An introduction to the logarithmic Laplacian: what it is and what are some of its applications

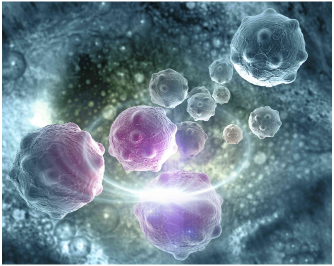
Alberto Saldaña

Lisbon WADE

May, 2022.

Motivation

Diffusion with jumps



The role of the parameter s to model diffusion strategies

Some questions

How to study the s-dependency?

For example:

- What can be said about the map $s \mapsto (-\Delta)^s \varphi$?
- If u_s is a solution of

$$(-\Delta)^s u_s = f,$$

  What can be said about the solution map $s \mapsto u_s$?, Is it continuous?, differentiable?, if so, how is the derivative $v_s := \partial_s u_s$ analyzed?
- How is this influenced by a nonlinear component?, for instance, if u_s is a solution of

$$(-\Delta)^s u_s = |u_s|^{p_s - 2} u_s,$$

  What can be said about the map $s \mapsto u_s$?, What happens as $s \to 0^+$?

What can be said about

$$s \mapsto (-\Delta)^s \varphi?$$

The logarithmic Laplacian: a pseudodifferential operator

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\xi \in \mathbb{R}^N \backslash \{0\}$, then

$$\widehat{(-\Delta)^s \varphi}(\xi) = |\xi|^{2s}\widehat{\varphi}(\xi) = (1 + s\ln(|\xi|^2) + o(s))\widehat{\varphi}(\xi)$$
$$= \widehat{\varphi}(\xi) + s\ln(|\xi|^2)\widehat{\varphi}(\xi) + o(s)$$

as $s \to 0^+$. Let

$$\widehat{\ln(-\Delta)\varphi}(\xi) = \ln(|\xi|^2)\widehat{\varphi}(\xi).$$

Other notations

$$\ln(-\Delta), \quad \log(-\Delta), \quad L_\Delta, \quad (-\Delta)^{\log}, \quad (-\Delta)^L.$$

An integrodifferential operator

$$L_\Delta \varphi(x) = \mathcal{F}^{-1}(\ln(|\cdot|^2)\widehat{\varphi})(x).$$

(Chen, Weth; 2019)

$$L_\Delta \varphi(x) = c_N \int_{B_1(x)} \frac{\varphi(x) - \varphi(y)}{|y|^N}\, dy - c_N \int_{\mathbb{R}^N \setminus B_1(x)} \frac{\varphi(y)}{|y|^N}\, dy + \rho_N \varphi(x),$$

where $c_N = \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2}) > 0$, $\rho_N = 2\ln 2 + \Gamma'(\frac{N}{2})/\Gamma(\frac{N}{2}) + \Gamma'(1) \in \mathbb{R}$.

Derivative of $(-\Delta)^s$

$$L_\Delta \varphi(x) = c_N \int_{B_1(x)} \frac{\varphi(x)-\varphi(y)}{|y|^N}\,dy - c_N \int_{\mathbb{R}^N \setminus B_1(x)} \frac{\varphi(y)}{|y|^N}\,dy + \rho_N \varphi(x),$$

(Chen, Weth; 2019)

For $1 < p \leq \infty$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\lim_{s\to 0^+} \left\| \frac{(-\Delta)^s \varphi - \varphi}{s} - L_\Delta \varphi \right\|_p = 0.$$

Namely,

$$L_\Delta \varphi(x) = \frac{d}{ds}\Big|_{s=0} (-\Delta)^s \varphi(x).$$

Variational structure

$$L_\Delta \varphi(x) = c_N \int_{B_1(x)} \frac{\varphi(x)-\varphi(y)}{|y|^N}\,dy - c_N \int_{\mathbb{R}^N\setminus B_1(x)} \frac{\varphi(y)}{|y|^N}\,dy + \rho_N \varphi(x),$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let

$$\mathcal{E}(\varphi,\psi) := \frac{c_N}{2}\int_{\mathbb{R}^N}\int_{B_1(x)} \frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^N}\,dy\,dx.$$

