Infinite Staircases in Symplectic Embeddings

Joint Work with Dusa McDuff, Ana Rita Pires, and Morgan Weiler

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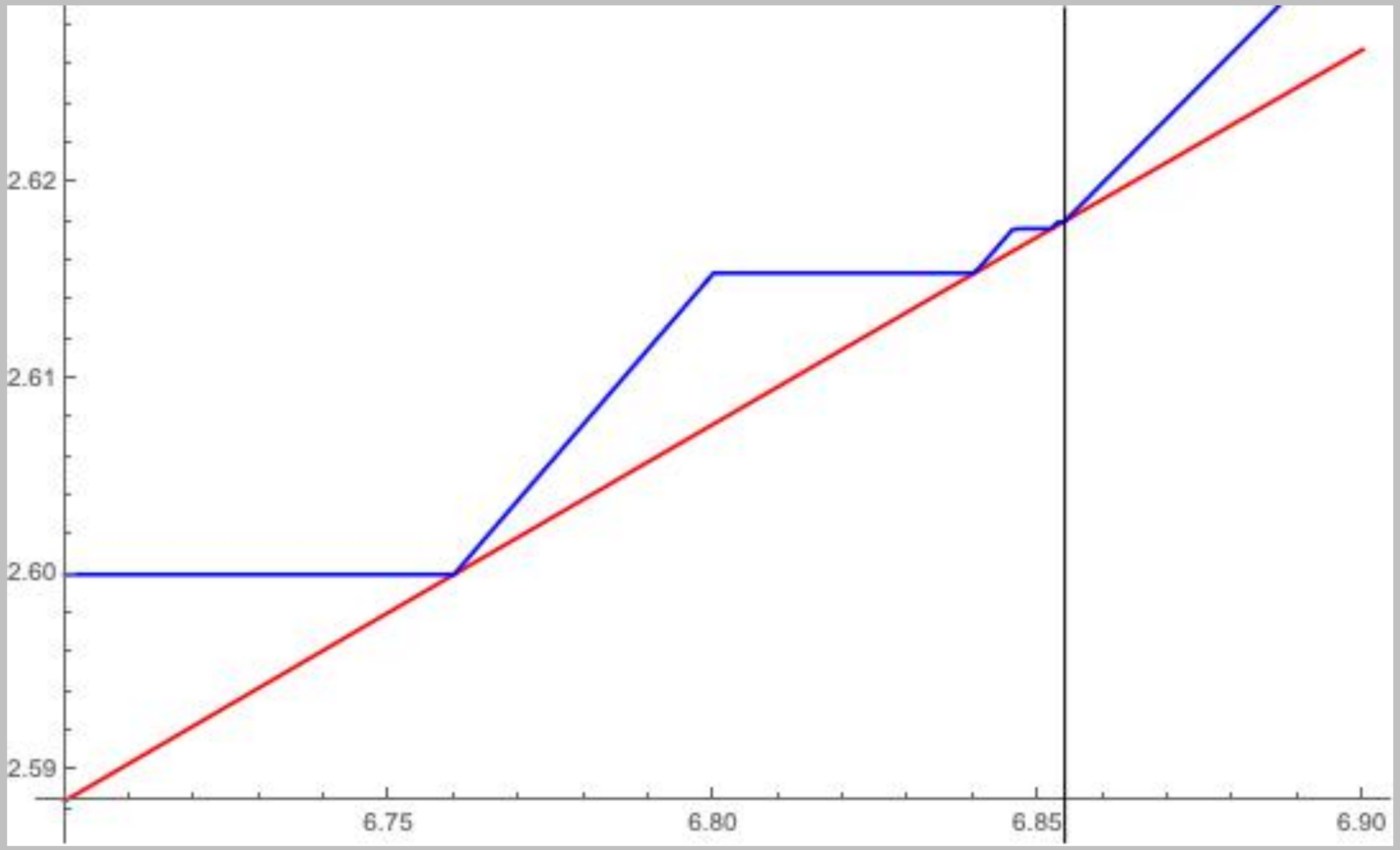
Summary of Main Result

We give a complete classification to which Hirzebruch surfaces have a property called an infinite staircase.



Symplectic Embeddings

A **symplectic manifold** is an even dimensional manifold with a non degenerate closed 2-form ω.

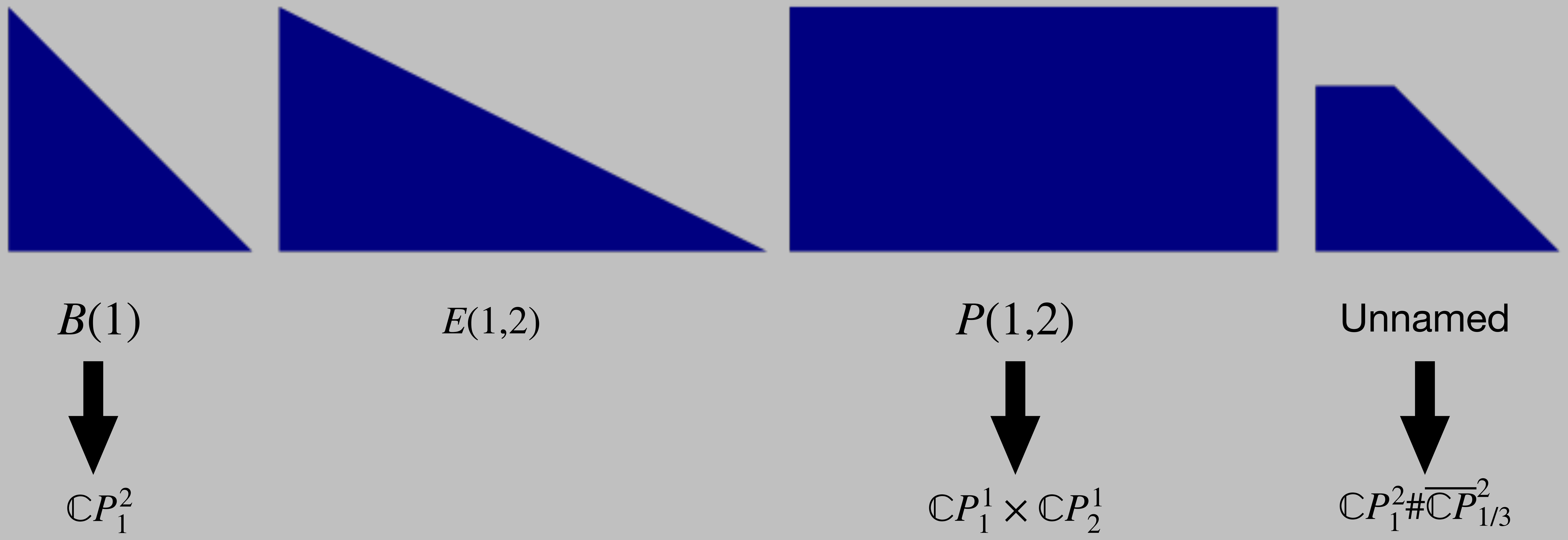
$(\mathbb{R}^{2n}, \sum_i dx_i \wedge dy_i)$ is a symplectic manifold.

A **symplectic embedding** $\varphi : (X_1, \omega_1) \overset{s}{\hookrightarrow} (X_2, \omega_2)$ is a smooth embedding such that $\varphi^*(\omega_2) = \omega_1$.

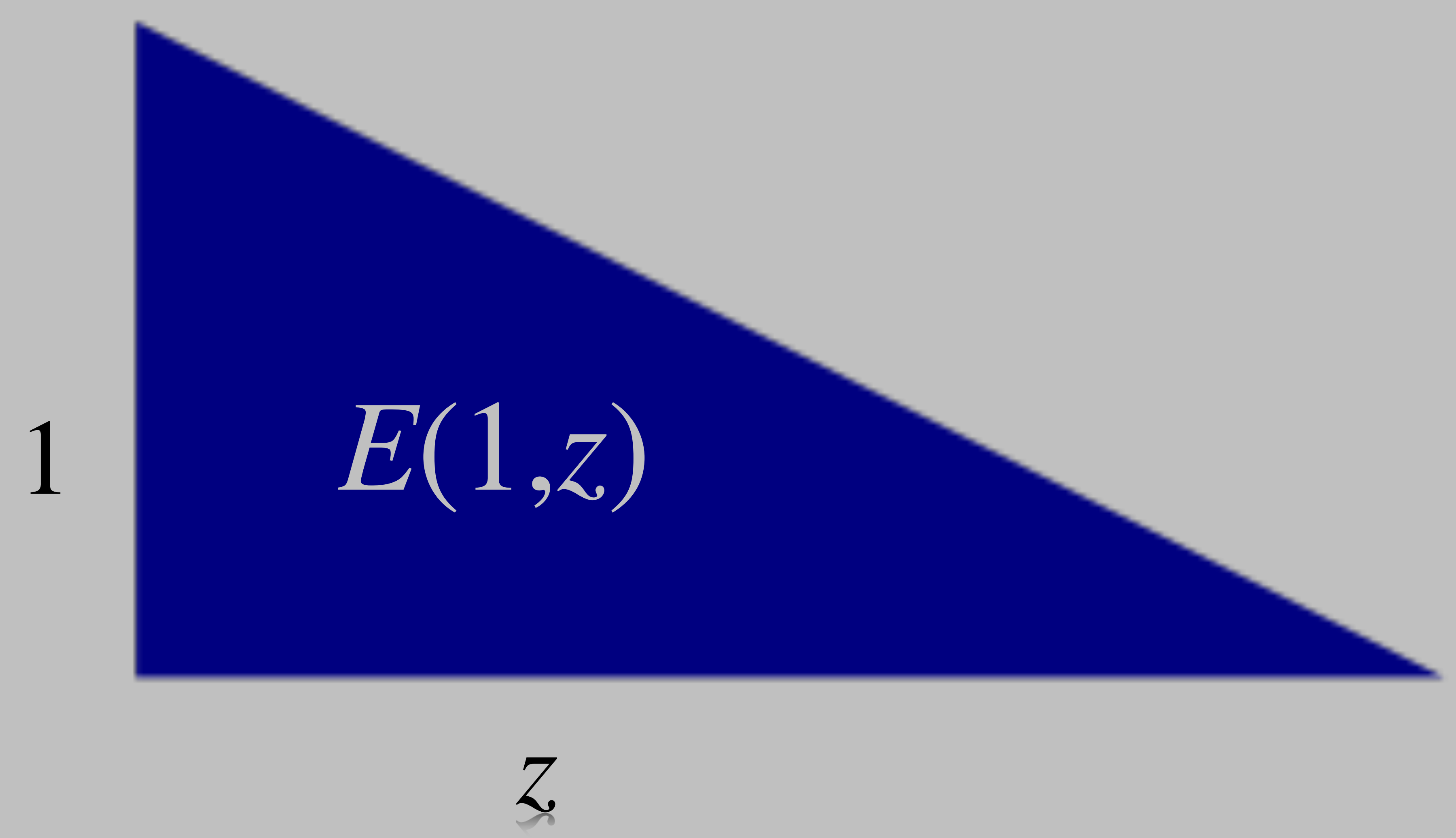
If $\varphi : (X_1, \omega_1) \overset{s}{\hookrightarrow} (X_2, \omega_2)$, then $vol(X_1) \leq vol(X_2)$.

Toric Domains

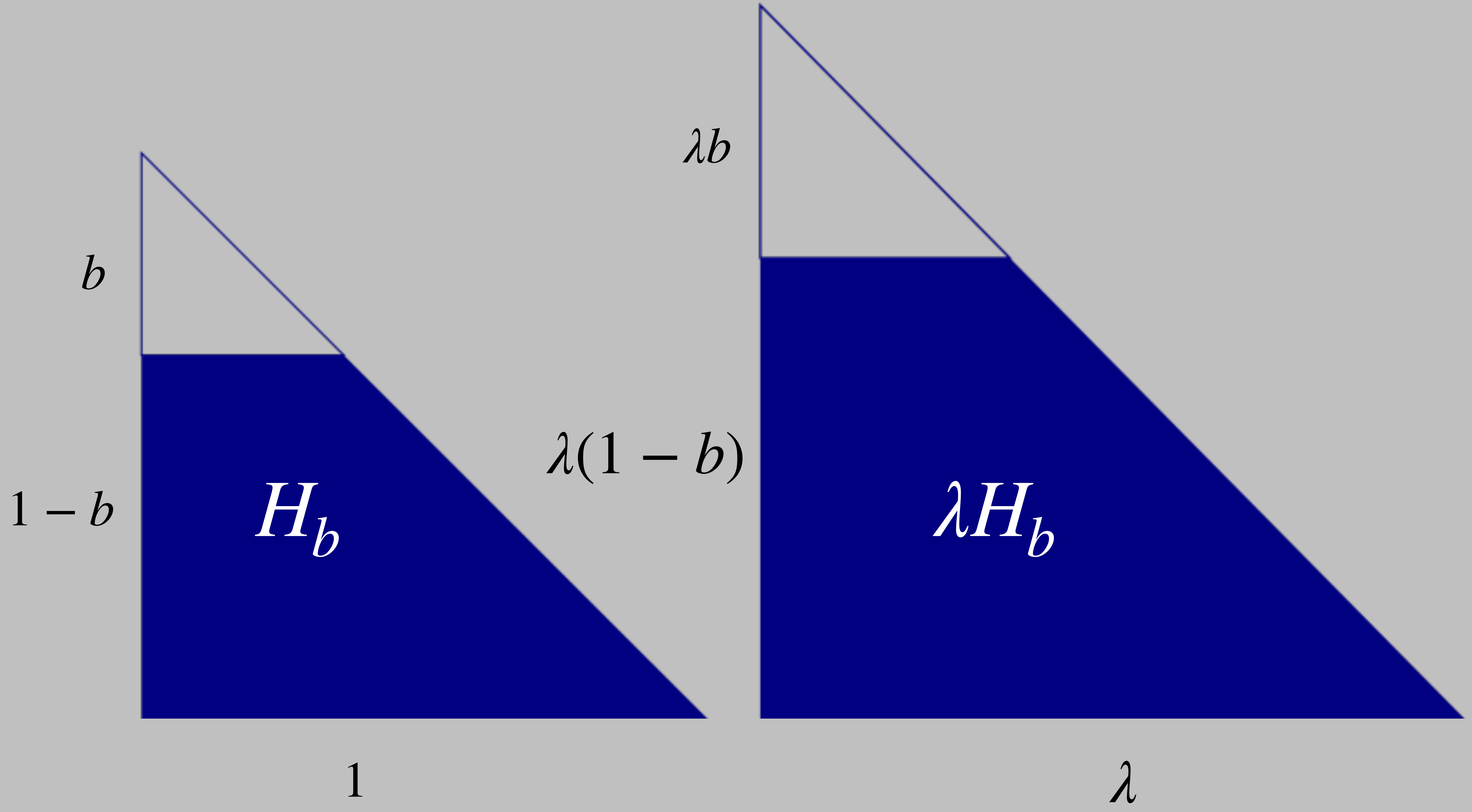
A toric domain in $\mathbb{C}^2$ is the preimage of a region $\Omega \subset \mathbb{R}^2$ under the map $(z_1, z_2) \mapsto (\pi |z_1|^2, \pi |z_2|^2)$



