

A first attempt to understand the phenomenon of critical dimension for Hamiltonian systems was presented in [11].

However, the asymmetric perturbation in [11] makes the problem more like a nonlinear version of the biharmonic equation (1.5), as the counterpart of the $p-$Laplacian version for (1.1).

In the case with $\lambda > 0$ and $\mu > 0$ in (HS), the natural symmetric perturbation of the critical Lane-Emden system, we recover that the only critical dimension is $N = 3$, as it happens to the scalar problem (1.1), unveiling the notions of critical and noncritical regions of the critical hyperbola for $N = 3$.

[11] - Melo-MS

Variational structure

To treat (HS), define

$$f_\lambda(t) = \lambda|t|^{r-1}t + |t|^{p-1}t, \quad \overline{F}_\lambda(t) = \int_0^t f_\lambda^{-1}(t)dt,$$
$$g_\mu(t) = \mu|t|^{s-1}t + |t|^{q-1}t, \quad \overline{G}_\mu(t) = \int_0^t g_\mu^{-1}(t)dt,$$

and rewrite (HS) as

$$\begin{cases} \Delta(f_\lambda^{-1}(\Delta u)) = \mu|u|^{s-1}u + |u|^{q-1}u \text{ in } \Omega, \\ u, \Delta u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

$$\begin{cases} \Delta(g_\mu^{-1}(\Delta v)) = \lambda|v|^{r-1}v + |v|^{p-1}v \text{ in } \Omega, \\ v, \Delta v = 0 \text{ on } \partial\Omega. \end{cases} \tag{P'}$$

Associated to (P) and (P'), consider the $C^1(E_p,\mathbb{R})$ and $C^1(E_q,\mathbb{R})$

$$I_F(u) = \int_\Omega \overline{F}_\lambda(\Delta u)dx - \frac{\mu}{s+1}\int_\Omega |u|^{s+1}dx - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx,$$

$$I_G(v) = \int_\Omega \overline{G}_\mu(\Delta v)dx - \frac{\lambda}{r+1}\int_\Omega |v|^{r+1}dx - \frac{1}{p+1}\int_\Omega |v|^{p+1}dx,$$

where $E_t := W^{2,\frac{t+1}{t}}(\Omega)\cap W_0^{1,\frac{t+1}{t}}(\Omega)$ with the norm $\|u\| = |\Delta u|_{\frac{t+1}{t}}$.

This approach is usually named as reduction by invertion.

Some extra difficulties: f_λ e g_μ are not pure power.

To capture the contribution of $\lambda|u|^{r-1}u$ to downsize the Mountain Pass level, we make some integral estimates on rings involving the "ground state" solutions of the critical Lane-Emden system on $\mathbb{R}^N$.

Mountain Pass geometry

We show next that I_F e I_G have the Mountain Pass Geometry.

For the case with $rs = 1$ we introduce the condition

$$\lambda^{1/r}\mu \leq \frac{(2|\Omega|)^{\frac{r-p}{r(p+1)}}}{2^{\frac{r+1}{r}}}\mathscr{C}_{r,\Omega}^{\frac{r+1}{r}}, \tag{2.1}$$

$$\lambda\mu^{1/s} \leq \frac{(2|\Omega|)^{\frac{s-q}{s(q+1)}}}{2^{\frac{s+1}{s}}}\mathscr{C}_{s,\Omega}^{\frac{s+1}{s}}, \tag{2.2}$$

where

$$\mathscr{C}_{r,\Omega} = \inf\{\|u\|;\ u \in E_p \text{ and } |u|_{\frac{r+1}{r}} = 1\},$$
$$\mathscr{C}_{s,\Omega} = \inf\{\|v\|;\ v \in E_q \text{ and } |v|_{\frac{s+1}{s}} = 1\}.$$

Remark

Conditions (2.1) and (2.2) for the case $rs = 1$ are natural and correspond to λ and μ in [8, Theorem 2] to treat (HS) with $r = s = 1$, and to the hypothesis $\lambda < \lambda_1$ in [3] to study (1.2).

Recall the spectral curve $\lambda^{1/r}\mu = constant$

Proposition 2.1.

Let (p, q) and (r, s) be as in (1.3) and (1.4).