Then,

$$\mathbb{H}(\Omega) := \left\{\varphi \in L^2(\Omega) \;:\; \mathcal{E}(\varphi,\varphi) < \infty,\ \varphi = 0 \text{ in } \mathbb{R}^N\setminus\Omega\right\}$$

is a Hilbert space with scalar product $\mathcal{E}(\cdot,\cdot)$. Let

$$\mathcal{E}_L(u,v) := \mathcal{E}(\varphi,\psi) - c_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N\setminus B_1(x)} \frac{\varphi(x)\psi(y)}{|x-y|^N}\,dy\,dx + \rho_N \int_{\mathbb{R}^N} \varphi(y)\psi(y)\,dy.$$

Weak solutions

$$L_\Delta u = f \qquad \iff \qquad \mathcal{E}_L(u,\varphi) = \int_\Omega f\varphi\,dx \quad \forall \varphi \in C_c^\infty(\Omega),$$

$$L_\Delta u = \lambda u \qquad \iff \qquad \mathcal{E}_L(u,\varphi) = \lambda\int_\Omega u\varphi\,dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

(Chen, Weth; 2019) There is a sequence of eigenvalues $\lambda_1^L < \lambda_2^L \leq \ldots \leq \lambda_k^L \leq \ldots$ and eigenfunctions $\xi_k^L \in \mathbb{H}(\Omega)$ such that

- $\{\xi_k^L\}$ is an orthonormal basis of $L^2(\Omega)$,
- $\xi_1^L > 0$,

Eigenfunctions

$$L_\Delta \xi_1^L = \lambda_1^L \xi_1^L$$

(Chen, Weth; 2019) If ξ_1^s is the first eigenfunction of $(-\Delta)^s$ and

$$(-\Delta)^s \xi_1^s = \lambda_1^s \xi_1^s,$$

then $\lim_{s\to 0^+} \lambda_1^s = 1$,

- $\lambda_1^L = \frac{d}{ds}\Big|_{s=0} \lambda_1^s$,
- $\xi_1^s \to \xi_1^L$ as $s \to 0^+$.

Maximum principle

$$L_\Delta \xi_1^L = \lambda_1^L \xi_1^L$$

(Chen, Weth; 2019)

- The operator L_Δ satisfies the maximum principle if and only if $\lambda_1^L > 0$.
- $\lambda_1^L \leq \ln(\lambda_1)$, where $\lambda_1 = \lambda_1(\Omega) > 0$ is the first eigenvalue of the Laplacian.
- $\lambda_1^L \leq \ln(\lambda_1) < 0$ if $\lambda_1 < 1$ (for Ω big, for instance).

Regularity

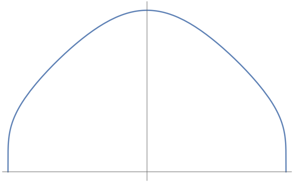
$$L_\Delta u = f$$

(Kassman, Mimica; 2013 - Chen, Weth; 2019)

- $f \in L^\infty(\Omega)$ implies that $u \in C(\overline{\Omega})$ and

$$|u(x)| \leq C_\tau \frac{1}{(-\ln \rho(x))^\tau} \qquad \text{as } \rho(x) = \text{dist}(x, \partial\Omega) \to 0$$

  for all $\tau \in (0, \frac{1}{2})$.



If u_s is a solution of

$$(-\Delta)^s u_s = f,$$

what can be said of $s \mapsto u_s$?