Domain of Embeddings: Ellipsoids

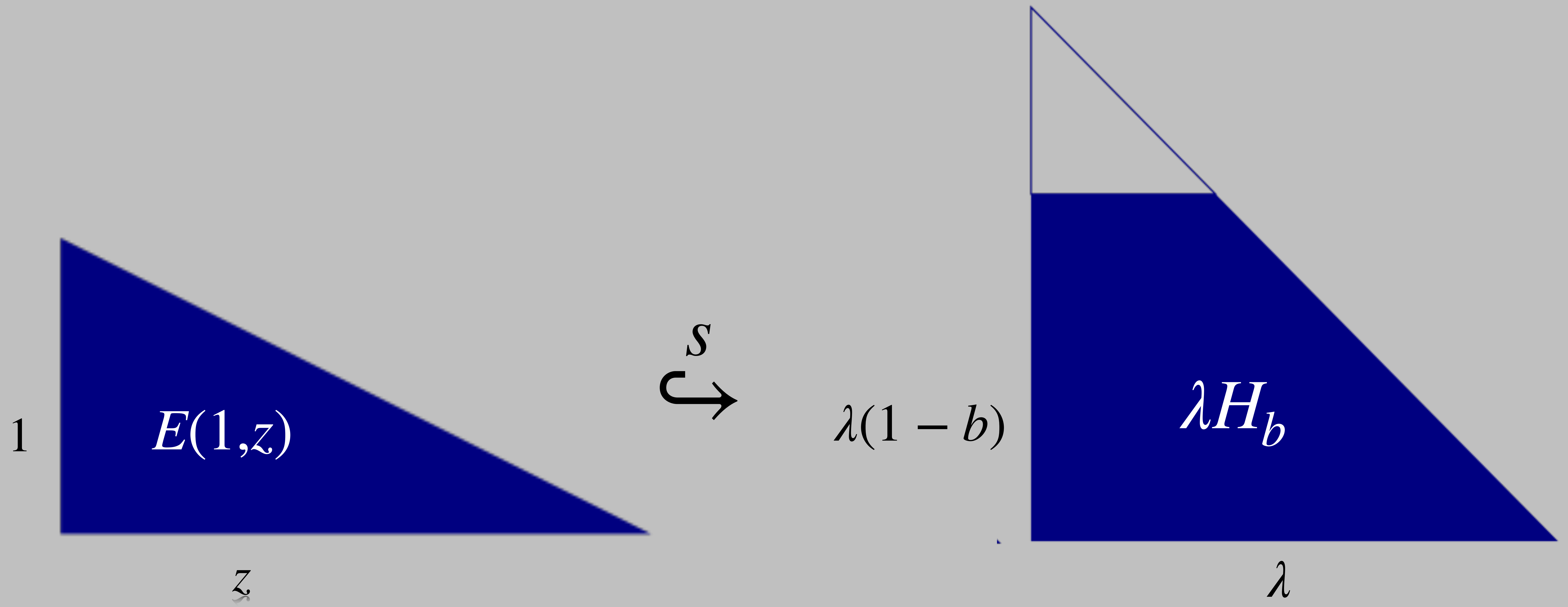


Targets of Embeddings: $H_b := \mathbb{C}P^2_1 \# \overline{\mathbb{C}P}^2_b$



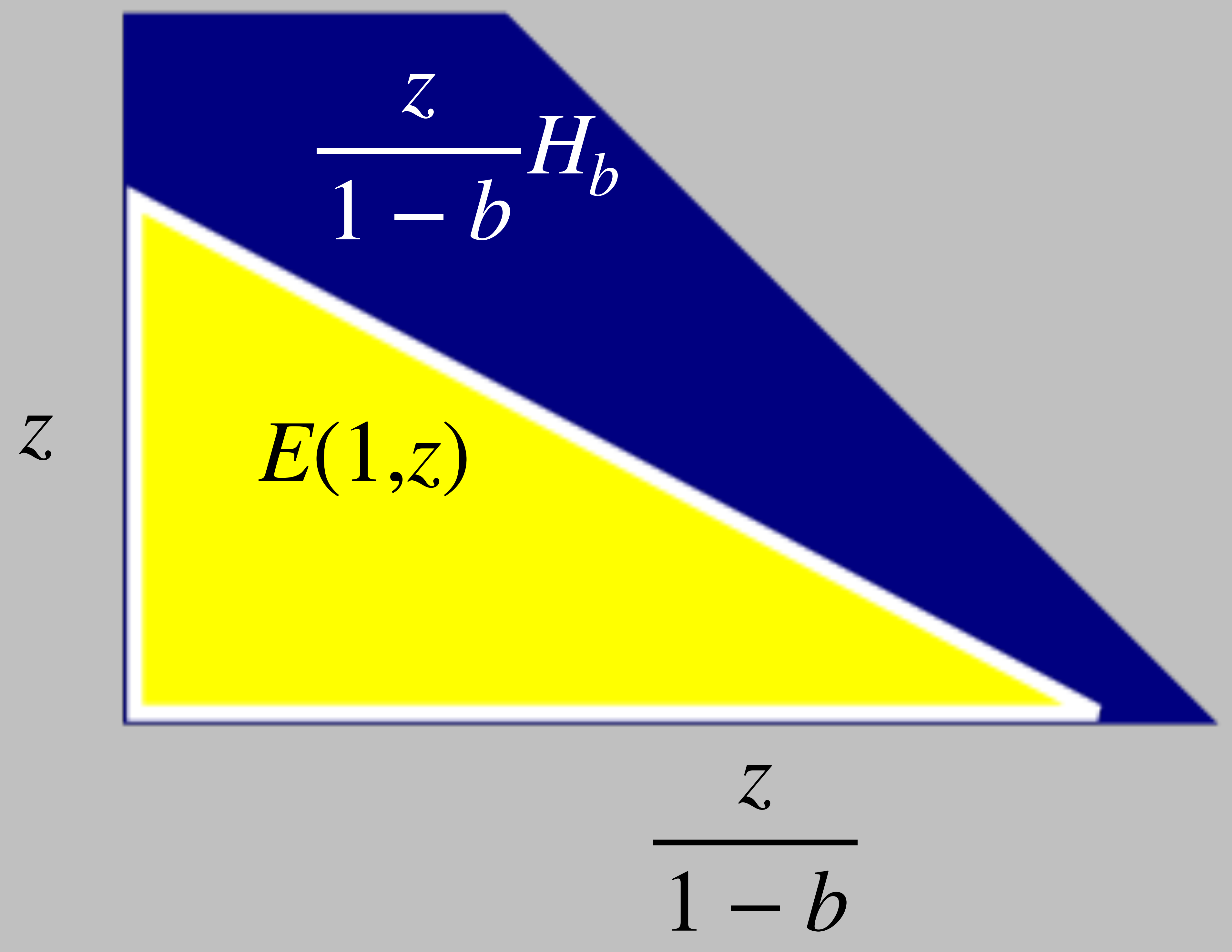
Main Question

For what λ, does $E(1,z) \overset{s}{\hookrightarrow} \lambda H_b$?



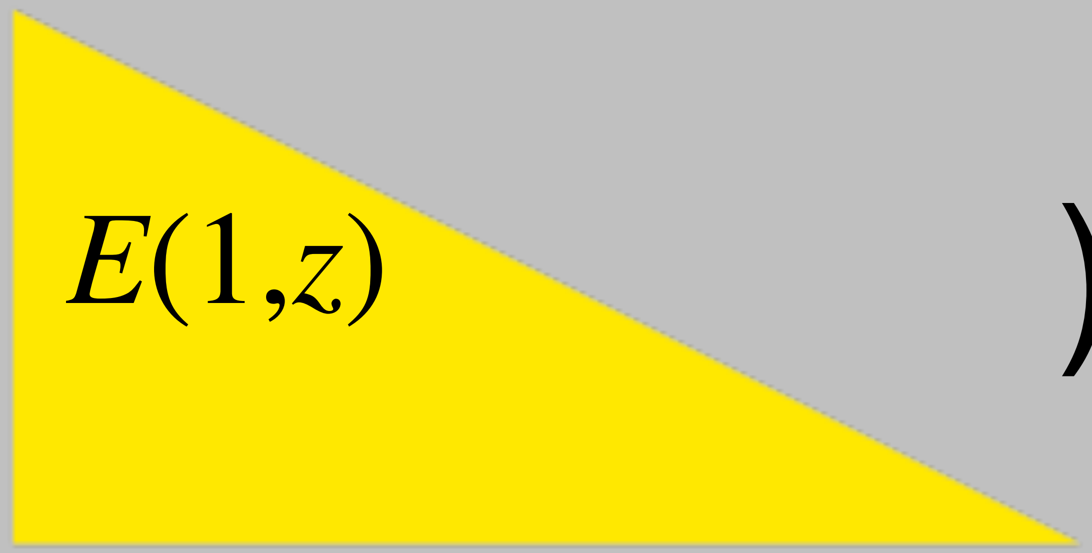
The Inclusion Embedding

$$E(1,z) \overset{s}{\hookrightarrow} \frac{z}{1-b} H_b$$



The Volume Obstruction

If $E(1,z) \overset{s}{\hookrightarrow} \lambda H_b$, then $\mathrm{vol}(E(1,z)) \leq \mathrm{vol}(\lambda H_b)$.

Area (

) $\leq$ Area (

)

This implies $\lambda \geq \sqrt{\dfrac{z}{1-b^2}}$

Minimizing the target

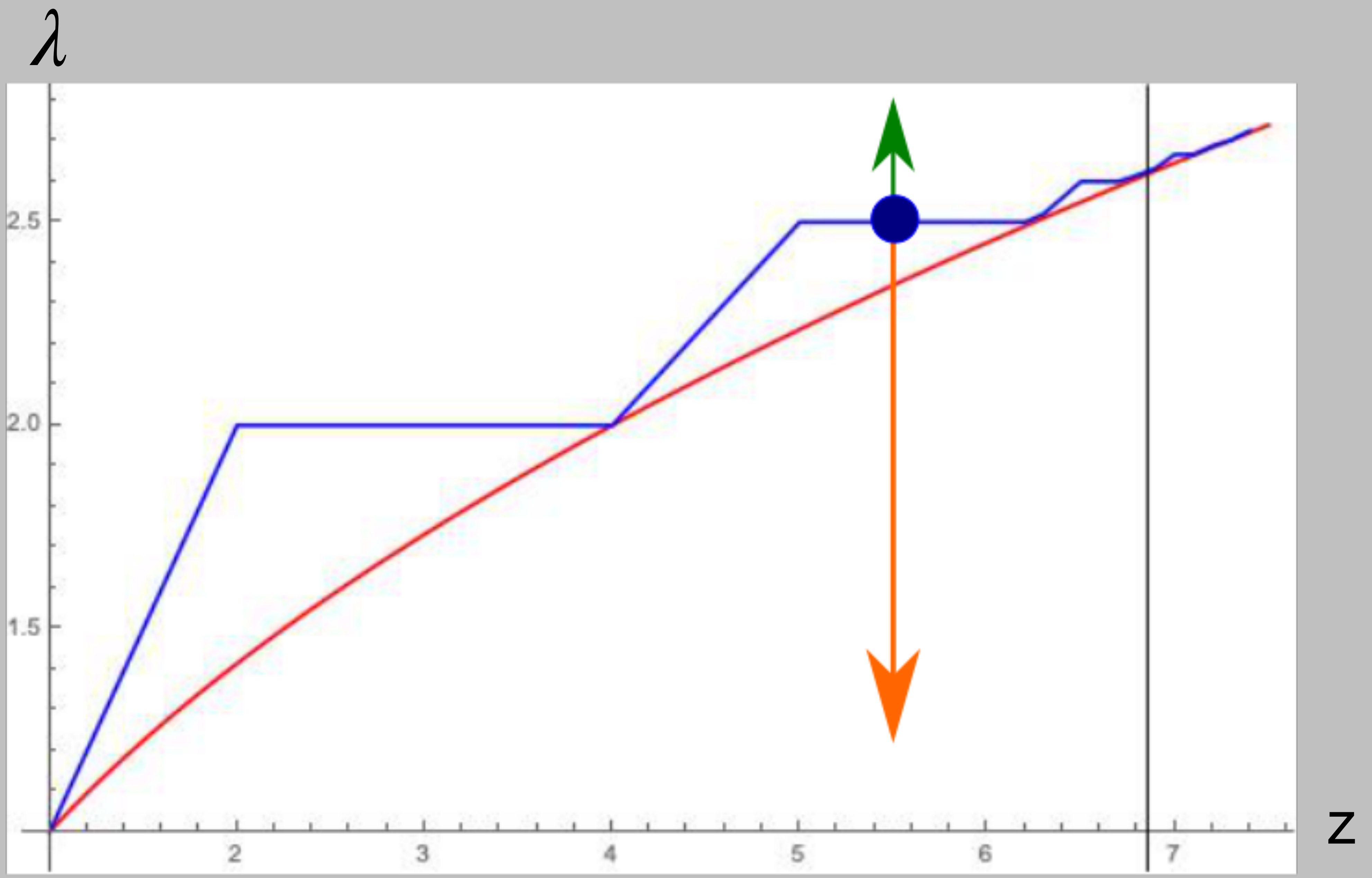
If we fix z and b, what is the smallest λ such that $E(1,z) \overset{s}{\hookrightarrow} \lambda H_b$?