- *Then I_F has the Mountain Pass geometry with a local minimum at zero, under the additional condition (2.1) when $rs = 1$.*
- *Then I_G has the Mountain Pass geometry with a local minimum at zero, under the additional condition (2.2) when $rs = 1$.*

$(\mathrm{PS})_c$ condition

Critical growth implies that the embedding $E_p \hookrightarrow L^{q+1}(\Omega)$ is not compact.

We localize the levels c to which the $(PS)_c$ condition holds.

Let S be the constant for the Sobolev embedding $E_p \hookrightarrow L^{q+1}(\Omega)$, i.e.

$$S = \inf_{u \in E_p, |u|_{q+1}=1} \|u\|.$$

Proposition 2.2.

I_F satisfies the $(PS)_c$ condition for all $c < \frac{2}{N} S^{\frac{pN}{2(p+1)}}$.

The Mountain Pass level

Proposition 2.3.

Suppose (1.3) *and* (1.4), $\mu > 0$, $\lambda > 0$ *and in the case with* $rs = 1$ *also suppose* (2.1). *If* $N \geq 4$ *and* $p \leq (N+2)/(N-2)$, *or* $N = 3$ *and* $p \leq 7/2$, *then the MP-level* c_F *of* I_F *is such that* $c_F \in (0, \frac{2}{N} S^{\frac{pN}{2(p+1)}})$.

This involves very delicate estimates regarding the asymptotic decay of the ground state solutions of the critical system on $\mathbb{R}^N$

Open problems and ongoing work

Open problems

a) To determine the critical region of the critical hyperbola (1.3) for $N = 3$.

b) A simpler problem is to find the optimal values $7/2 < p_* \leq p^* < 8$ s.t. (HS) has no solution for $p_* \leq p \leq p^*$ with $r = s = 1$, $\lambda = \mu$.

c) To investigate some "Remainder terms for second order Sobolev inequalities" – not only in the Hilbert case (see [5] Gazzola-Grunau and [6] Gazzola-Weth).

For question b), due to the results in [3, Theorem 1.2] and some estimates from Angelo's thesis, we know that

$$4 \leq p_* \leq 5 \leq p^* \leq 13/2.$$

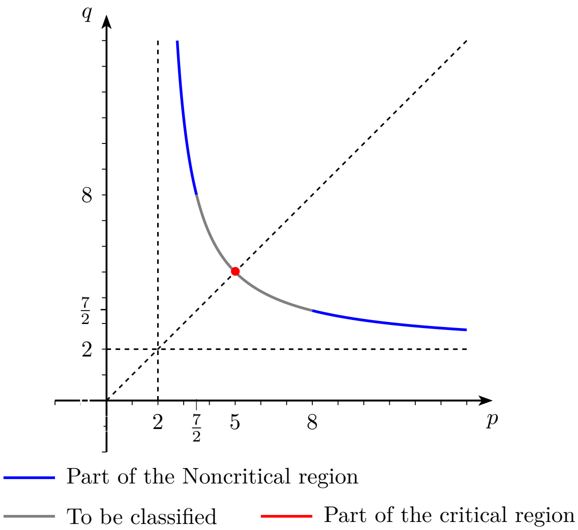

Figura: Critical hyperbola for $N = 3$

- In view of [9, Teorema 1.1], that consider $r = s = 1$, it could be interesting to study "blow up" for (HS), with $rs = 1$, as $\lambda = \mu \to 0$.

[9] Kim-Pistoia

Sobolev inequalities with remainder terms

Brezis-Lieb-1985

$$\|\nabla u\|_2^2 \geqslant S_n\|u\|_{2^*}^2 + C(\Omega)\|u\|_{p,w}^2, \quad \forall u \in H_0^1(\Omega)$$

with $p = 2^*/2 = N/(N-2)$ and

$$\|u\|_{p,w} = \sup_A |A|^{-1/p'} \int_A |u(x)|dx$$

Gazzola-Weth-2010

$$\|\Delta u\|_2^2 \geqslant S_n\|u\|_{2^*}^2 + C(\Omega)\|u\|_{p,w}^2 \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega)$$

with $p = 2_*/2 = N/(N-4)$

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Obrigado!