Differentiability of $s \mapsto u_s$

Let $z > 0$, $s \in [0,1]$, and $h \in C^1(0,1)$, then

$$\frac{d}{ds}(z^s h(s)) = \ln(z) z^s h(s) + z^s h'(s).$$

If

$$(-\Delta)^s u_s = f \quad \text{en } \Omega, \qquad u_s = 0 \quad \text{on } \mathbb{R}^N \backslash \Omega,$$

then $s \mapsto u_s$ is differentiable in $(0,1)$ (Jarohs, S., Weth; 2020) and

$$0 = \frac{d}{ds}\Big((-\Delta)^s u_s\Big) = \ln(-\Delta)\Big((-\Delta)^s u_s\Big) + (-\Delta)^s\Big(\frac{d}{ds} u_s\Big)$$

Then $v_s := \frac{d}{ds} u_s$ is the solution of

$$(-\Delta)^s v_s = -L_\Delta\Big((-\Delta)^s u_s\Big) \quad \text{in } \Omega, \qquad v_s = 0 \quad \text{on } \partial\Omega.$$

See also (Burkovska, Gunzburger, 2020).

Some results for $s \in [0,1]$

$v_s = \frac{d}{ds} u_s$, $u_s = v_s = 0$ in $\mathbb{R}^N \backslash \Omega$,

$$(-\Delta)^s u_s = f \geq 0 \quad \text{in } \Omega, \qquad (-\Delta)^s v_s = -L_\Delta \Big((-\Delta)^s u_s \Big) \quad \text{in } \Omega.$$

(Jarohs, S., Weth; 2020)

- $s \mapsto u_s(x)$ is decreasing in $[0,1]$ for all $x \in \Omega \iff L_\Delta f \geq 0$ in Ω.
- if $f \equiv 1$ $\exists\, r_N > 0 : \Omega \subset B_{r_N}(0) \implies s \mapsto u_s$ is decreasing for $s \in [0,1]$.
- if $f \equiv 1$ and $B_{r_N}(0) \subset \Omega$, then $s \mapsto u_s(0)$ is not decreasing in $[0,1]$.

Note: $\frac{r_N}{\sqrt{N}} \to c > 0$ as $N \to \infty$.

If u_s is a solution of

$$(-\Delta)^s u_s = |u_s|^{p_s-2}u_s,$$

What can be said about $s \mapsto u_s$?,
What happens when $s \to 0^+$?

A superlinear subcritical problem

Let $p \in C^1([0, \frac{1}{4}])$ be such that

$$2 < p(s) < 2_s^* := \frac{2N}{N-2s} \quad \text{para } s \in (0, \tfrac{1}{4}), \qquad p'(0) \notin \{0, \frac{4}{N}\}.$$

Let $p_s := p(s)$ and let u_s be the solution of

$$(-\Delta)^s u_s = |u_s|^{p_s - 2} u_s \quad \text{in } \Omega, \qquad u_s = 0 \text{ on } \mathbb{R}^N \backslash \Omega.$$

(S., Hernández Santamaría; 2022)

Let $s_k \to 0$ as $k \to \infty$ and let u_{s_k} be a least-energy solution of

$$(-\Delta)^{s_k} u_{s_k} = |u_{s_k}|^{p_{s_k}-2} u_{s_k} \quad \text{in } \Omega, \qquad u_{s_k} = 0 \text{ on } \mathbb{R}^N \backslash \Omega.$$

Then there is $u_0 \in \mathbb{H}(\Omega)\backslash\{0\}$ such that (passing to a subsequence)

$$u_{s_k} \to u_0 \qquad \text{en } L^2(\Omega)$$

and u_0 is a least-energy solution of

$$L_\Delta u_0 = p'(0) \ln(|u_0|) u_0 \quad \text{in } \Omega, \qquad u_0 = 0 \quad \text{on } \mathbb{R}^N \backslash \Omega.$$

Some difficulties

- Lack of compactness (no embedding such as $\mathcal{H}_0^s(\Omega) \hookrightarrow L^{p_s}(\Omega)$).
- It only holds that $\mathbb{H}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, but this is not enough.
- An important tool: the logarithmic Sobolev inequality.