$$\sqrt{\frac{z}{1-b^2}} \leq \lambda \leq \frac{z}{1-b}$$

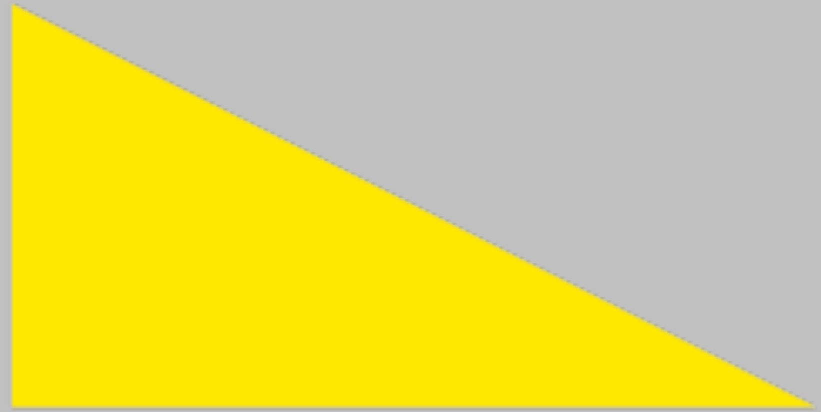
Depending on z and b, sometimes λ is volume bound (floppy), inclusion bound (rigid), and sometimes in between.

Object of Study

Embedding function: $c_b(z) := \inf\{\lambda \mid E(1,z) \overset{s}{\hookrightarrow} \lambda H_b\}$



Minimum scaling of 

Size E(1,z) of ellipsoid 

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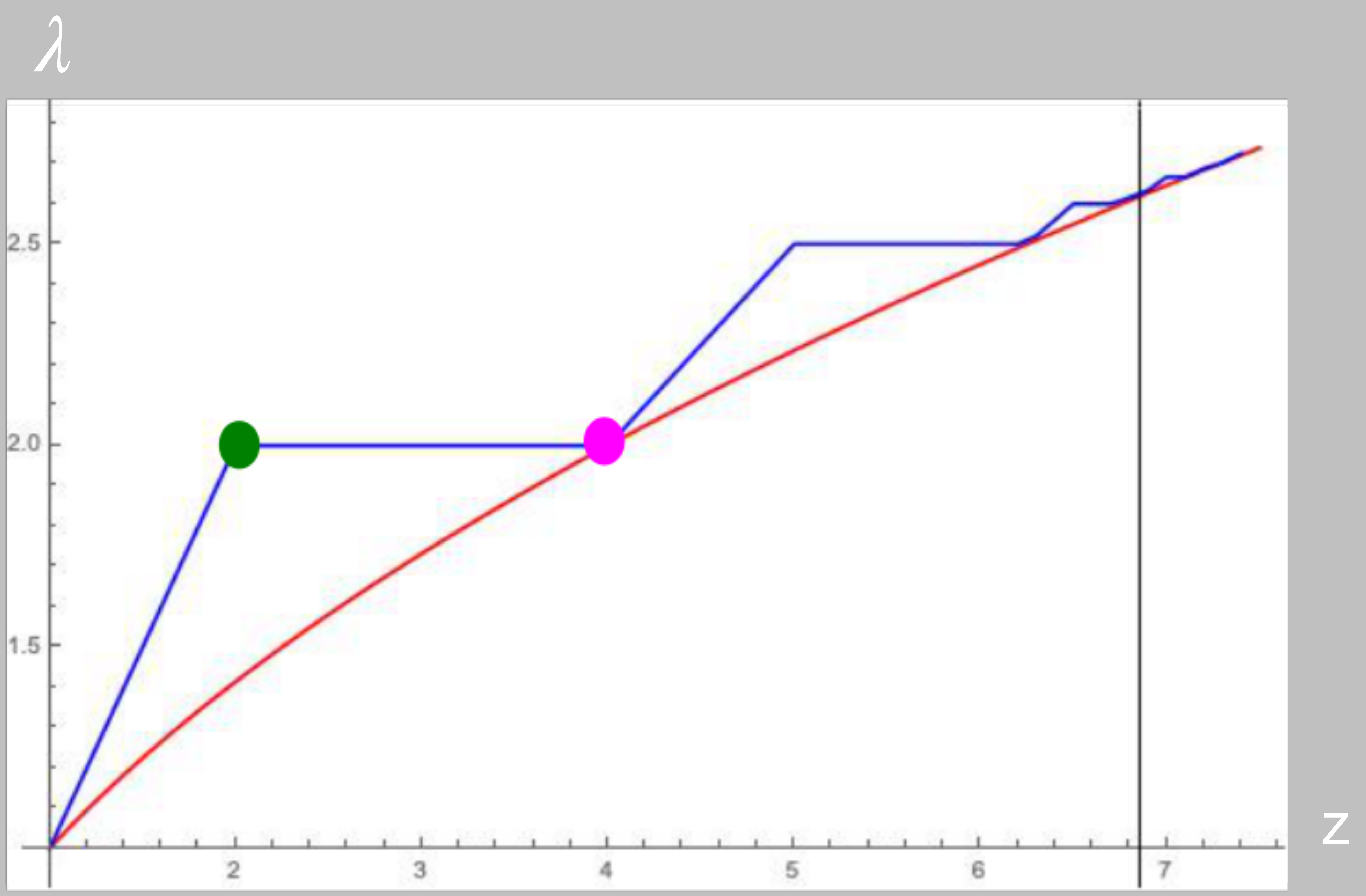
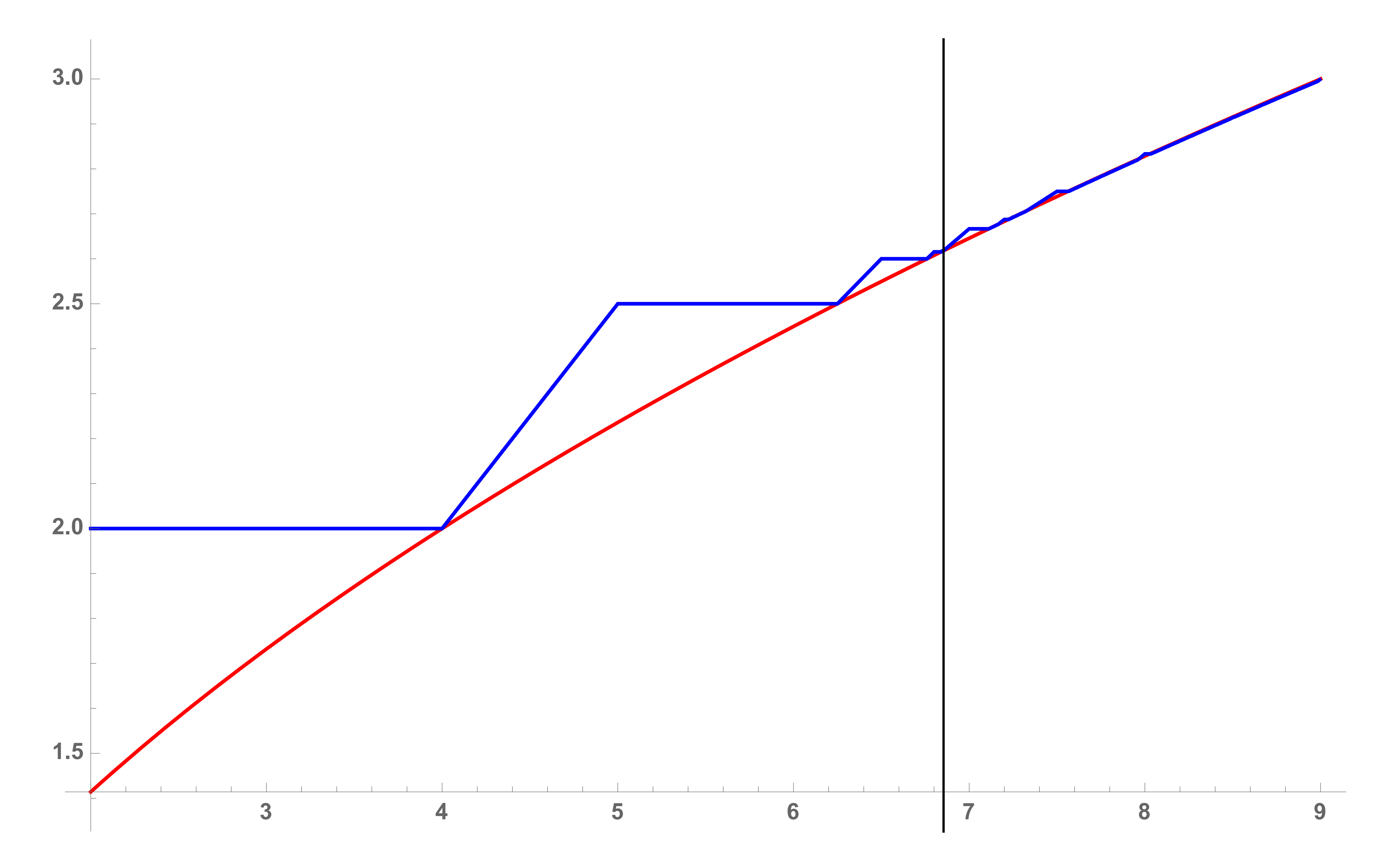


Minimum scaling of

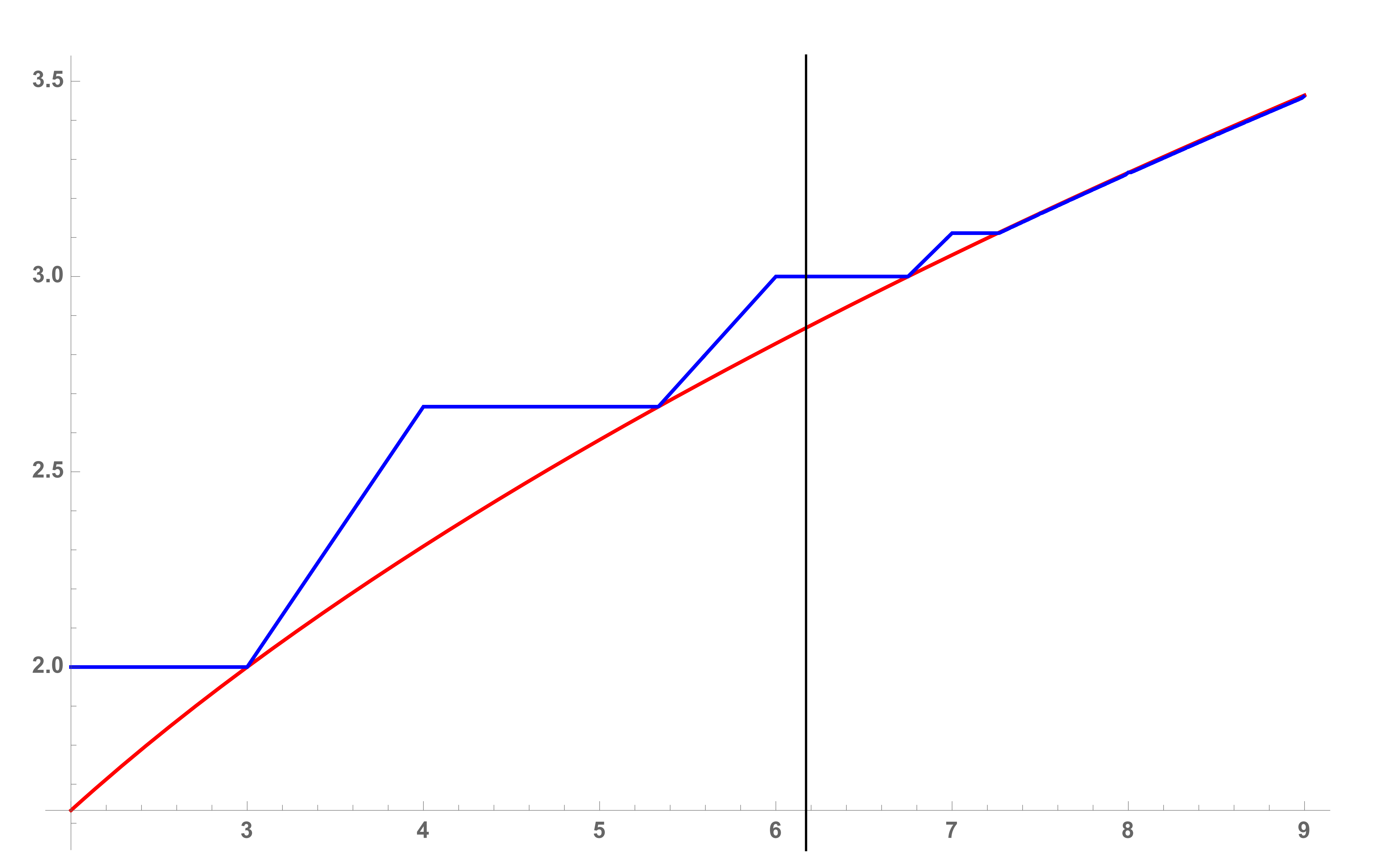
Size E(1,z) of ellipsoid

Properties of Embedding Function

(Cristofaro Gardiner-Holm-Mandini-Pires, 2020)



Infinite Staircase (infinitely many steps)



No Infinite Staircase (finitely many steps)

The Main Question

Which $b \in [0,1)$ values does the embedding function $c_b(z)$ have an infinite staircase?

