Cheeger constants and Partition problems

Matthias Hofmann

October 18, 2017

Google and the Network problem

Given a search query, how do you sort the results of the search.

Google and the Network problem

Given a search query, how do you sort the results of the search.

L. Page, S. Brin, R. Motwani, and T. Winograd.
The pagerank citation ranking: Bringing order to the web.
Technical report, 1999.

Google and the Network problem

Given a search query, how do you sort the results of the search.

L. Page, S. Brin, R. Motwani, and T. Winograd.
The pagerank citation ranking: Bringing order to the web.
Technical report, 1999.

A website will be associated to vertices v and we associate edges between those vertices when they are linked. General idea: How do we balance between inherent importance and importance between each other. Their answer: PageRank.

Google and the Network problem

Given a search query, how do you sort the results of the search.

L. Page, S. Brin, R. Motwani, and T. Winograd.
The pagerank citation ranking: Bringing order to the web.
Technical report, 1999.

A website will be associated to vertices v and we associate edges between those vertices when they are linked. General idea: How do we balance between inherent importance and importance between each other. Their answer: PageRank. Associate to each website a relevance R. Given $\vec{b}$, they introduced a PageRank associated to an equation

$$\vec{R} = A\vec{R} + \vec{b}.$$

An Application: Network of Thrones

An Application: Network of Thrones

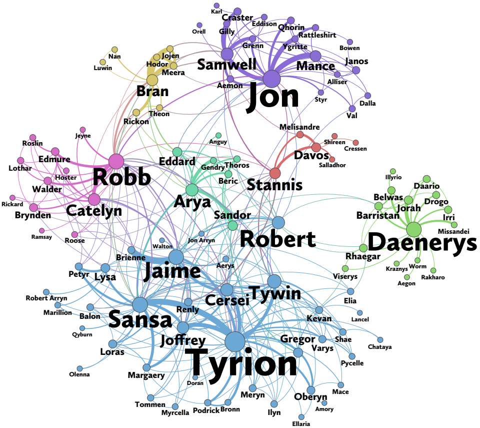
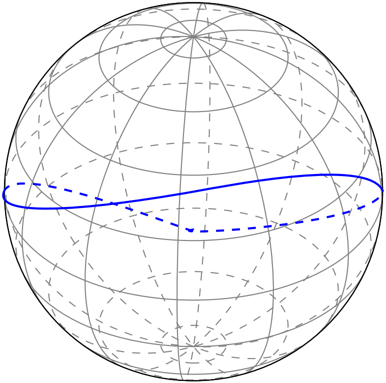


Figure: The social network generated from A Storm of Swords. Source: www.maa.org/mathhorizons : : Math Horizons : : April 2016.

A partition problem on the Sphere $\mathbb{S}^2$

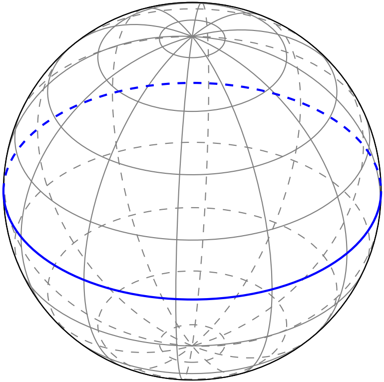
A partition problem on the Sphere $\mathbb{S}^2$

How do we divide a sphere into two open subsets of equal volume with minimal boundary length.



A partition problem on the Sphere $\mathbb{S}^2$

How do we divide a sphere into two open subsets of equal volume with minimal boundary length.



Is the hemisphere the optimal choice?

Isoperimetric inequalities

Isoperimetric inequalities

The isoperimetric inequality in the Euclidean space $\mathbb{R}^n$ says, that among subsets S with given volume the ball B with this volume has minimal boundary measure:

$$|\partial S| \geq |\partial B|, \qquad \forall S : |S| = |B|.$$

Restating this result a little differently

$$\frac{|\partial S|}{|S|} \geq \frac{|\partial B|}{|B|}, \qquad |S| \leq |B|.$$

Isoperimetric inequalities

The isoperimetric inequality in the Euclidean space $\mathbb{R}^n$ says, that among subsets S with given volume the ball B with this volume has minimal boundary measure:

$$|\partial S| \geq |\partial B|, \qquad \forall S : |S| = |B|.$$

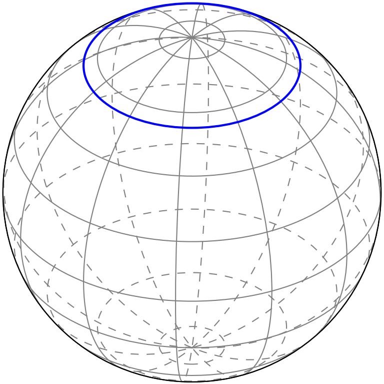
Restating this result a little differently

$$\frac{|\partial S|}{|S|} \geq \frac{|\partial B|}{|B|}, \qquad |S| \leq |B|.$$

Isoperimetric inequalities on the sphere

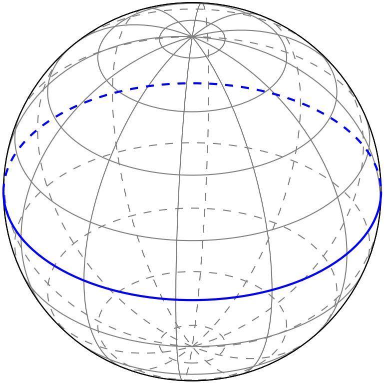
Isoperimetric inequalities on the sphere

One can generalize this result also for the sphere. Among all open subsets on the sphere $S \subset \mathbb{S}^2$ with fixed area the circular cap C with such area has smallest perimeter.



Isoperimetric inequalities on the sphere

In conclusion, as we increase the size of the cap, the quotient $\frac{|\partial S|}{|S|}$ becomes smaller until we reach the equator.



Cheeger constant

Cheeger constant

We define the Cheeger constant of a compact n-dimensional Riemannian manifold M^n to be

Cheeger constant

We define the Cheeger constant of a compact n-dimensional Riemannian manifold M^n to be

- $\partial M^n = \emptyset$:

$$h(M^n) = \inf \frac{|\partial A^n|}{|A^n|}$$

where A^n are open subsets with $|A^n| \leq \frac{1}{2}|M^n|$.

Cheeger constant

We define the Cheeger constant of a compact n-dimensional Riemannian manifold M^n to be

- $\partial M^n = \emptyset$:

$$h(M^n) = \inf \frac{|\partial A^n|}{|A^n|}$$

  where A^n are open subsets with $|A^n| \leq \frac{1}{2}|M^n|$.

- $\partial M^n \neq \emptyset$:

$$h(M^n) = \inf \frac{|\partial A^n|}{|A^n|},$$

  s.t. $\partial A^n \cap \partial M^n = \emptyset$.

Cheeger constant

We define the Cheeger constant of a compact n-dimensional Riemannian manifold M^n to be

- $\partial M^n = \emptyset$:

$$h(M^n) = \inf \frac{|\partial A^n|}{|A^n|}$$

  where A^n are open subsets with $|A^n| \leq \frac{1}{2}|M^n|$.

- $\partial M^n \neq \emptyset$:

$$h(M^n) = \inf \frac{|\partial A^n|}{|A^n|},$$

  s.t. $\partial A^n \cap \partial M^n = \emptyset$.

We call subsets minimizing the volume quotient Cheeger sets.

An eigenvalue problem

An eigenvalue problem

We define the Laplace-Beltrami operator

$$-\Delta = -\operatorname{div}\left(\nabla \,\cdot\,\right).$$

Consider the eigenvalue problem on $M = M^n$:

$$\begin{cases} -\Delta u = \lambda u, & \text{on } M \\ u = 0, & \text{on } \partial M, \end{cases}$$

where zero boundary conditions are only imposed if $\partial M \neq \emptyset$.

An eigenvalue problem

We define the Laplace-Beltrami operator

$$-\Delta = -\operatorname{div}\left(\nabla \,\cdot\,\right).$$

Consider the eigenvalue problem on $M = M^n$:

$$\begin{cases} -\Delta u = \lambda u, & \text{on } M \\ u = 0, & \text{on } \partial M, \end{cases}$$

where zero boundary conditions are only imposed if $\partial M \neq \emptyset$. The eigenvalues (counting multiplicity) can be then arranged as a nondecreasing sequence

$$0 \leq \lambda_1(M) \leq \lambda_2(M) \leq \dots$$

Cheeger inequality

Cheeger inequality

The following inequality is referred to as the Cheeger inequality.

Theorem (Cheeger 1970)

For all eigenvalues $\lambda > 0$ to the Laplace-Beltrami operator

$$\lambda(M) \geq \frac{1}{4} h^2(M).$$

Cheeger inequality

The following inequality is referred to as the Cheeger inequality.

