Dynamical implications of convexity beyond dynamical convexity

Leonardo Macarini
(ongoing joint work with Miguel Abreu)

Basic setup

- $(\mathbb{R}^{2n+2}, \omega)$, $\omega = \sum_i dq_i \wedge dp_i = d\lambda$ where $\lambda = \frac{1}{2}\sum_i (q_i dp_i - p_i dq_i)$.

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- A contact form on S^{2n+1} supporting ξ is a 1-form α given by $f\lambda|_{S^{2n+1}}$ for some positive function $f : S^{2n+1} \to \mathbb{R}$. Its Reeb vector field is the unique vector field R_α s.t. $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$.

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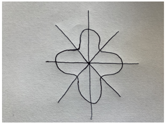
- We want to study the dynamics of Reeb flows on the standard contact sphere (S^{2n+1}, ξ).

- There is a bijection between contact forms α on (S^{2n+1}, ξ) and starshaped hypersurfaces Σ_α in $\mathbb{R}^{2n+2}$:

$$\alpha = f\lambda|_{S^{2n+1}} \longleftrightarrow \Sigma_\alpha = \{\sqrt{f(x)}x;\ x \in S^{2n+1}\}.$$



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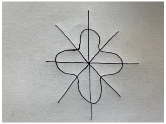

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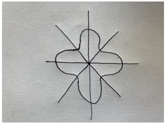

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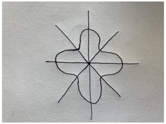

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- The Hamiltonian flow on a regular energy level of H is equivalent to the Reeb flow of α.
- Therefore, the study of Reeb flows on (S^{2n+1},ξ) is equivalent to the study of Hamiltonian flows of proper homogeneous of degree two Hamiltonians on $\mathbb{R}^{2n+2}$.

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- Let $\mathcal{P}_e \subset \mathcal{P}_{nh} \subset \mathcal{P}$ denote the set of simple elliptic and non-hyperbolic orbits.

General problem:

Study the **multiplicity** and **stability** of periodic orbits of α. More precisely, try to get lower bounds for $\#\mathcal{P}$, $\#\mathcal{P}_{nh}$ and $\#\mathcal{P}_e$.

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- Note that irrational ellipsoids in $\mathbb{R}^{2n+2}$ carry precisely $n+1$ periodic orbits. Moreover, all these orbits are elliptic. (An irrational ellipsoid is given by $\sum_i r_i \|z_i\|^2 = 1$ with $r_0, \ldots, r_n$ rationally indepedent.)

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- No general lower bound for $\#\mathcal{P}_e$ or $\#\mathcal{P}_{nh}$ is known.

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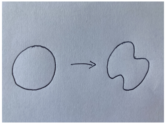
- The hypothesis of convexity is used in several ways.

Dynamical convexity

- The hypothesis of convexity is not natural from the point of view of Contact Topology since it is not a condition invariant by contactomorphisms.



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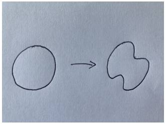

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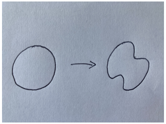

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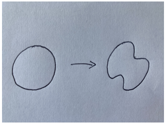

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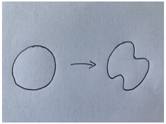

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- Clearly, dynamical convexity is a condition invariant by contactomorphisms.

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- Abreu-M.'2017: $\#\mathcal{P}_e \geq 1$ for any n.
- Ginzburg-M.'2019: $\#\mathcal{P} \geq n+1$ for any n if α is **strongly dynamically convex**.

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Theorem. (Ginzburg-M.'2019)

Given $n \geq 2$ there exists a contact form on S^{2n+1} that is DC but it is not equivalent to a strictly convex contact form via a contactomorphism that commutes with the antipodal map.

- This gives a partial answer to the following important question: Are there examples of DC contact forms that are not contactomorphic to convex ones? This is part of the general question on how to understand convexity from the symplectic point of view.