- McDuff-Schlenk (2010) showed $b = 0$ has an infinite staircase.
- Cristofaro Gardiner-Holm-Mandini-Pires (2020) showed $b = \frac{1}{3}$ has an infinite staircase and conjectured this is the only rational value other than 0 with an infinite staircase.
- Bertozzi-Holm-Maw-McDuff-Mwakyoma-Pires-Weiler (2021) found three infinite families of irrational b values that have infinite staircases.

The Accumulation Theorem

Cristofaro Gardiner-Holm-Mandini-Pires

If $c_b(z)$ has an infinite staircase, then it must accumulate at the larger solution z_b to the quadratic equation

$$z^2 + \left(\frac{(3-b)^2}{1-b^2} - 2 \right) z + 1 = 0 \text{ and } c_b(z_b) = \sqrt{\frac{z_b}{1-b^2}} = vol(z_b).$$

This implies there is a well defined notion of accumulation point.

Complete classification

M., McDuff, Pires, and Weiler

Define the two sets:

Block $:= \{b \in [0,1) \mid c_b(z_b) > vol(z_b)\}$

Stair $:= \{b \in [0,1) \mid c_b(z)$ has infintie staircase$\}$.

Then,

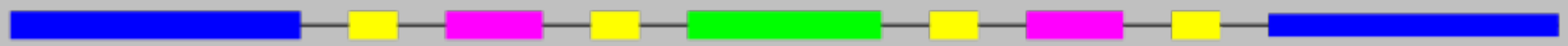
$[0,1) =$ Block $\sqcup$ Stair $\sqcup X$

where X is a countable set of rational b determined by the sequence $1,6,35,204,\ldots,6p_i - p_{i-1}$

Block and Stair

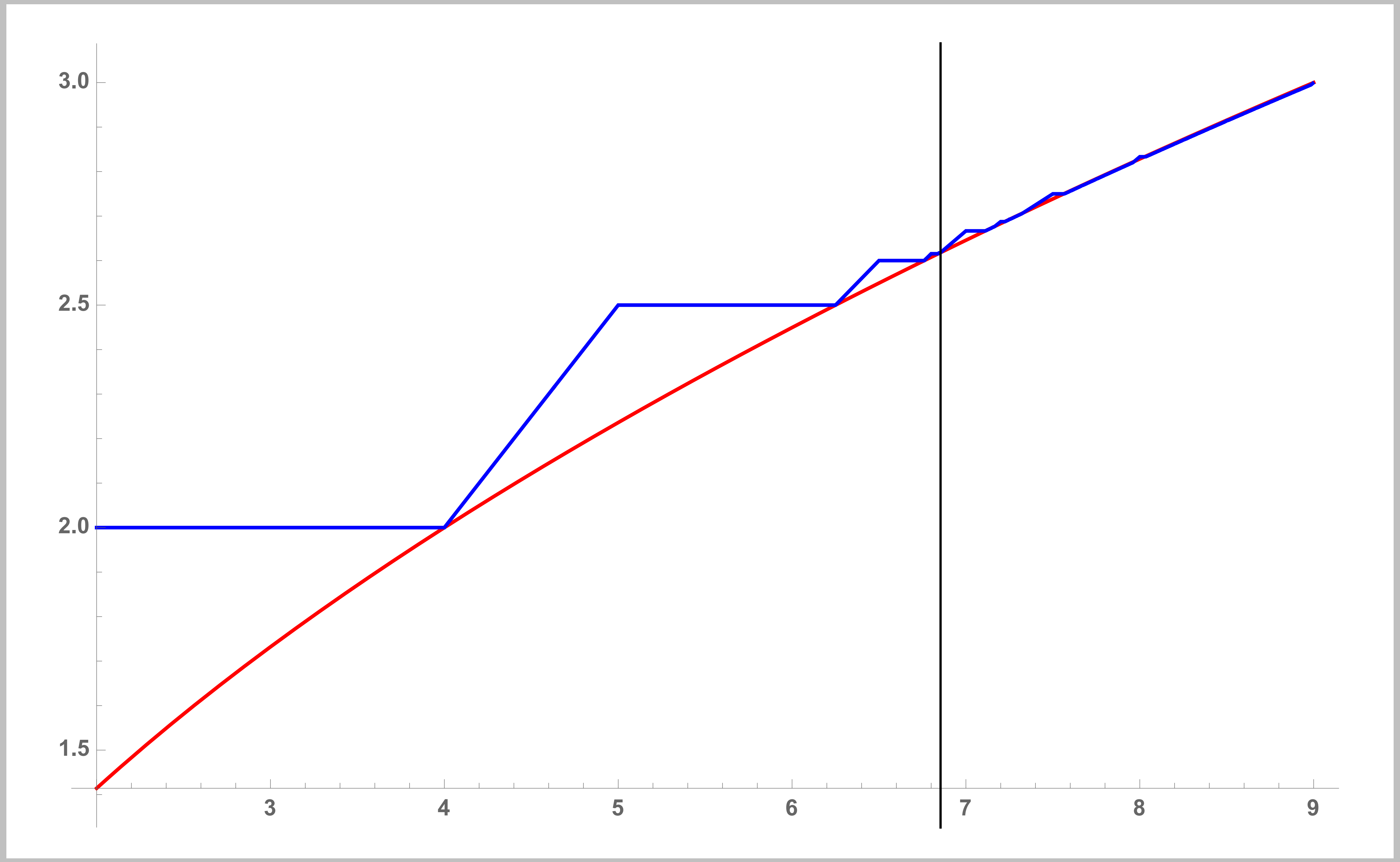
M., McDuff, Weiler

- Block is an open dense set in (0,1).
- For each n, Block $\cap \left(\frac{n}{n+1}, \frac{n+1}{n+2}\right)$ is homeomorphic to the complement of the middle third Cantor set.
- There are infinite staircases at the endpoints of these blocked intervals.

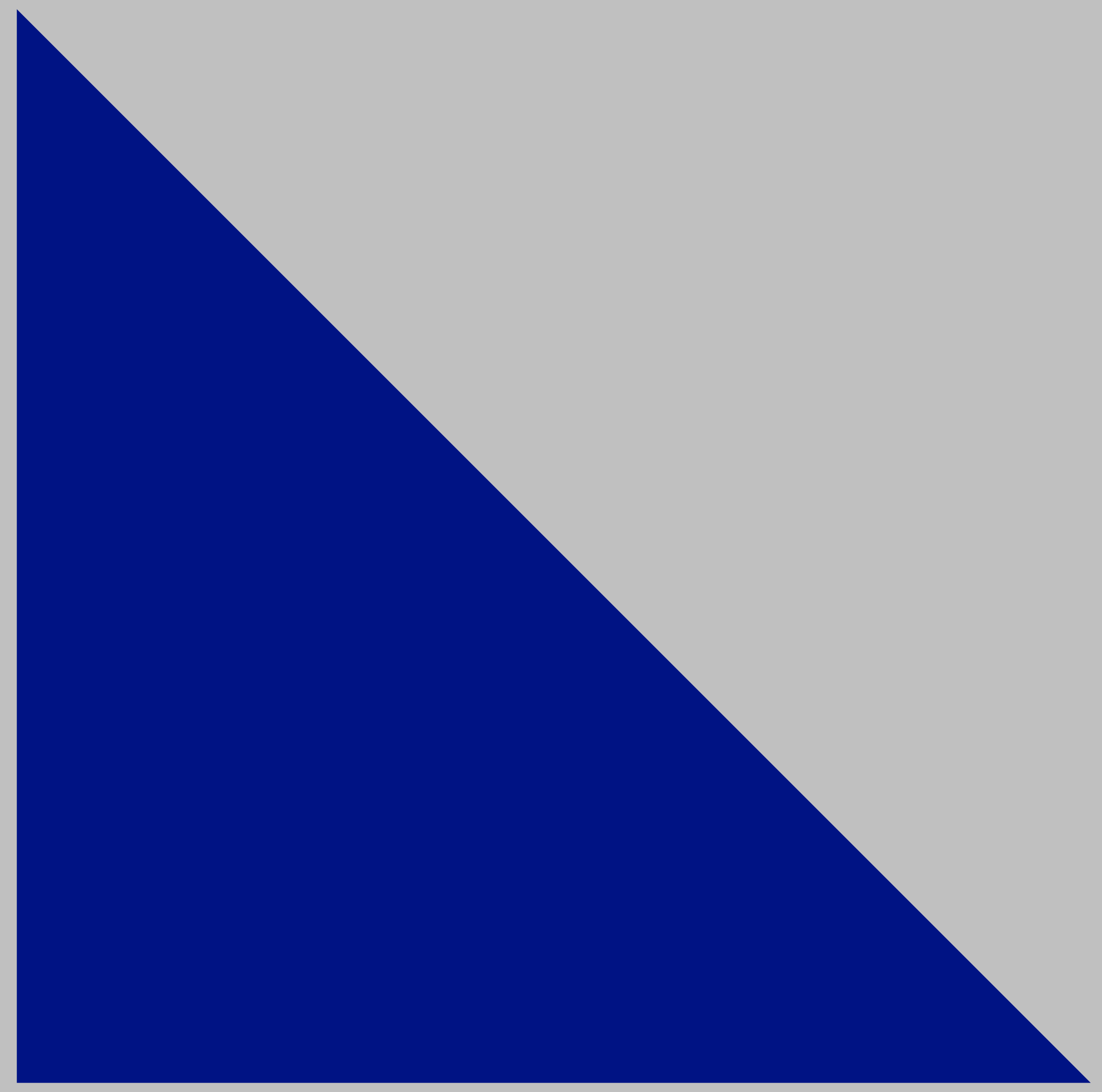
$\frac{1}{2} < b$  $b < \frac{2}{3}$

The Focus of the Talk:
M.

For the b-values at the left endpoint of blocked intervals, $c_b(z_b) = vol(z_b)$ where z_b is the accumulation point.