Theorem (Cheeger 1970)

For all eigenvalues $\lambda > 0$ to the Laplace-Beltrami operator

$$\lambda(M) \geq \frac{1}{4} h^2(M).$$

Remark

If $\partial M = \emptyset$ then constant functions are always eigenfunctions to the Laplace-Beltrami operator $-\Delta$ and thus $\lambda_1(M) = 0$.

Cheeger inequalities for p-Laplacian

Cheeger inequalities for p-Laplacian

Consider the eigenvalue problem on an open, bounded, connected domain $\Omega \subset \mathbb{R}^n$ with $p \geq 1$

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, & \text{on } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Cheeger inequalities for p-Laplacian

Consider the eigenvalue problem on an open, bounded, connected domain $\Omega \subset \mathbb{R}^n$ with $p \geq 1$

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, & \text{on } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

One can show that there exists a unique positive eigenfunction to this eigenvalue problem.

Cheeger inequalities for p-Laplacian

Consider the eigenvalue problem on an open, bounded, connected domain $\Omega \subset \mathbb{R}^n$ with $p \geq 1$

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, & \text{on } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

One can show that there exists a unique positive eigenfunction to this eigenvalue problem. We call the associated eigenvalue $\lambda_p(\Omega)$ the first eigenvalue of the p-Laplacian.

Cheeger inequalites for p-Laplacian (cont.)

For $p > 1$ one can characterize $\lambda_p(\Omega)$ by the variational principle

$$\lambda_p(\Omega) = \inf_{0 \neq v \in C_c^1(\Omega)} \frac{\int_\Omega |\nabla v|^p \,\mathrm{d}x}{\int_\Omega |v|^p \,\mathrm{d}x}.$$

Cheeger inequalites for p-Laplacian (cont.)

For $p > 1$ one can characterize $\lambda_p(\Omega)$ by the variational principle

$$\lambda_p(\Omega) = \inf_{0 \neq v \in C_c^1(\Omega)} \frac{\int_\Omega |\nabla v|^p \, \mathrm{d}x}{\int_\Omega |v|^p \, \mathrm{d}x}.$$

Then one may extend the Cheeger inequality for the p-Laplacian:

Theorem

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p}\right)^p, \qquad \lambda_p(\Omega) \to h(\Omega) \qquad (p \to 1+).$$

In particular we can relate $\lambda_1(\Omega) := h(\Omega)$.

Proof of Theorem

Proof of Theorem

Suppose $w \in C_c^1(\Omega)$ and set $A(t) := \{x \in \Omega |\, w(x) > t\}$. By the coarea formula

$$\int_\Omega |\nabla w| \,\mathrm{d}x = \int_{-\infty}^{\infty} |\partial A(t)| \,\mathrm{d}t$$

Proof of Theorem

Suppose $w \in C_c^1(\Omega)$ and set $A(t) := \{x \in \Omega |\, w(x) > t\}$. By the coarea formula

$$\int_\Omega |\nabla w| \,\mathrm{d}x = \int_{-\infty}^{\infty} |\partial A(t)| \,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{|\partial A(t)|}{|A(t)|} |A(t)| \,\mathrm{d}t$$

Proof of Theorem

Suppose $w \in C_c^1(\Omega)$ and set $A(t) := \{x \in \Omega |\, w(x) > t\}$. By the coarea formula

$$\begin{aligned} \int_\Omega |\nabla w| \,\mathrm{d}x = \int_{-\infty}^{\infty} |\partial A(t)| \,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{|\partial A(t)|}{|A(t)|} |A(t)| \,\mathrm{d}t \\ \geq h(\Omega) \int_{-\infty}^{\infty} |A(t)| \,\mathrm{d}t = h(\Omega) \int_\Omega |w| \,\mathrm{d}x. \end{aligned} \tag{1}$$

For $p > 1$ take any $v \in C_c^1(\Omega)$ and define $\Phi(v) = |v|^{p-1} v \in C_c^1(\Omega)$.