A proof of the logarithmic Sobolev inequality

Step 1: Consider the fractional Sobolev inequality

$$|u|_{2_s^*}^2 \leq \kappa_{N,s} \|u\|_s^2 \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^N),$$

where $s \in (0, \frac{N}{2})$, $2_s^* := \frac{2N}{N-2s}$,

$$|u|_{2_s^*}^2 = \left(\int_{\mathbb{R}^N} |u|^{2_s^*} \, dx \right)^{\frac{2}{2_s^*}},$$

$$\|u\|_s^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi,$$

$$\kappa_{N,s} = 2^{-2s} \pi^{-s} \frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})} \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right)^{\frac{2s}{N}}.$$

A proof of the logarithmic Sobolev inequality

Step 1: Consider the fractional Sobolev inequality

$$|u|_{2_s^*}^2 \le \kappa_{N,s} \|u\|_s^2 \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^N),$$

where $s \in (0, \frac{N}{2})$, $2_s^* := \frac{2N}{N-2s} = 2 + s\frac{4}{N} + o(s)$,

$$|u|_{2_s^*}^2 = \left(\int_{\mathbb{R}^N} |u|^{2_s^*}\, dx\right)^{\frac{2}{2_s^*}}$$

$$= \int_{\mathbb{R}^N} |u|^2\, dx + s\frac{4}{N}\left(\int |u|^2 \ln|u|\, dx - \int_{\mathbb{R}^N} |u|^2\, dx \ln\left(\int_{\mathbb{R}^N} |u|^2\, dx\right)\right) + o(s),$$

$$\|u\|_s^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2\, d\xi$$

$$= \int |u|^2\, dx + s\int_{\mathbb{R}^N} \ln(|\xi|^2)|\widehat{u}(\xi)|^2\, d\xi + o(s),$$

$$\kappa_{N,s} = 2^{-2s}\pi^{-s}\frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})}\left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}\right)^{\frac{2s}{N}}$$

$$= (1 + s\, a_N + o(s))$$

for some $a_N \in \mathbb{R}$ as $s \to 0^+$.

A proof of the logarithmic Sobolev inequality

Step 2: An first order expansion on s:

$$|u|_{2_s^*}^2 \leq \kappa_{N,s}\|u\|_s^2 \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^N),$$

becomes

$$\int_{\mathbb{R}^N} |u|^2\,dx + s\frac{4}{N}\left(\int_{\mathbb{R}^N} |u|^2 \ln|u|\,dx - \int_{\mathbb{R}^N} |u|^2\,dx \ln\left(\int_{\mathbb{R}^N} |u|^2\,dx\right)\right) + o(s)$$
$$\leq (1 + s\,a_N + o(s))\left(\int_{\mathbb{R}^N} |u|^2\,dx + s\int_{\mathbb{R}^N} \ln(|\xi|^2)|\widehat{u}(\xi)|^2\,d\xi + o(s)\right)$$
$$= \int |u|^2\,dx + s\left(a_N\int_{\mathbb{R}^N} |u|^2\,dx + \int_{\mathbb{R}^N} \ln(|\xi|^2)|\widehat{u}(\xi)|^2\,d\xi\right) + o(s)$$

Step 3: Simplify:

$$\frac{4}{N}\left(\int_{\mathbb{R}^N}|u|^2\ln|u|\,dx-|u|_2^2\ln|u|_2^2\right)\leq a_N|u|_2^2+\int_{\mathbb{R}^N}\ln(|\xi|^2)|\widehat{u}(\xi)|^2\,d\xi+o(s),$$

where

$$|u|_2^2=\int_{\mathbb{R}^N}|u|^2\,dx.$$

A proof of the logarithmic Sobolev inequality

Step 4: Let $s \to 0^+$:

$$\frac{2}{N}\int_{\mathbb{R}^N} |u|^2 \ln(|u|^2)\, dx \leq \int_{\mathbb{R}^N} \ln(|\xi|^2)|\widehat{u}(\xi)|^2\, d\xi + \frac{2}{N}|u|_2^2 \ln|u|_2^2 + a_N|u|_2^2,$$

or

$$\frac{2}{N}\int_{\mathbb{R}^N} |u|^2 \ln(|u|^2)\, dx \leq \mathcal{E}_L(u,u) + \frac{2}{N}|u|_2^2 \ln|u|_2^2 + a_N|u|_2^2.$$

This is the Sobolev logarithmic inequality with optimal constant (Beckner, 1995).