The Moment Polytope

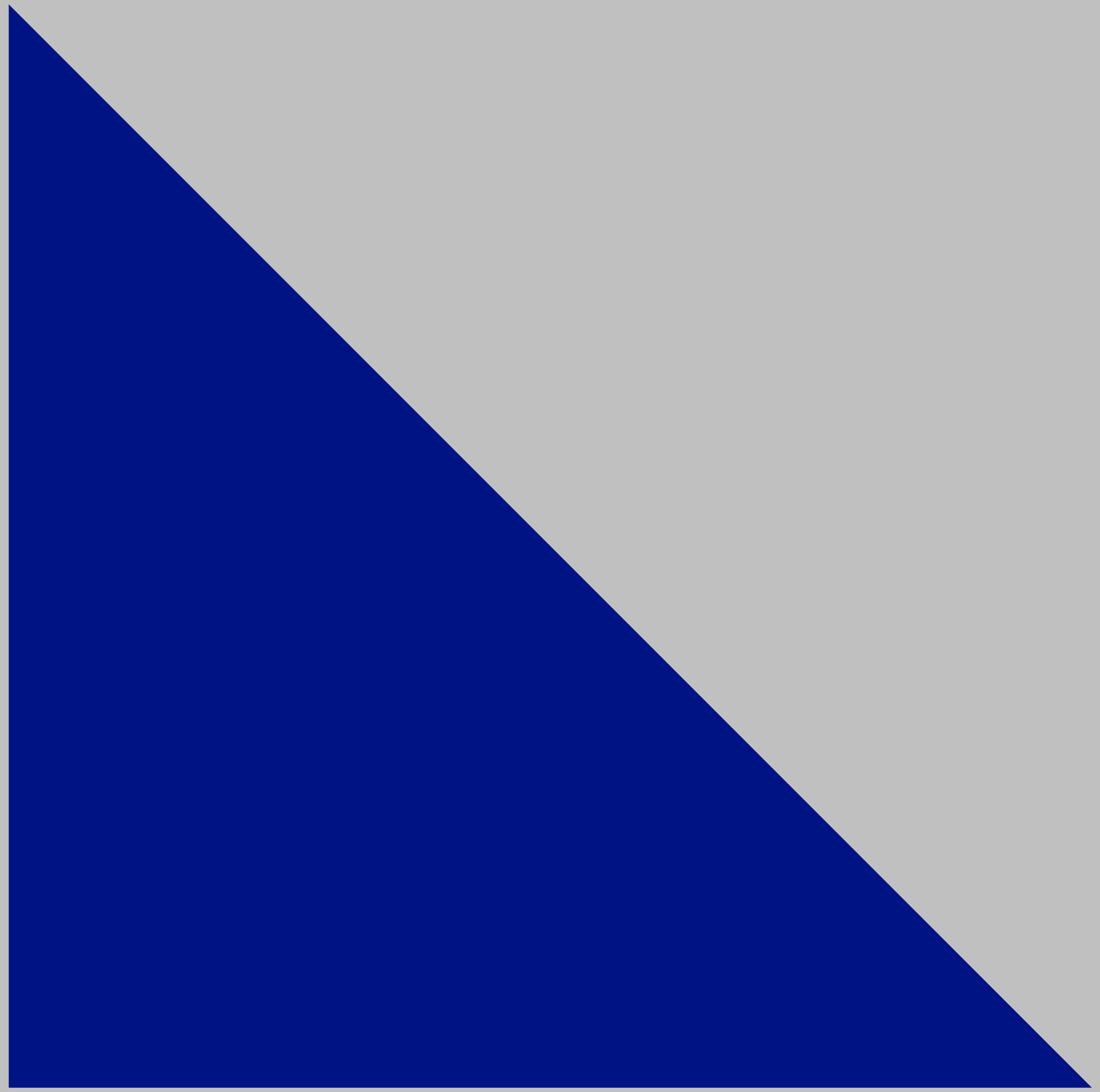


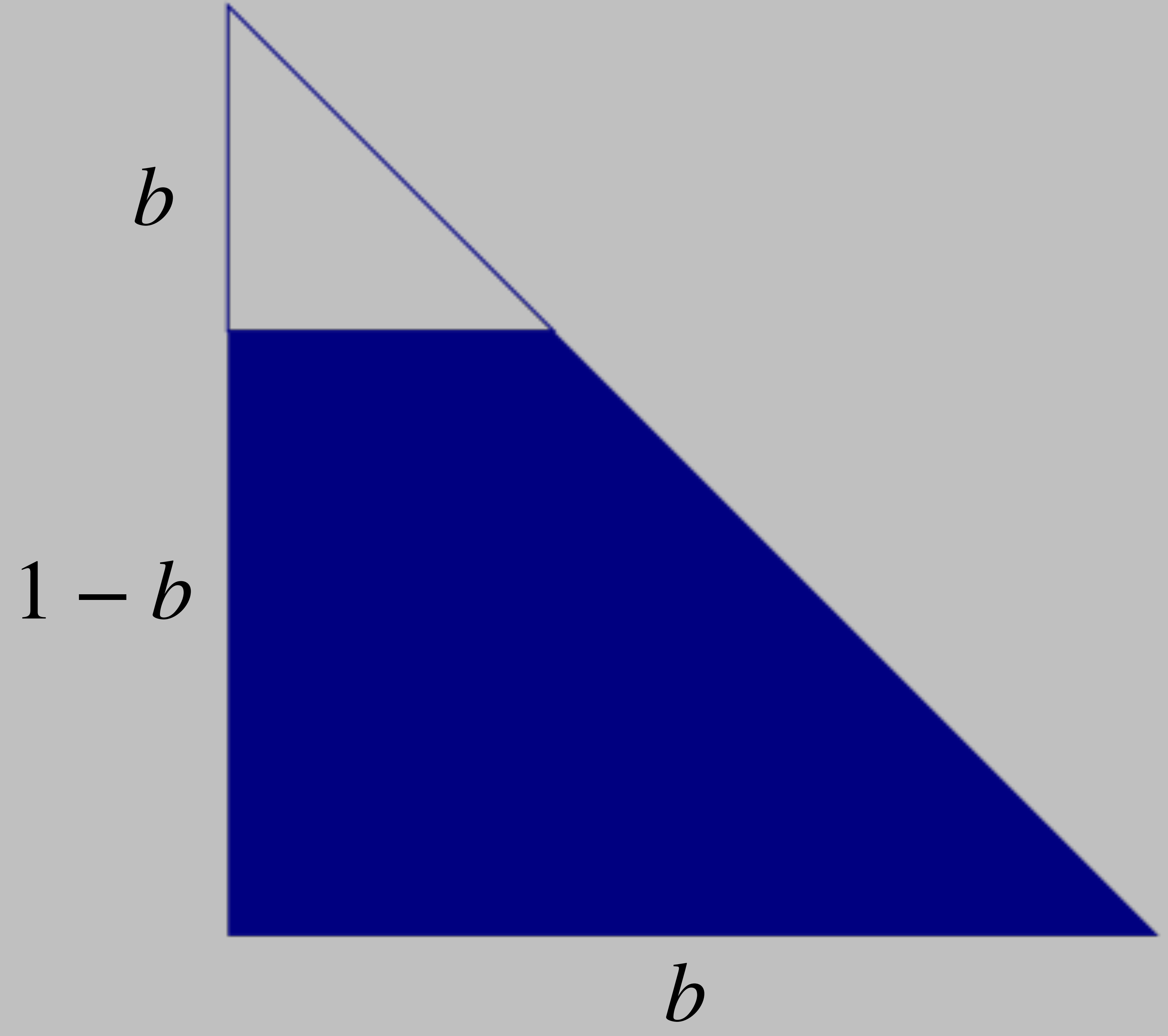
$$\mathbb{C}P^2 \to \mathbb{R}^2$$

$$[z_0 : z_1 : z_2] \mapsto \left(\frac{|z_0|^2}{|z|^2}, \frac{|z_1|^2}{|z|^2} \right)$$

- Fiber over a point on the interior of the triangle is a torus
- Fiber over a point on the interior of an edge is a circle
- Fiber over a vertex is a point
- Fiber of an edge is a $\mathbb{C}P^1$. Each edge corresponds to some $z_i = 0$

The Moment Polytope





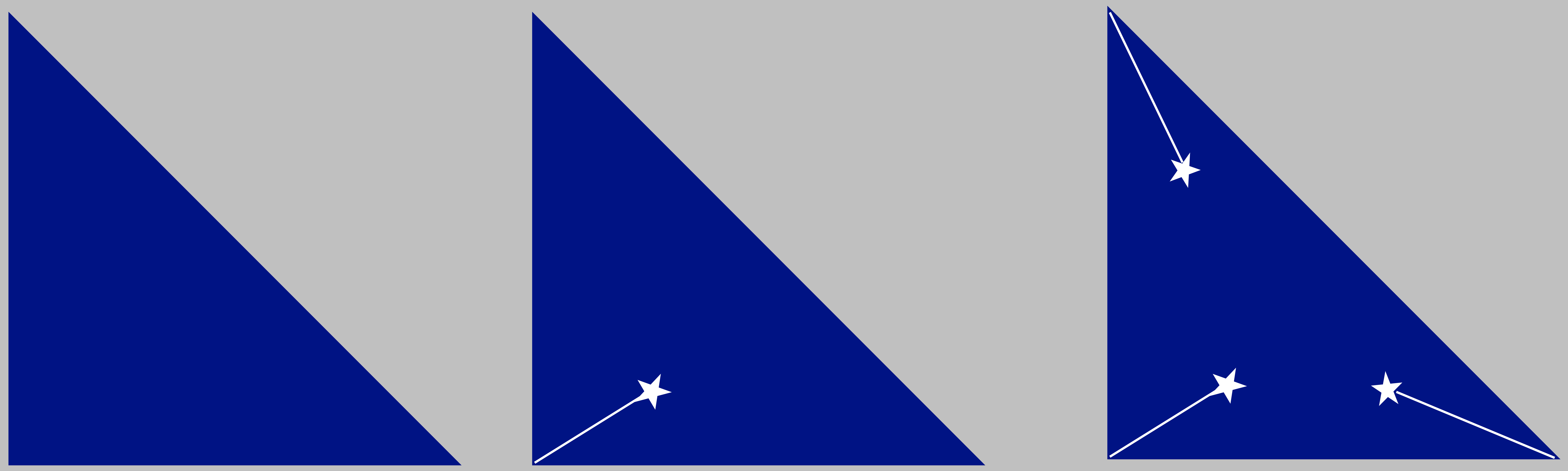
$$\mathbb{C}P^2 \to \mathbb{R}^2$$

$$[z_0 : z_1 : z_2] \mapsto \left(\frac{|z_0|^2}{|z|^2}, \frac{|z_1|^2}{|z|^2} \right)$$

$$\mathbb{C}P^2_1 \# \overline{\mathbb{C}P^2}_b \to \mathbb{R}^2$$

Almost Toric Fibrations

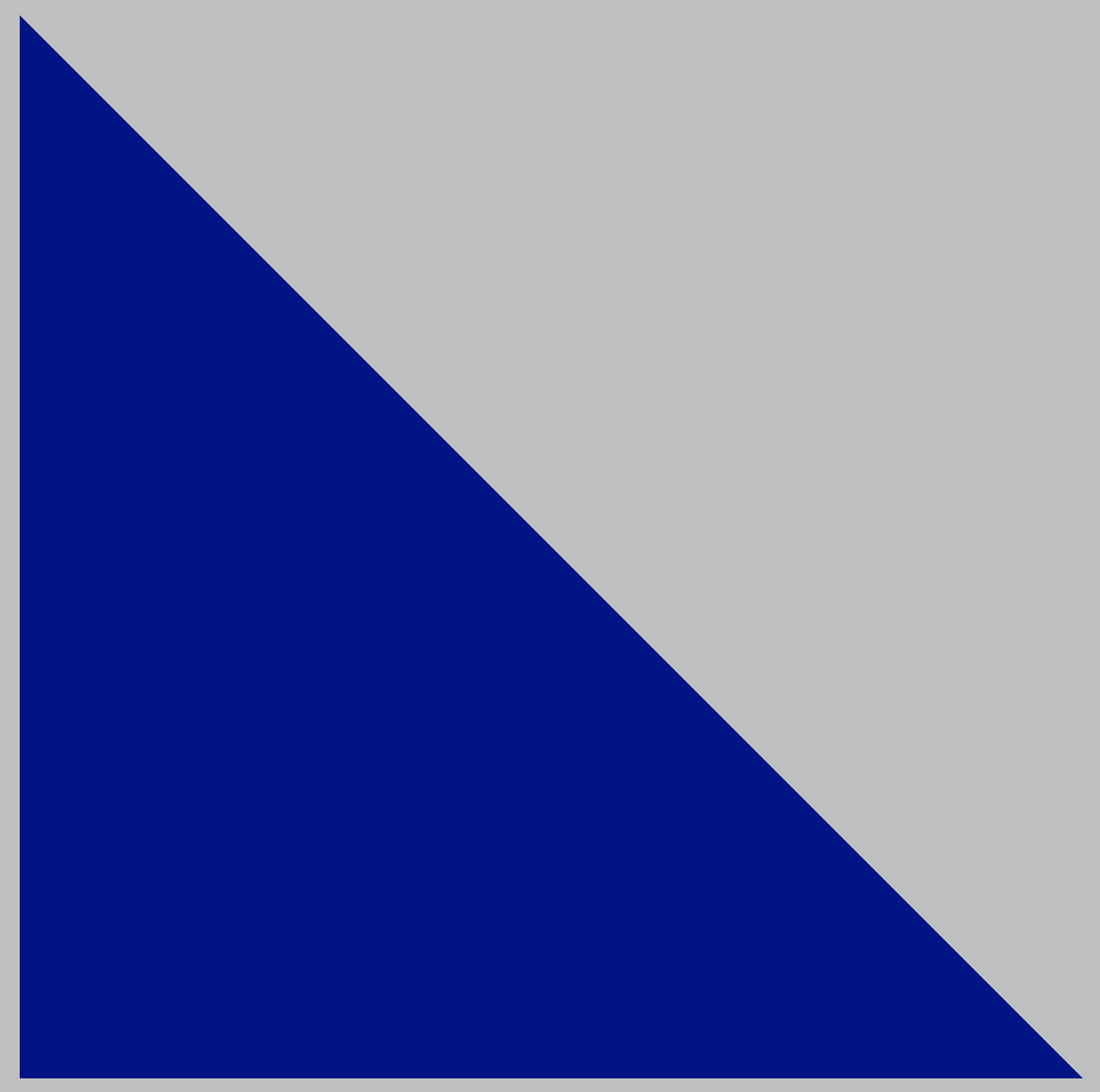
Symington and Leung classified closed almost toric manifolds in terms of the base diagrams with decorations for the various singularities.