Proof of Theorem

Suppose $w \in C_c^1(\Omega)$ and set $A(t) := \{x \in \Omega | \, w(x) > t\}$. By the coarea formula

$$\begin{aligned} \int_\Omega |\nabla w| \,\mathrm{d}x &= \int_{-\infty}^{\infty} |\partial A(t)| \,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{|\partial A(t)|}{|A(t)|} |A(t)| \,\mathrm{d}t \\ &\geq h(\Omega) \int_{-\infty}^{\infty} |A(t)| \,\mathrm{d}t = h(\Omega) \int_\Omega |w| \,\mathrm{d}x. \end{aligned} \tag{1}$$

For $p > 1$ take any $v \in C_c^1(\Omega)$ and define $\Phi(v) = |v|^{p-1} v \in C_c^1(\Omega)$. Indeed, Hölder's inequality implies

$$\begin{aligned} \int_\Omega |\nabla \Phi(v)| \,\mathrm{d}x &= p \int_\Omega |v|^{p-1} |\nabla v| \,\mathrm{d}x \\ &\leq p \left(\int_\Omega |v|^p \,\mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \end{aligned} \tag{2}$$

Proof of Theorem (cont.)

Then with (1) we infer

$$\int_\Omega |\nabla \Phi(v)| \,\mathrm{d}x \geq h(\Omega) \int_\omega |v|^p \,\mathrm{d}x.$$

Proof of Theorem (cont.)

Then with (1) we infer

$$\int_\Omega |\nabla \Phi(v)| \,\mathrm{d}x \geq h(\Omega) \int_\omega |v|^p \,\mathrm{d}x.$$

Using (2) we obtain

$$h(\Omega) \leq \frac{\int_\Omega |\nabla \Phi(v)| \,\mathrm{d}x}{\int_\Omega |\Phi(v)| \,\mathrm{d}x} \leq \frac{p \left(\int_\Omega |v|^p \,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v|^p \,\mathrm{d}x\right)^{\frac{1}{p}}}{\int_\Omega |v|^p \,\mathrm{d}x}$$

Proof of Theorem (cont.)

Then with (1) we infer

$$\int_\Omega |\nabla \Phi(v)| \, \mathrm{d}x \geq h(\Omega) \int_\omega |v|^p \, \mathrm{d}x.$$

Using (2) we obtain

$$h(\Omega) \leq \frac{\int_\Omega |\nabla \Phi(v)| \, \mathrm{d}x}{\int_\Omega |\Phi(v)| \, \mathrm{d}x} \leq p \left(\frac{\int_\Omega |\nabla v|^p \, \mathrm{d}x}{\int_\Omega |v|^p \, \mathrm{d}x} \right)^{\frac{1}{p}}$$

Since $v \in C_c^1(\Omega)$ is arbitrary, we conclude the inequality. This concludes the proof.

Buser inequality

Buser inequality

Let $p = 2$ and $n \in \mathbb{N}$. Buser proved for $n = 2$ in 1979:

Theorem

If the curvatures of n-dimensional compact surfaces A_m^n without boundary are uniformly bounded from below, then $\lambda_p(A_m^n) \to 0$ iff $h(A_m^n) \to 0$ as $m \to \infty$.

Buser inequality

Let $p = 2$ and $n \in \mathbb{N}$. Buser proved for $n = 2$ in 1979:

Theorem

If the curvatures of n-dimensional compact surfaces A_m^n without boundary are uniformly bounded from below, then $\lambda_p(A_m^n) \to 0$ iff $h(A_m^n) \to 0$ as $m \to \infty$.

Generalizations of this theorem for general $n \in \mathbb{N}$ and non-compact surfaces can be found in

P. Buser.
A note on the isoperimetric constant.
In *Annales scientifiques de l'École Normale Supérieure*, volume 15, pages 213–230, 1982.

Preliminaries: Combinatorial Graphs

Preliminaries: Combinatorial Graphs

Let $G = (V, E)$ be an undirected, finite graph with a set of vertices V and a set of edges E.

Preliminaries: Combinatorial Graphs

Let $G = (V, E)$ be an undirected, finite graph with a set of vertices V and a set of edges E. Some notations:

Preliminaries: Combinatorial Graphs

Let $G = (V, E)$ be an undirected, finite graph with a set of vertices V and a set of edges E. Some notations:

- $u \sim v \overset{\text{def}}{\iff} u$ and v are vertices connected by an edge.