The Nehari manifold method

$$L_\Delta u = \mu \ln(|u|)u, \qquad \mu \in (0, \tfrac{4}{N}).$$

The energy functional $J : \mathbb{H}(\Omega) \to \mathbb{R}$ is given by

$$J(u) := \frac{1}{2}\mathcal{E}_L(u,u) - \frac{\mu}{4}\int_\Omega u^2(\ln(|u|^2) - 1).$$

The Nehari manifold is

$$\mathcal{N} = \left\{ u \in \mathbb{H}(\Omega)\backslash\{0\} \ : \ \mathcal{E}_L(u,u) = \mu \int_\Omega \ln(|u|)u^2 \right\}$$

Given $\varphi \in C_c^\infty(\Omega)\backslash\{0\}$,

$$t_\varphi \varphi \in \mathcal{N}, \qquad \text{where } t_\varphi := \exp\left(\frac{\mathcal{E}_L(\varphi,\varphi) - \mu \int_\Omega \ln|\varphi|\varphi^2}{\mu|\varphi|_2^2} \right)$$

$$L_\Delta u = \mu \ln(|u|)u \qquad \Longleftrightarrow \qquad J(u) = \inf_{\mathcal{N}} J.$$

If $u \in \mathcal{N}$, then

$$J(u) := \frac{1}{2}\mathcal{E}_L(u,u) - \frac{\mu}{4}\int_\Omega u^2(\ln(|u|^2) - 1) = \frac{\mu}{4}\int_\Omega u^2.$$

Proposition

Let $(u_n) \subset \mathcal{N}_0$ *be such that* $\sup_{n\in\mathbb{N}} J(u_n) \le C$. *Then* (u_n) *is bounded in* $\mathbb{H}(\Omega)$.

$$J(u_n) = \frac{\mu}{4}|u_n|_2^2 \implies \sup_{n\in\mathbb{N}} |u_n|_2^2 \le \frac{4}{\mu}C =: C_1.$$

By the Sobolev logarithmic inequality,

$$\begin{aligned} 2C > 2J(u_n) &= \mathcal{E}_L(u,u) - \frac{\mu}{2}\int_\Omega u^2 \ln(|u|^2) + \frac{\mu}{2}\int_\Omega u^2 \\ &\ge \left(1 - \frac{N\mu}{4}\right)\mathcal{E}_L(u_n,u_n) - \frac{\mu}{2}\ln(|u_n|_2^2)|u_n|_2^2 + \left(\frac{\mu}{2} - a_N\frac{N\mu}{4}\right)|u_n|_2^2. \end{aligned}$$

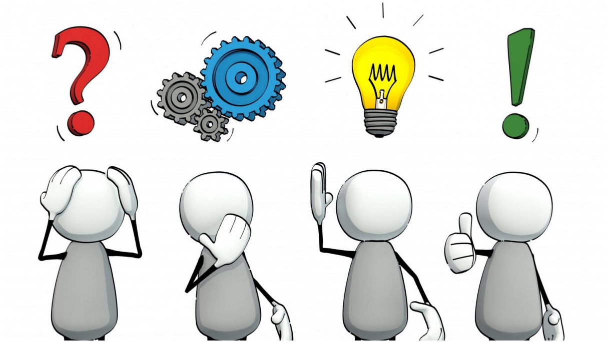
Then,

$$\sup_{n\in\mathbb{N}} \mathcal{E}_L(u_n,u_n) \le \widetilde{C}\left(1 + \sup_{t\in[0,C_1]}(|\ln(t)|+1)t\right) =: C_2.$$