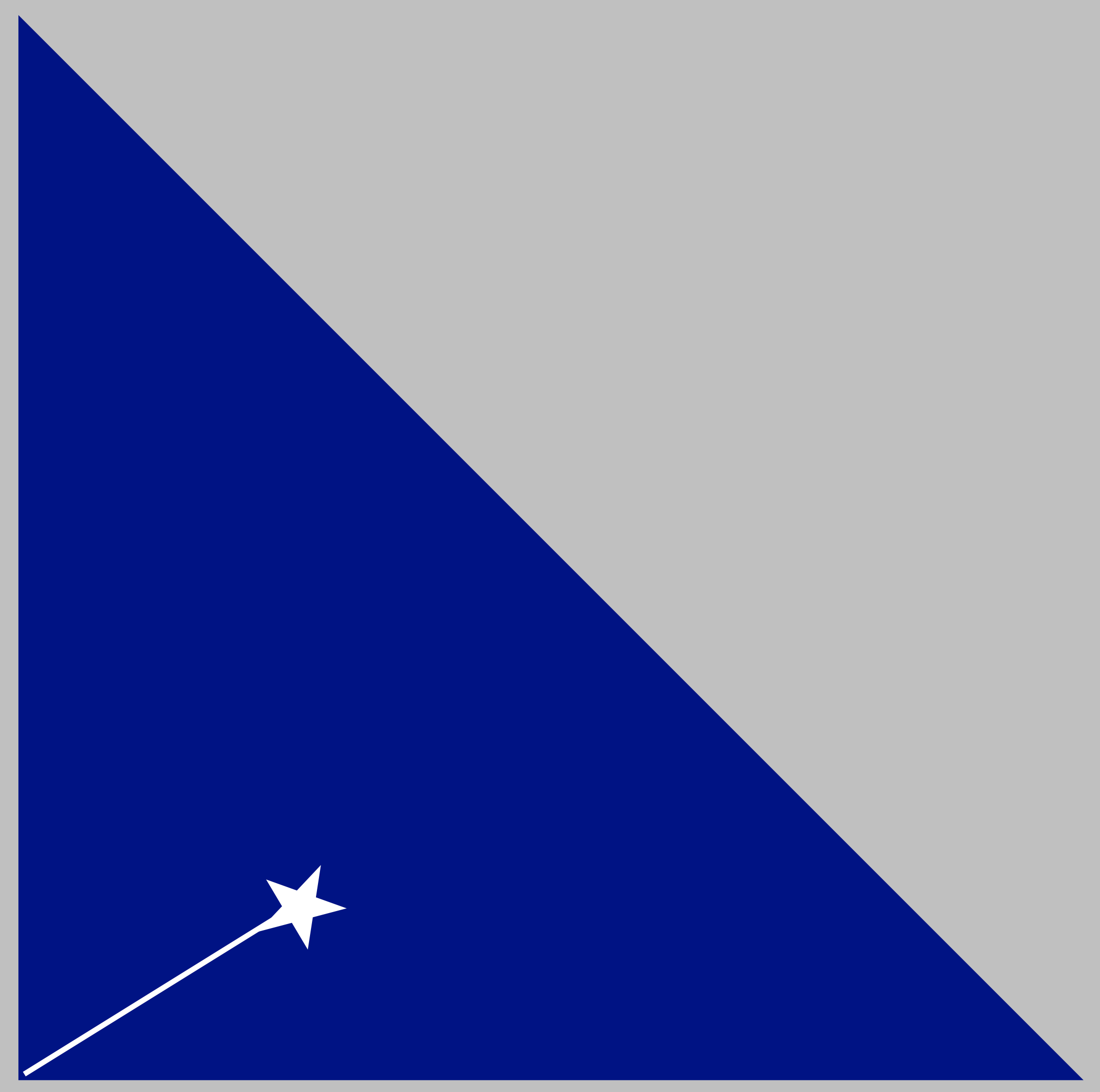


$$\mathbb{C}P^2 \to \mathbb{R}^2$$

$$[z_0 : z_1 : z_2] \mapsto \left(\frac{|z_0|^2}{|z|^2}, \frac{|z_1|^2}{|z|^2} \right)$$

Two different fibrations for symplectomorphic $\mathbb{C}P^2$





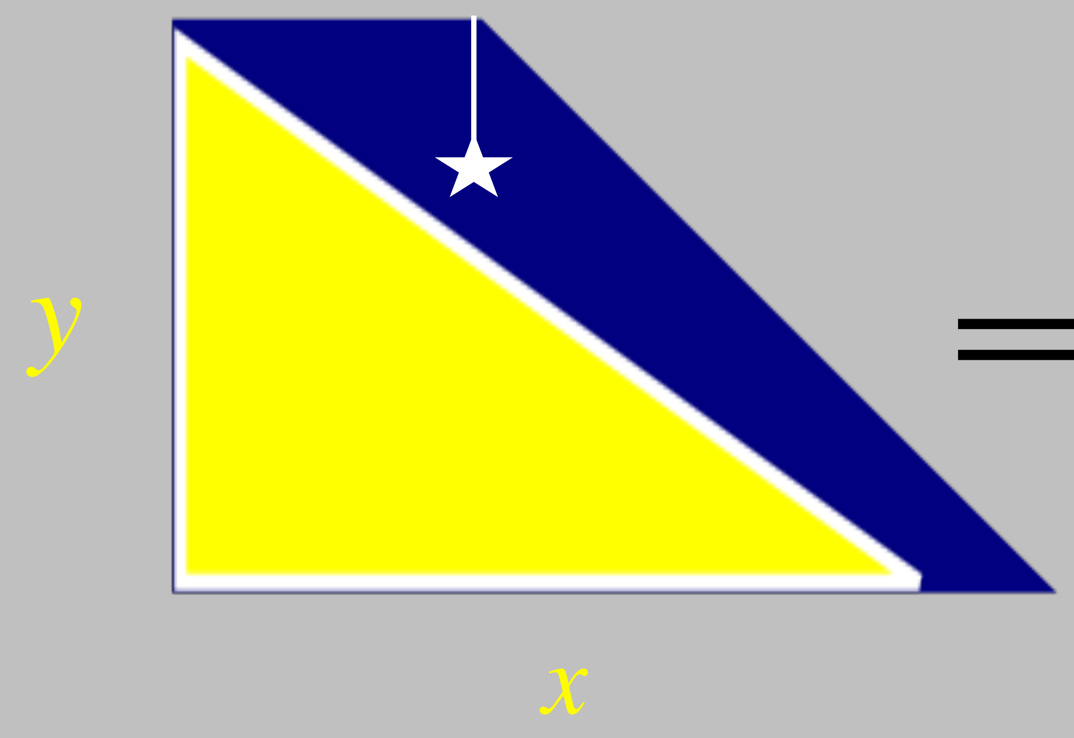
- Near origin, looks like $z_0 z_1 = 0$
- Fiber over interior points are Lagrangian tori
- Fiber over interior of an edge is a circle
- Fiber over vertices are points

- Near origin, looks like $z_0 z_1 = c$
- Fiber of star is pinched torus
- Fiber of vertex ray is emanating from is a circle
- Nodal ray is a branch cut whose direction is eigenvector of monodromy

Away from the ray, these are symplectomorphic fibrations of $\mathbb{C}P^2$

Almost Toric Pictures and Embeddings

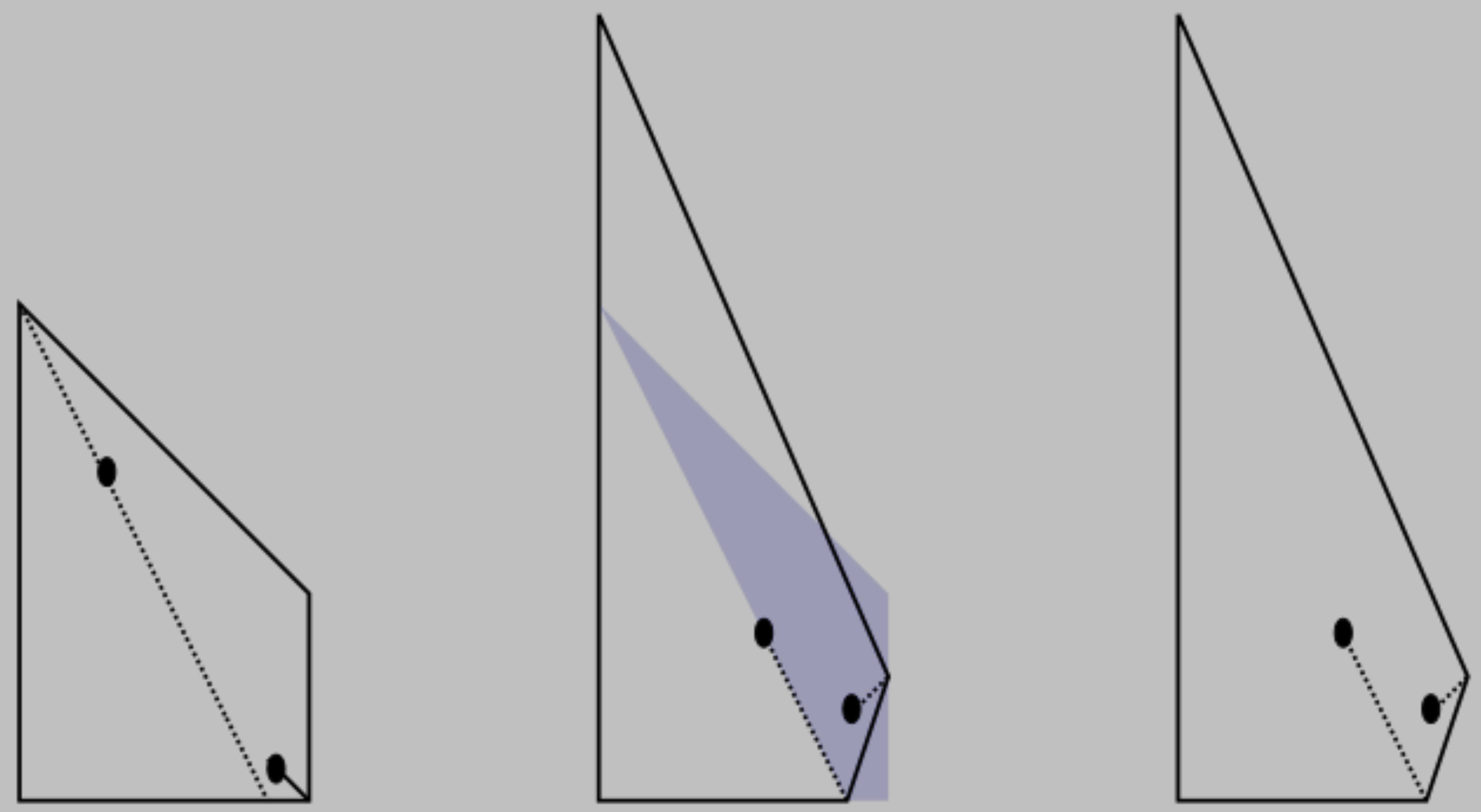
Idea used in Casals-Vianna and CG-HMP



$$\Longrightarrow (1-\epsilon)E(x,y) \overset{s}{\hookrightarrow} H_b \Longrightarrow c_b\left(\frac{y}{x}\right) \leq \frac{1}{x}$$