Preliminaries: Combinatorial Graphs

Let $G = (V, E)$ be an undirected, finite graph with a set of vertices V and a set of edges E. Some notations:

- $u \sim v \overset{\text{def}}{\iff} u$ and v are vertices connected by an edge.
- The degree of a vertex $d(v)$ is defined as the number of edges that have v as an endpoint, and we define the volume of a subset of vertices $S \subset V$ as

$$|S| = \sum_{v \in S} d(v)$$

Preliminaries: Combinatorial Graphs

Let $G = (V, E)$ be an undirected, finite graph with a set of vertices V and a set of edges E. Some notations:

- $u \sim v \overset{\text{def}}{\iff} u$ and v are vertices connected by an edge.
- The degree of a vertex $d(v)$ is defined as the number of edges that have v as an endpoint, and we define the volume of a subset of vertices $S \subset V$ as
$$|S| = \sum_{v \in S} d(v)$$
- We define the set $E(S)$ of a subset of vertices $S \subset V$ as the subset of edges, that connect S with its complement $S^c = V \setminus S$ and we write $|E(S)| := \text{card}(E(S))$.

Normalized Discrete Laplacian

Normalized Discrete Laplacian

Introduce the discretized Rayleigh quotient

$$\mathcal{R}_G(f) = \frac{\sum_{u \sim v} |f(u) - f(v)|^2}{\sum_{v \in V} d(v) f(v)^2}.$$

Normalized Discrete Laplacian

Introduce the discretized Rayleigh quotient

$$\mathcal{R}_G(f) = \frac{\sum_{u\sim v} |f(u) - f(v)|^2}{\sum_{v\in V} d(v)f(v)^2}.$$

Consider the variational problem

$$\lambda_1 = \min_{\substack{f_1\in\mathbb{R}^{|V|} \\ f\neq 0}} \mathcal{R}_G(f)$$

Then we can associate similarly as before an operator, the so called normalized Laplacian $\mathcal{L}_G$ to the variational problem.

Higher order eigenvalues and Minmax principle

That is, $\lambda_1 = 0$ is the smallest eigenvalue for an associated eigenvalue problem

$$\mathcal{L}_G f = \lambda f$$

with constant functions as eigenfunctions.

Higher order eigenvalues and Minmax principle

That is, $\lambda_1 = 0$ is the smallest eigenvalue for an associated eigenvalue problem

$$\mathcal{L}_G f = \lambda f$$

with constant functions as eigenfunctions. The other eigenvalues can be characterized in terms of a minmax problem

$$\lambda_k = \min_{f_1,\dots,f_k \in \mathbb{R}^{|V|}} \max_{f \neq 0} \{\mathcal{R}_G(f) : f \in \operatorname{span}\{f_1, \dots, f_k\}\},$$

where the minimum is over sets of k linearly independent functions

Cheeger constant on graphs

Cheeger constant on graphs

Given a subset $S \subset V$, we denote the conductance of S by

$$\phi_G(S) := \frac{|E(S)|}{|S|}.$$

Cheeger constant on graphs

Given a subset $S \subset V$, we denote the conductance of S by

$$\phi_G(S) := \frac{|E(S)|}{|S|}.$$

Then we define the Cheeger constant

$$h := \min_{S:|S|\leq \frac{|G|}{2}} \frac{E(S)}{|S|} = \min_S \max\left\{\frac{|E(S)|}{|S|}, \frac{|E(S^c)|}{|S^c|}\right\}$$

where the minimum is over subsets of vertices $S \subset V$.

Higher-order Cheeger constants

Higher-order Cheeger constants

We define higher Cheeger constants

$$h_k(G) = \min_{S_1,\dots,S_k} \max_i \phi_G(S_i),$$

where the minimium is over a k-partition of k nonempty vertices sets.

Higher-order Cheeger constants

We define higher Cheeger constants

$$h_k(G) = \min_{S_1,\dots,S_k} \max_i \phi_G(S_i),$$

where the minimium is over a k-partition of k nonempty vertices sets.

Theorem (Higher-order Cheeger inequalities)

For any finite, connected graph G and every $k \in \mathbb{N}$ we have

$$\frac{\lambda_k}{2} \leq h_k(G) \leq Ck^4\sqrt{\lambda_k}$$

Proof of the Buser inequality $\lambda_k \leq 2h_k(G)$

Proof of the Buser inequality $\lambda_k \leq 2h_k(G)$

Recall the minmax principle

$$\lambda_k = \min_{f_1,\dots,f_k \in \mathbb{R}^{|V|}} \max_{f \neq 0} \{\mathcal{R}_G(f) : f \in \operatorname{span}\{f_1, \dots, f_k\}\},$$

where the minimum is over sets of k non-zero linearly independent vectors in $\mathbb{R}^{|V|}$ and

$$\mathcal{R}_G(f) = \frac{\sum_{u \sim v} |f(u) - f(v)|^2}{\sum_{v \in V} d(v) f(v)^2}.$$