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Goal of this talk:

Show new dynamical implications of convexity that do not follow from dynamical convexity. In this way, we will furnish new examples of DC contact forms that are not equivalent to (strictly or not) convex ones via contactomorphisms preserving the symmetry. Moreover, we will also establish the multiplicity of symmetric non-hyperbolic closed Reeb orbits without assuming that $\#\mathcal{P} < \infty$ and the existence of symmetric elliptic orbits.

Lens spaces

- Given an integer $p \geq 1$, consider the $\mathbb{Z}_p$-action on S^{2n+1}, regarded as a subset of $\mathbb{C}^{n+1} \setminus \{0\}$, generated by the map

$$\psi(z_0, \ldots, z_n) = \left(e^{\frac{2\pi i \ell_0}{p}} z_0, e^{\frac{2\pi i \ell_1}{p}} z_1, \ldots, e^{\frac{2\pi i \ell_n}{p}} z_n \right),$$

where $\ell_0, \ldots, \ell_n$ are integers called the weights of the action. Such an action is free when the weights are coprime with p and in that case we have a lens space obtained as the quotient of S^{2n+1} by the action of $\mathbb{Z}_p$. We denote this lens space by $L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$.

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- We consider on $L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$ the induced contact structure ξ. We say that a contact form on this lens space is (strictly) convex if so is its lift to S^{2n+1}.

Equivariant symplectic homology

- The positive equivariant symplectic homology $\mathrm{ESH}_*(L_p^{2n+1}(\ell_0,\dots,\ell_n))$ is an invariant of the contact structure ξ that can be obtained as the homology of a chain complex generated the periodic orbits of the Reeb flow graded by the CZ index. It has a filtration given by the homotopy classes of $L_p^{2n+1}(\ell_0,\dots,\ell_n)$.

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- It allows us to give a fractional grading to ESH_*.
- Although a fractional grading may seem unnatural at first (since the differential decreases the degree by 1) it can be thought of as a way of keeping track the filtration of ESH_* in the homotopy classes. Indeed, given two homotopic orbits γ_1, γ_2 we have that $\mu_{CZ}(\gamma_1) - \mu_{CZ}(\gamma_2) \in \mathbb{Z}$.

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- Given $a \in \pi_1(L_p^{2n+1}(\ell_0, \ldots, \ell_n))$, let $j_a \in \{1, \ldots, p\}$ be such that ψ^{j_a} is the deck transformation corresponding to a. Let $\ell_0^a, \ell_1^a, \ldots, \ell_n^a$ be the *homotopy weights* given by the (unique) integers such that

$$\psi^{j_a}(z_0, \ldots, z_n) = \left(e^{\frac{2\pi i \ell_0^a}{p}} z_0, e^{\frac{2\pi i \ell_1^a}{p}} z_1, \ldots, e^{\frac{2\pi i \ell_n^a}{p}} z_n\right)$$

satisfying $-p/2 < \ell_i^a \leq p/2$ for every i, $\ell_0^a = j_a$ if $j_a \leq p/2$, $\ell_0^a = j_a - p$ if $j_a > p/2$.

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- **Example**: Let a be a non-trivial homotopy class of $L_p^{2n+1}(1, \dots, 1)$. It is easy to see that $k_a = \frac{2j_a(n+1)}{p} - n$. In particular, $k_a \neq k_b$ whenever $a \neq b$.

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- If $j_a > p/2$ then $h_a = \tilde{h}_a = k_a - 1 = \frac{2j_a(n+1)}{p} - (n+1)$.

Theorem 1. (Abreu-M.'2020)

Let α be a convex (resp. strictly convex) contact form on a lens space $L_p^{2n+1}(\ell_0, \dots, \ell_n)$ and γ a closed Reeb orbit of α with non-trivial homotopy class a. Then the following assertions hold:

1. $\mu_{CZ}(\gamma) \geq k_a$;
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- When γ is contractible, $\mu_{CZ}(\gamma) \geq k_0 = n + 2$ is precisely DC.
- This result is sharp: we must have an orbit γ such that $\mu_{CZ}(\gamma) = k_a$ and we have convex examples with an hyperbolic orbit γ such that $\mu_{CZ}(\gamma) = h_a$ and with a non-elliptic orbit γ s.t. $\mu_{CZ}(\gamma) = k_a + 1$ and whose homotopy class a satisfies $\ell_i^a > 0$ and $\ell_i^a \neq p/2$.