Embeddings gives upper bounds of the embedding function

Almost Toric Fibration Mutations



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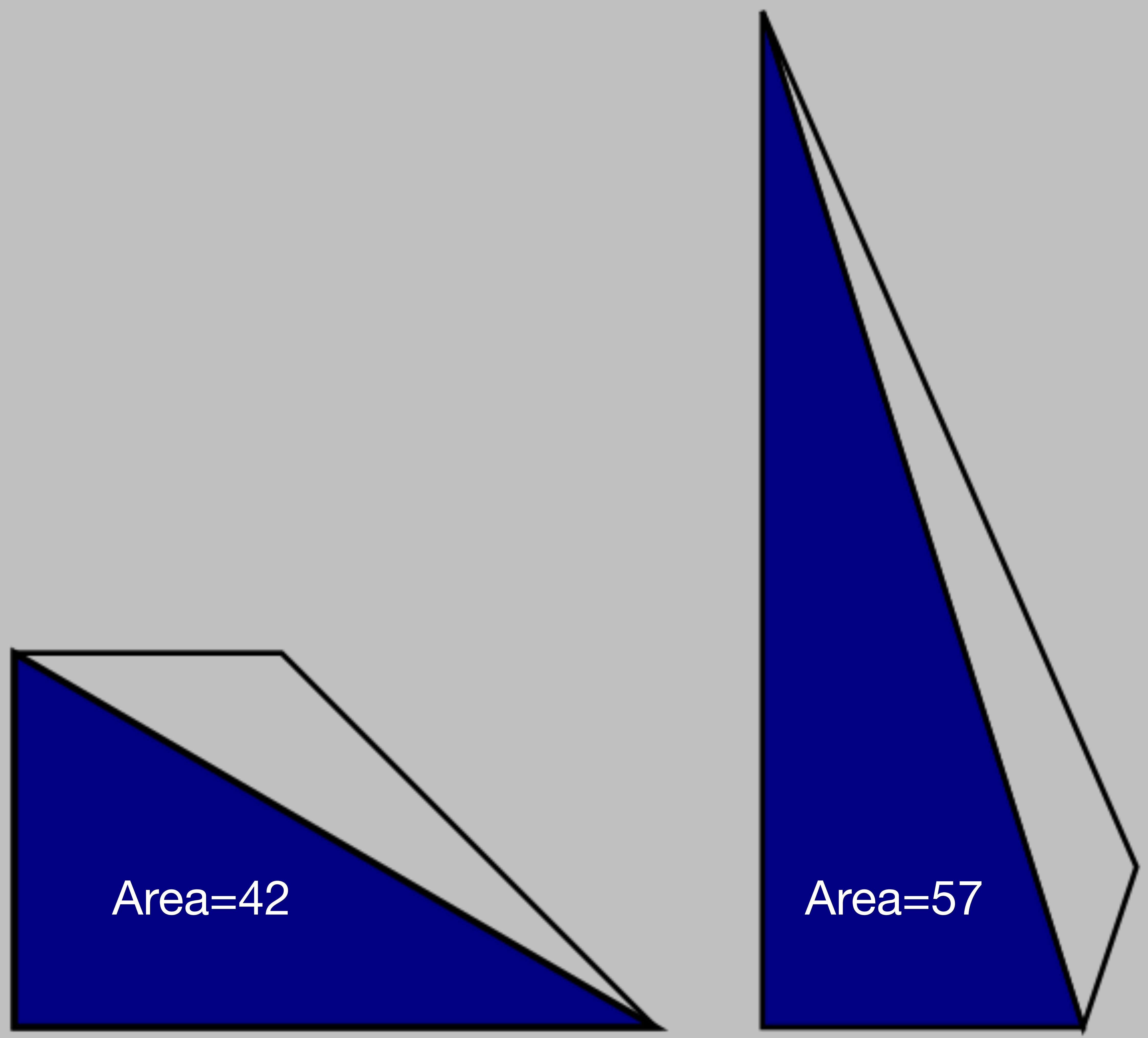
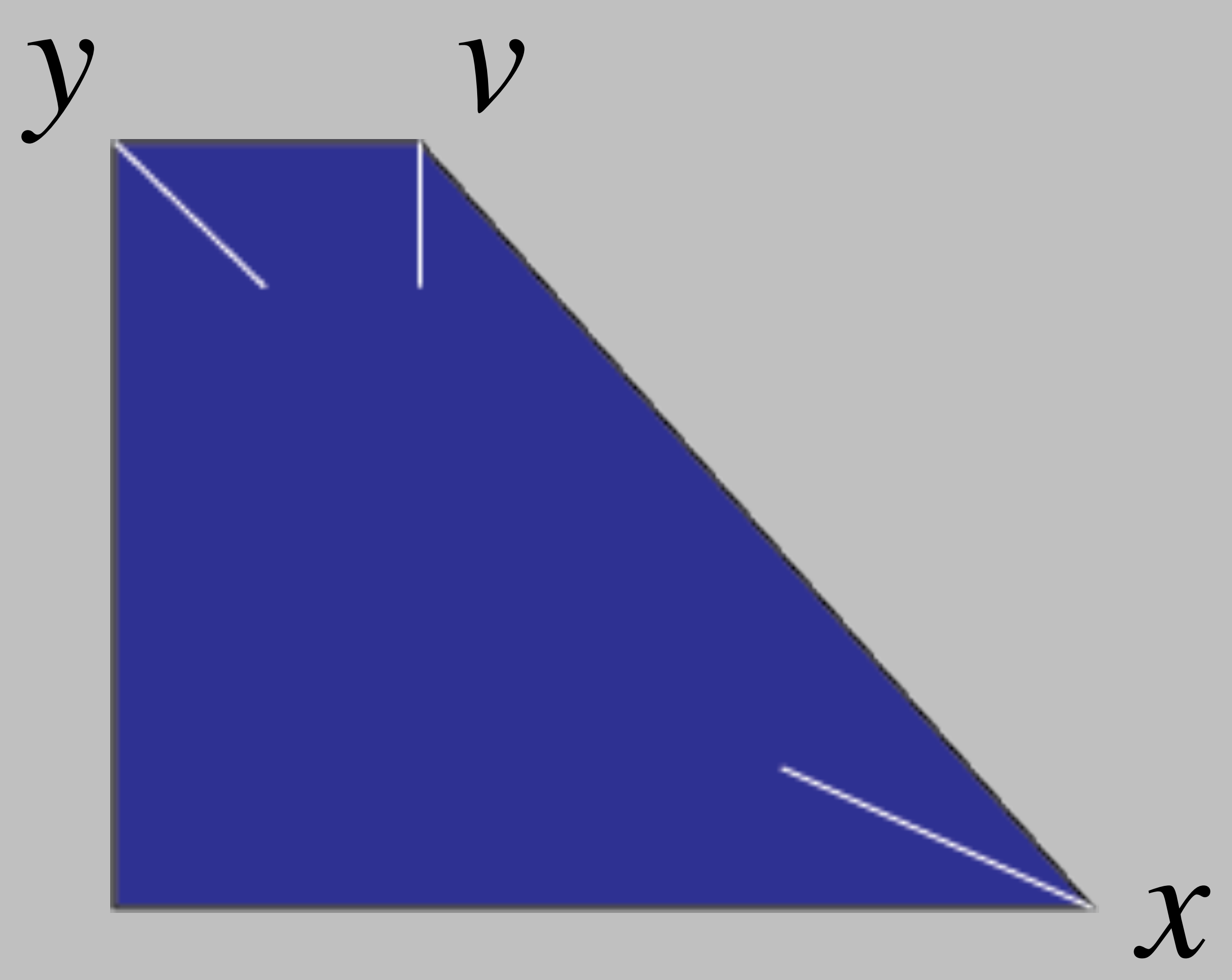
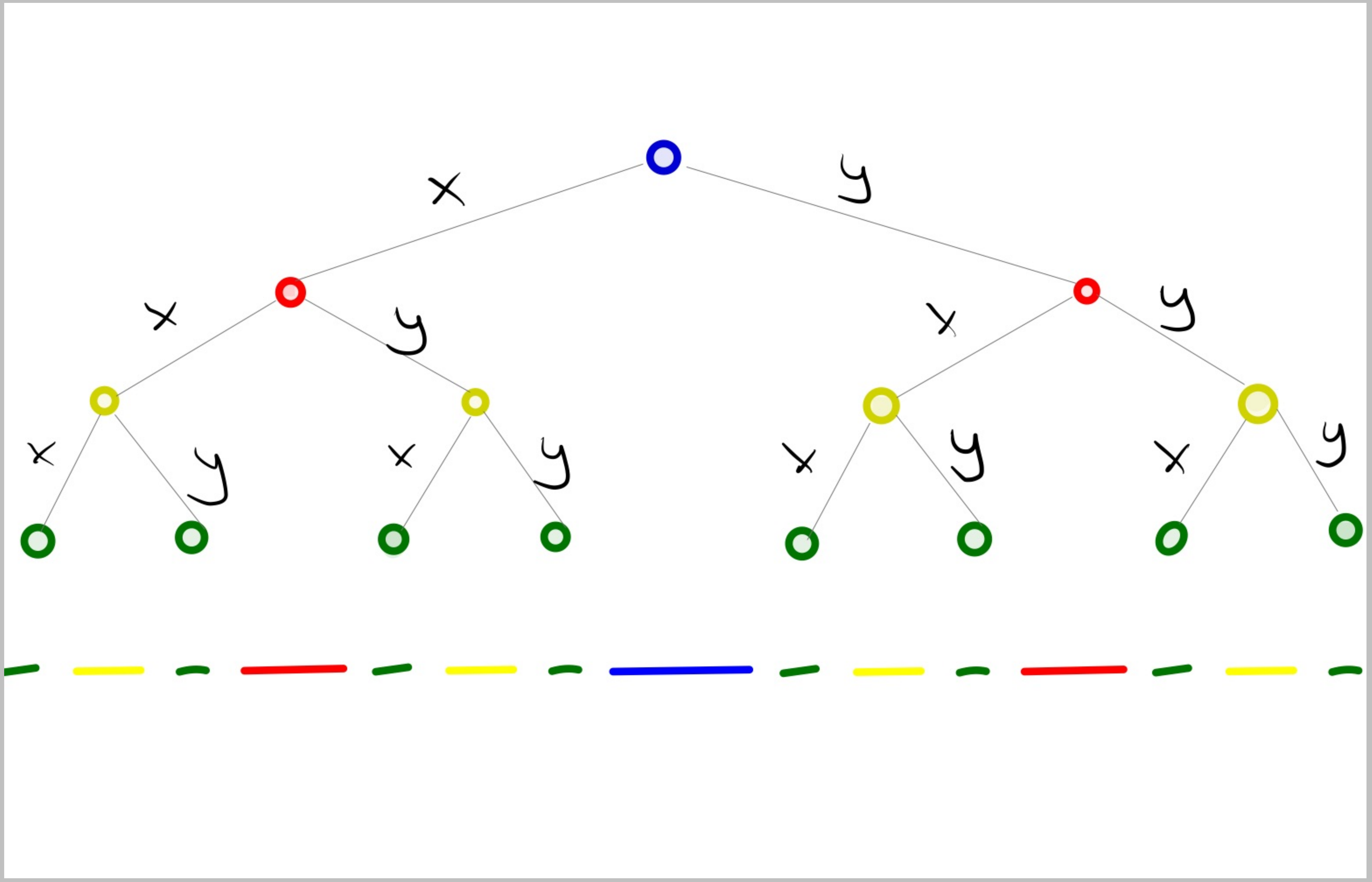
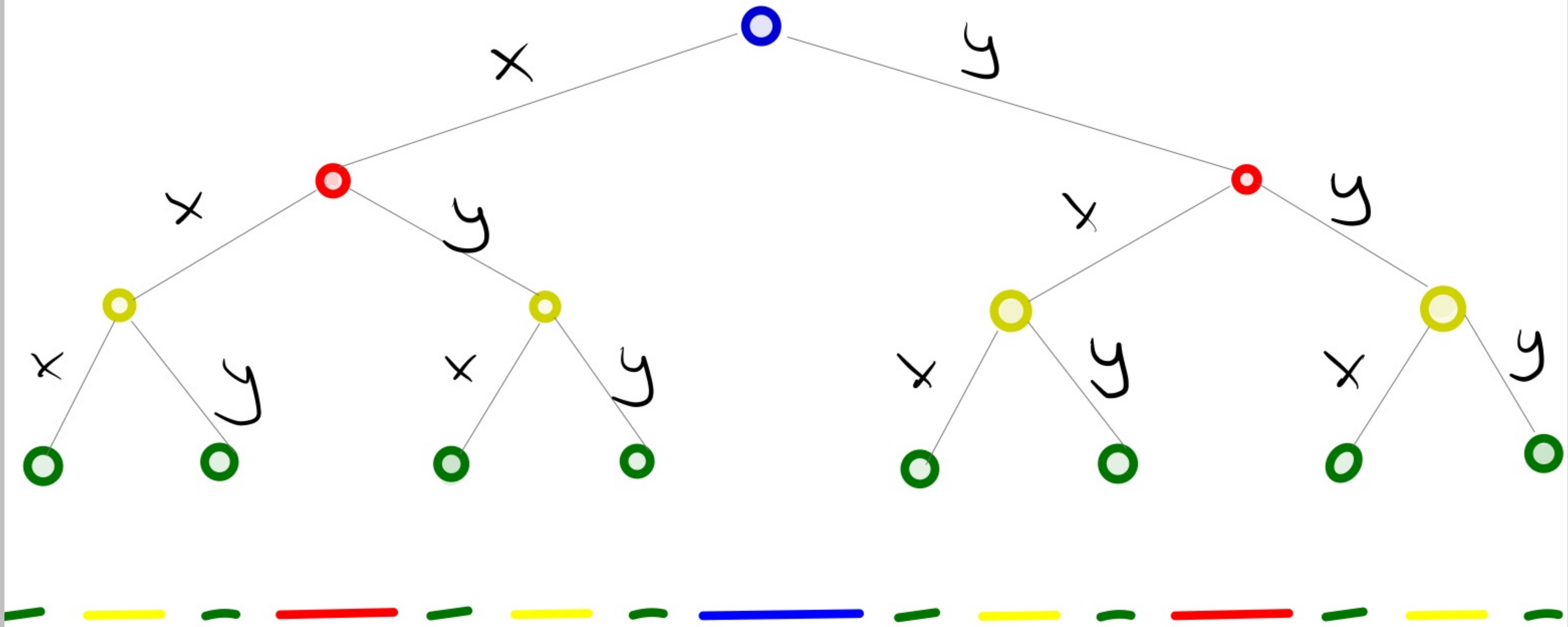