Proof of the Buser inequality $\lambda_k \leq 2h_k(G)$

Recall the minmax principle

$$\lambda_k = \min_{f_1,\ldots,f_k \in \mathbb{R}^{|V|}} \max_{f \neq 0} \{\mathcal{R}_G(f) : f \in \operatorname{span}\{f_1, \ldots, f_k\}\},$$

where the minimum is over sets of k non-zero linearly independent vectors in $\mathbb{R}^{|V|}$ and

$$\mathcal{R}_G(f) = \frac{\sum_{u \sim v} |f(u) - f(v)|^2}{\sum_{v \in V} d(v) f(v)^2}.$$

Given $S_1, \cdots, S_k$ we define

$$f_i(v) = \begin{cases} 1, & v \in S_i \\ 0, & \text{else.} \end{cases}$$

Proof of Buser inequality (cont.)

Proof of Buser inequality (cont.)

Then

$$\lambda_k \leq \max_i \frac{\sum_{u\sim v} |f_i(u) - f_i(v)|^2}{\sum_{v\in V} d(v) f_i(v)^2}$$

Proof of Buser inequality (cont.)

Then

$$\lambda_k \leq \max_i \frac{\sum_{u \sim v} |f_i(u) - f_i(v)|^2}{\sum_{v \in V} d(v) f_i(v)^2} \leq 2 \max_i \frac{|E(S_i)|}{|S_i|}.$$

Proof of Buser inequality (cont.)

Then

$$\begin{aligned}\lambda_k &\leq \max_i \frac{\sum_{u\sim v} |f_i(u) - f_i(v)|^2}{\sum_{v\in V} d(v) f_i(v)^2} \\ &\leq 2 \max_i \frac{|E(S_i)|}{|S_i|}.\end{aligned}$$

In particular

$$\lambda_k \leq 2 \min_{S_1,\ldots,S_i} \max_i \frac{|E(S_i)|}{|S_i|} = 2h_k(G).$$

Proof of Buser inequality (cont.)

Then

$$\lambda_k \leq \max_i \frac{\sum_{u\sim v} |f_i(u) - f_i(v)|^2}{\sum_{v\in V} d(v) f_i(v)^2}$$
$$\leq 2 \max_i \frac{|E(S_i)|}{|S_i|}.$$

In particular

$$\lambda_k \leq 2 \min_{S_1,\ldots,S_i} \max_i \frac{|E(S_i)|}{|S_i|} = 2h_k(G).$$

For the other inequality we refer to

J. R. Lee, S. O. Gharan, and L. Trevisan.
Multiway spectral partitioning and higher-order Cheeger inequalities.
Journal of the ACM (JACM), 61(6):37, 2014.

Higher-order Cheeger inequalities for manifolds

Higher-order Cheeger inequalities for manifolds

Define the Cheeger-constant (connectivity spectrum) through

$$h_n := \min_{(A_1,\ldots,A_n)\in D_n} \max_k \frac{|\partial A_K|}{|A_k|},$$

where D_n is the set of n-tuples of disjoint and open subsets with smooth boundary.

Higher-order Cheeger inequalities for manifolds

Define the Cheeger-constant (connectivity spectrum) through

$$h_n := \min_{(A_1,\ldots,A_n)\in D_n} \max_k \frac{|\partial A_K|}{|A_k|},$$

where D_n is the set of n-tuples of disjoint and open subsets with smooth boundary.

Theorem (Miclo 2013)

There exists a universal constant $\eta > 0$ such that for any compact Riemannian manifold S, we have

$$\lambda_n \geq \frac{\eta}{n^6} h_n^2.$$

Recent extensions

Recent extensions

- Cheeger inequalities for magnetic Laplacian (on graphs and manifolds)

C. Lange, S. Liu, N. Peyerimhoff, and O. Post.
Frustration index and Cheeger inequalities for discrete and continuous magnetic laplacians.
Calculus of Variations and Partial Differential Equations, 54(4):4165–4196, 2015.

Recent extensions (cont.)

- Cheeger inequalities on Quantum Graphs

J. B. Kennedy and D. Mugnolo.
The Cheeger constant of a quantum graph.
PAMM, 16(1):875–876, 2016.

- For p-Laplacians (on combinatorial graphs)

F. Tudisco and M. Hein.
A nodal domain theorem and a higher-order Cheeger inequality for the graph p-laplacian.
arXiv preprint arXiv:1602.05567, 2016.

Thank you very much for your attention.

Source: www.hbo.com ; ©HBO 2017