- In particular, consider a strictly convex contact form α on $\mathbb{RP}^{2n+1}$. Then every closed Reeb orbit γ of α satisfying $\mu_{CZ}(\gamma) < n+1$ is non-hyperbolic (if γ is contractible then $\mu_{CZ}(\gamma) \geq n+2$).

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- When $n=1$ this result readily follows from the dynamical convexity of α (i.e. $\mu_{CZ}(\gamma) \geq n+2$ for every contractible orbit γ). Indeed, if γ is hyperbolic then $\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma)$ $\forall k$ (in any dimension). Thus, if γ is hyperbolic and $\mu_{CZ}(\gamma) < 2$ then $\mu_{CZ}(\gamma^2) < 3$ (on $\mathbb{RP}^3$, $\mu_{CZ}(\gamma) \in \mathbb{Z}$), contradicting the dynamical convexity. However, in higher dimensions it does not follow from DC:

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Theorem 2. (Abreu-M.'2020)

Given $n \geq 4$ there exists a dynamically convex contact form on $\mathbb{RP}^{2n+1}$ with a hyperbolic closed Reeb orbit γ satisfying $\mu_{CZ}(\gamma) = n+1-2$.

- The previous thm shows that the hypothesis of strict convexity cannot be relaxed to DC in the second assertion of Thm 1. It turns out that the assumption that α is convex in Thm 1 cannot be relaxed to the condition that α is DC at all:

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Theorem 3. (Abreu-M.'2020)

The following assertions hold:

1. Consider integers $n \geq 1$ and $p \geq 3$. Then there exists a DC contact form α on $L_p^{2n+1}(1, \dots, 1)$ carrying a closed Reeb orbit with non-trivial homotopy class a such that $\mu_{CZ}(\gamma) < k_a$.
2. There exists a DC contact form α on $L_3^{17}(1, \dots, 1)$ and a hyperbolic closed Reeb orbit γ of α with non-trivial homotopy class a such that $\mu_{CZ}(\gamma) < h_a$.
3. There exists a DC contact form α on $L_9^5(1, 1, 1)$ and a hyperbolic closed Reeb orbit γ of α with non-trivial homotopy class a such that $\ell_i^a > 0$ ($\ell_i^a \neq p/2$ since p is odd) for every i and $\mu_{CZ}(\gamma) = k_a$.

- Note that, by invariance of ESH, if $\varphi : L_p^{2n+1}(\ell_0, \dots, \ell_n) \hookleftarrow$ is a contactomorphism then $k_a = k_{\varphi_* a}$. In particular, if $k_a \neq k_b$ whenever $a \neq b$ then φ acts trivially on π_1. Hence, in this case, all the properties stated in Thm 1 are invariant by φ ($h_a = h_{\varphi_* a}$ and $\tilde{h}_a = \tilde{h}_{\varphi_* a}$). Therefore, since this property holds for $L_p^{2n+1}(1, \dots, 1)$, Thm 3 furnish new examples of DC contact forms on spheres that are not equivalent to convex ones via contactormophisms that commute with the symmetry. Actually, we have something better.

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Theorem 4. (Abreu-M.'2020)

Let α be one of the contact forms furnished by Thm 3 and consider its lift β to S^{2n+1}. Let $S \subset Cont(S^{2n+1})$ be the subset of contactomorphisms that commute with the corresponding $\mathbb{Z}_p$-action. Then there exists a C^1-neighborhood U of S such that β is not equivalent to a convex contact form via any $\varphi \in U$.

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- A periodic orbit γ of β is symmetric if $\psi(\gamma(\mathbb{R})) = \gamma(\mathbb{R})$.
- Note that the simple symmetric periodic orbits of β are in bijection with the simple closed orbits of α whose homotopy classes are generators of $\pi_1(L_p^{2n+1}(\ell_0,\dots,\ell_n))$.