Tree of Mutations

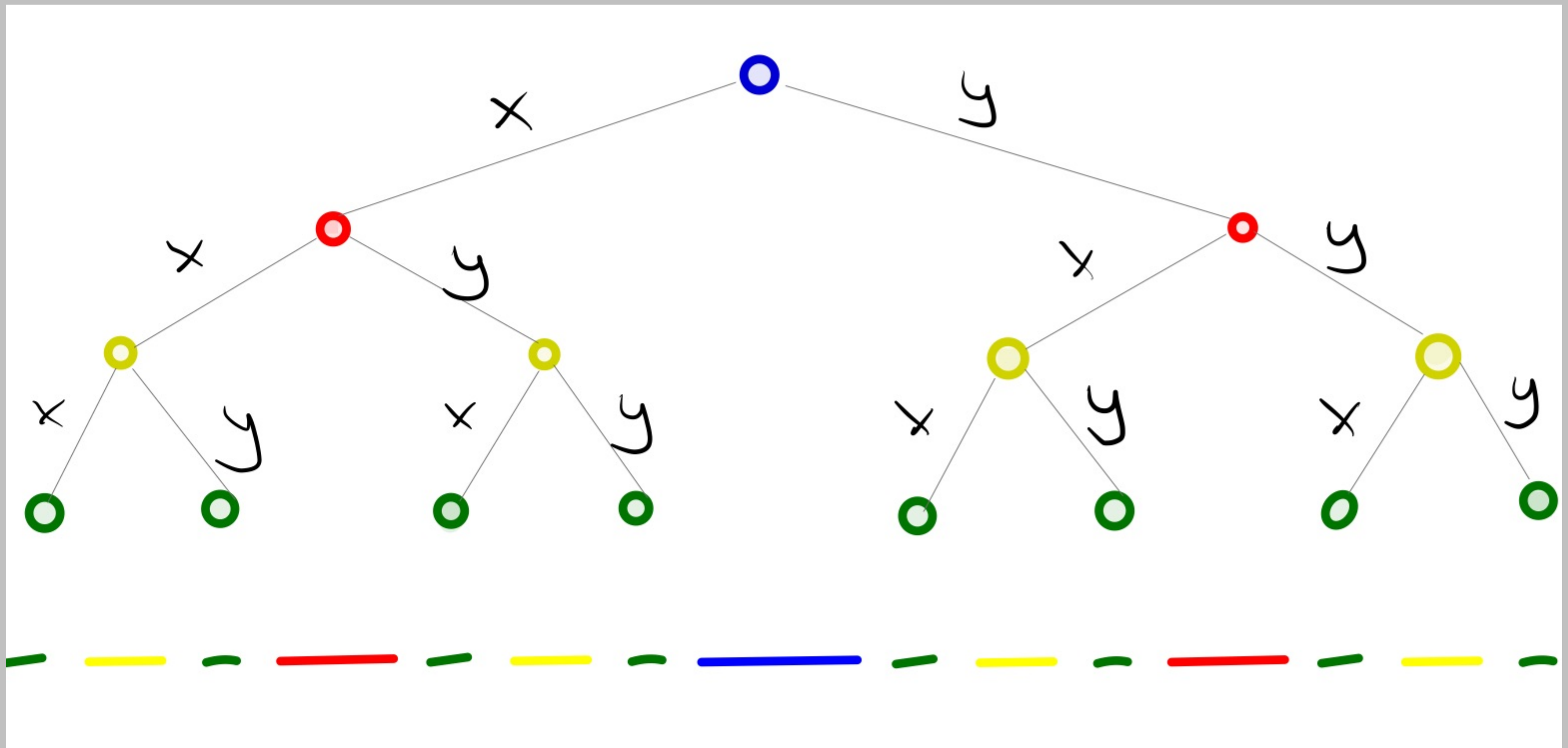


We can mutate from the x, v, or y corner





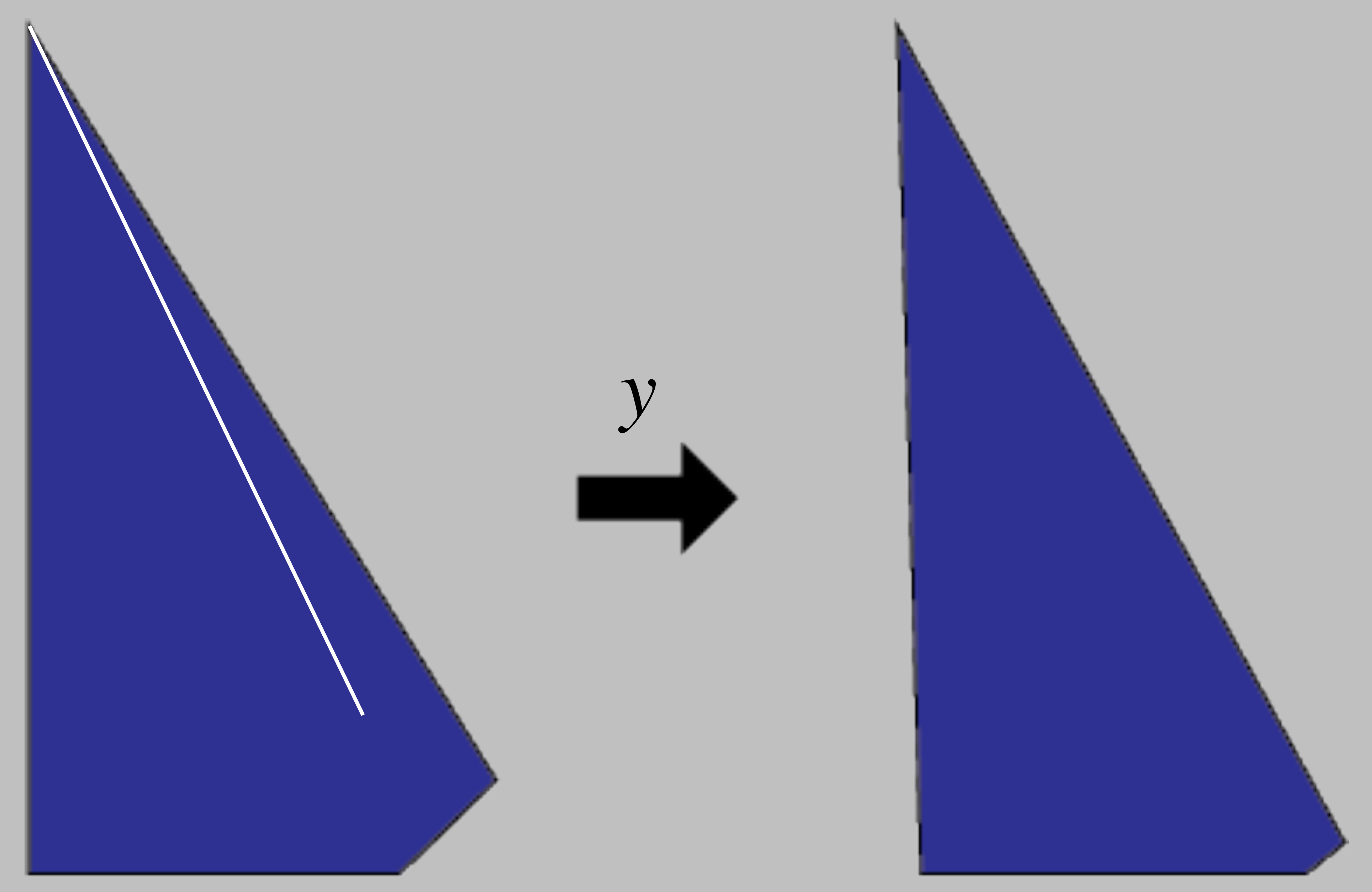
x
y
x
y
x
y
x
y
x
y
x
y
x
y

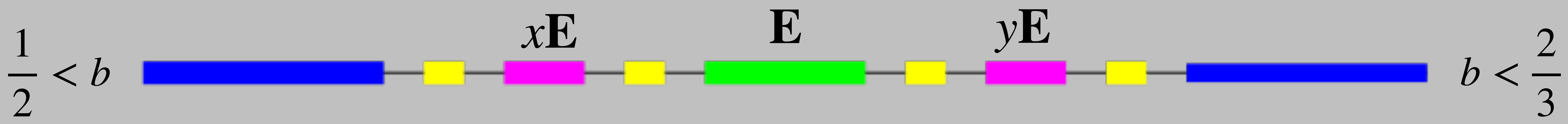


For an interval I in $(\frac{n}{n+1}, \frac{n+1}{n+2})$ to construct the embedding for the left endpoint of I,

- Perform $n+2$ mutations by v.
- Perform the mutations to get to the vertex in the graph corresponding to I.

Then, perform consecutive mutations by y.





- Each interval is described by a homology class $\mathbf{E} \in H_2(\mathbb{C}P^2 \#_k \overline{\mathbb{C}P}^2, \mathbb{Z})$ representing a symplectic sphere of self intersection -1.

- In [MMW], we constructed a mutation process on the homology classes $x\mathbf{E}$ and $y\mathbf{E}$.

- If we have an interval I, the classes that give the steps of the staircase that accumulate at the left endpoint of the interval are given by $y^k\mathbf{E}_\mu$ for some $\mathbf{E}_\mu$.

The Proof

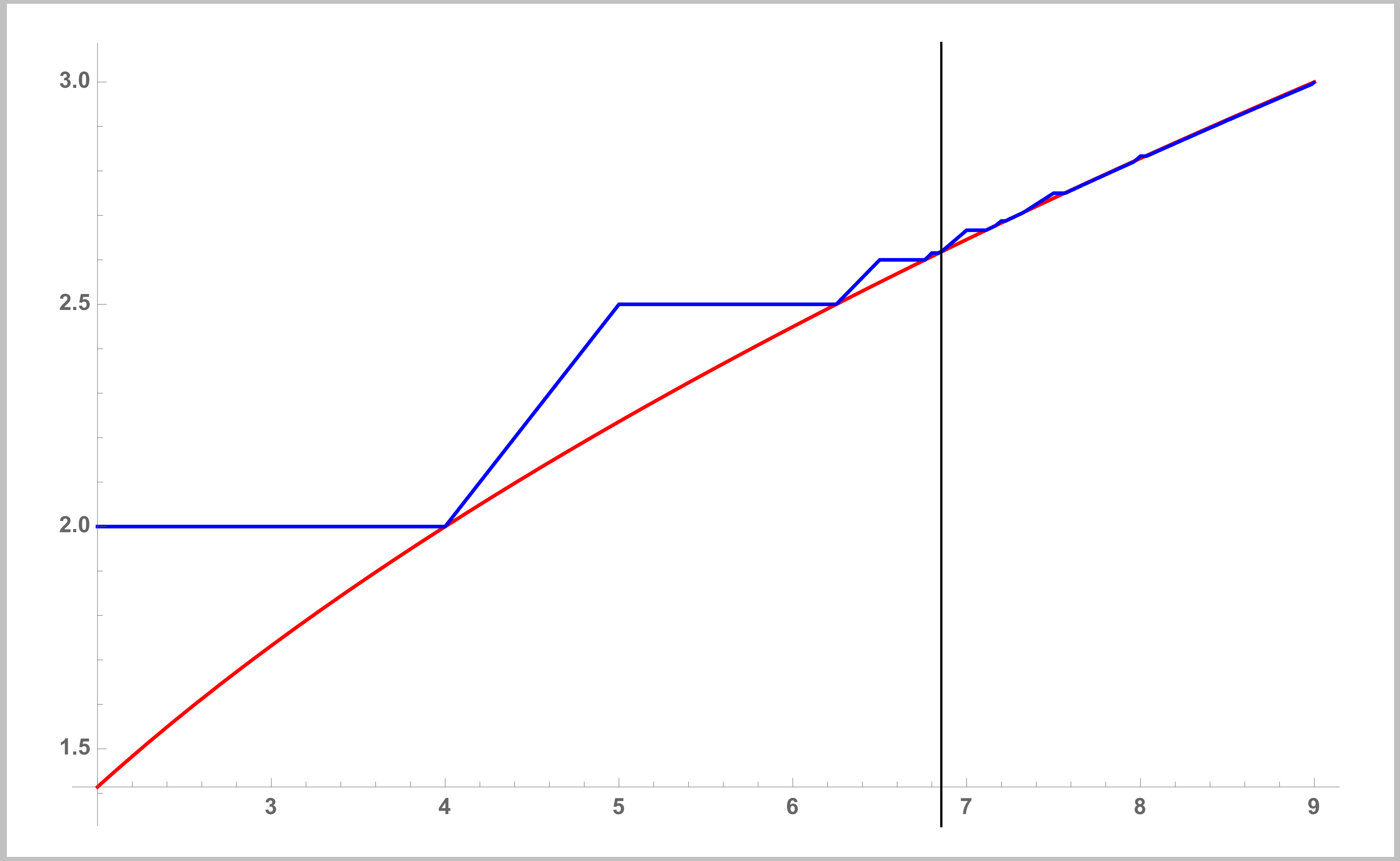
Look at mutations of the from $y^k w v^{2n+1} Q$ where w is any finite word of x, y and Q is the moment polygon of H_b.

Use the numerics of the classes $\mathbf{E}$ and the mutation process of the classes $\mathbf{E}$ to find nice formulas for the result of each mutation.

ATFs expected to compute all embeddings

The sequences of mutations we considered for each specific b only construct an optimal embedding at accumulation point.

Preliminary evidence suggests different sequence of mutations will compute all embeddings on the embedding function before the accumulation point (i.e. for increasing staircases)



Main Result

M., McDuff, Weiler

- Block is a disjoint union of open intervals that is dense in [0,1)
- For each n, Block $\cap\ (\frac{n}{n+1}, \frac{n+1}{n+2})$ is homeomorphic to the complement of the middle third Cantor set.
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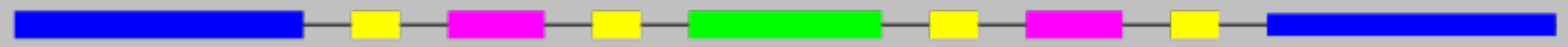
$\frac{1}{2} < b$  $b < \frac{2}{3}$