Thus,

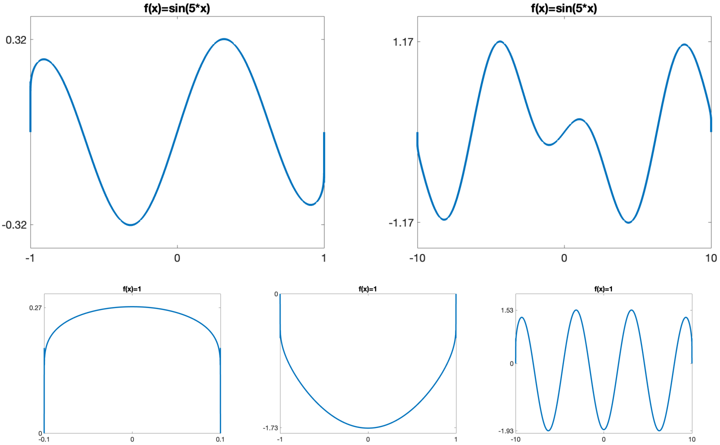
$$\begin{aligned} C_2 \ge \mathcal{E}_L(u_n,u_n) &= \|u_n\|_{\mathbb{H}(\Omega)}^2 - c_N\int_{\mathbb{R}^N}\int_{\mathbb{R}^N\setminus B_1(x)} \frac{u_n(x)u_n(y)}{|x-y|^N}\,dy\,dx + \rho_N|u_n|_2^2 \\ &\ge \|u_n\|_{\mathbb{H}(\Omega)}^2 - (|\Omega| + |\rho_N|)C_1. \end{aligned}$$

Work in progress



Existence of classical solutions (with Héctor Chang-Lara - CIMAT)

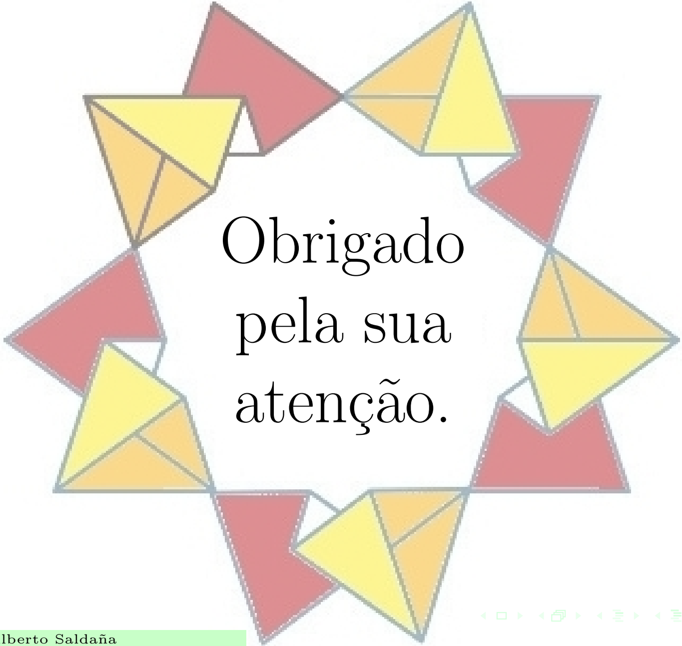
Numerical approximation of solutions (with Víctor Hernández-Santamaría - IMUNAM)


f(x)=sin(5*x)
0.32
-0.32
-1
0
1
f(x)=sin(5*x)
1.17
-1.17
-10
0
10
f(x)=1
0.27
0
-0.1
0
0.1
f(x)=1
0
-1.73
-1
0
1
f(x)=1
1.53
0
-1.93
-10
0
10

Asymptotic analysis in sublinear-type problems (with Felipe Angeles García - IMUNAM)

Some references (all available in arXiv)

- Chen, Huyuan, Tobias Weth. **"The Dirichlet problem for the logarithmic Laplacian."** Communications in Partial Differential Equations 44.11 (2019): 1100-1139.
- Jarohs, Sven, A.S., Tobias Weth. **"A new look at the fractional Poisson problem via the logarithmic Laplacian."** Journal of Functional Analysis 279.11 (2020): 108732.
- Laptev, Ari, Tobias Weth. **"Spectral properties of the logarithmic Laplacian."** Analysis and Mathematical Physics 11.3 (2021): 1-24.
- Hernández-Santamaría, Víctor, A.S. **"Small order asymptotics for nonlinear fractional problems."** To appear in Calculus of Variations and Partial Differential Equations. (2022)



Obrigado pela sua atenção.