- Given real numbers $0 < r \leq R$ we say that α is (r, R)-pinched if $R^{-2}\|v\| \leq d^2H_\beta(x)(v,v) \leq r^{-2}\|v\|$ for every $x \in \Sigma_\beta$ and $v \in \mathbb{R}^{2n+2}$.

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Theorem 5. (Abreu-M.'2020)

Let $n \geq 1$ and $p \geq 2$ be integers and $0 < r \leq R$ be real numbers such that $\frac{R}{r} < \sqrt{p+1}$. Given an (r,R)-pinched contact form α on $L_p^{2n+1}(1,\dots,1)$ we have that α carries at least $\lfloor \frac{n+1}{2} \rfloor$ simple non-hyperbolic closed Reeb orbits with homotopy class a such that a is a generator of $\pi_1(L_p^{2n+1}(\ell_0,\dots,\ell_n))$.

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- Liu obtained related results when $p = 2$.

Theorem 6. (Abreu-M.'2020)

Let α be a convex (resp. strictly convex) contact form on $L_p^{2n+1}(\ell_0,\dots,\ell_n)$. Assume that $\ell_i > 0$ and $\ell_i \neq p/2$ (resp. $\ell_i > 0$) for every i. Then α carries at least one elliptic closed orbit whose homotopy class is a generator of $\pi_1(L_p^{2n+1}(\ell_0,\dots,\ell_n))$.

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- When α is strictly convex it follows from a previous result due to Arnaud.

Very brief idea of the proof of Theorem 1

- Let α be convex (resp. strictly convex). Given a closed orbit γ of α with non-trivial homotopy class a we have its Bott function $\mathcal{B}_\gamma : S^1 \to \mathbb{Q}$.

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- If α is convex and $p \geq 2$ we can show that $\mathcal{B}_\gamma(1) \geq k_a$.
- Moreover, $\exists z \in S^1$ s.t. $\mathcal{B}_\gamma(z) \geq h_a$ (resp. $\mathcal{B}_\gamma(z) \geq \tilde{h}_a$).

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- We have that $\mu_{CZ}(\gamma^k) = \sum_{z^k=1} \mathcal{B}_\gamma(z)$ (Bott's formula).
- If α is convex and $p \geq 2$ we can show that $\mathcal{B}_\gamma(1) \geq k_a$.
- Moreover, $\exists z \in S^1$ s.t. $\mathcal{B}_\gamma(z) \geq h_a$ (resp. $\mathcal{B}_\gamma(z) \geq \tilde{h}_a$).
- Hence, if $\mathcal{B}_\gamma(1) = \mu_{CZ}(\gamma) < h_a$ (resp. $\mu_{CZ}(\gamma) < \tilde{h}_a$) then γ is non-hyperbolic.

Very brief idea of the proof of Theorem 1

- Let α be convex (resp. strictly convex). Given a closed orbit γ of α with non-trivial homotopy class a we have its Bott function $\mathcal{B}_\gamma : S^1 \to \mathbb{Q}$.
- This function is continuous except possibly at the eigenvalues of the linearized Poincaré map with modulus one.
- In particular, $\mathcal{B}_\gamma$ is constant whenever γ is hyperbolic.
- We have that $\mu_{CZ}(\gamma^k) = \sum_{z^k=1} \mathcal{B}_\gamma(z)$ (Bott's formula).
- If α is convex and $p \geq 2$ we can show that $\mathcal{B}_\gamma(1) \geq k_a$.
- Moreover, $\exists z \in S^1$ s.t. $\mathcal{B}_\gamma(z) \geq h_a$ (resp. $\mathcal{B}_\gamma(z) \geq \tilde{h}_a$).
- Hence, if $\mathcal{B}_\gamma(1) = \mu_{CZ}(\gamma) < h_a$ (resp. $\mu_{CZ}(\gamma) < \tilde{h}_a$) then γ is non-hyperbolic.
- Under the assumptions of item (3) of Thm 1, we can show that there exists $z \in S^1$ such that $\mathcal{B}_\gamma(z) \geq k_a + n$. If $\mathcal{B}_\gamma(1) = \mu_{CZ}(\gamma) = k_a$ this implies that γ is elliptic.