Characterization of Quasi-Abelian Surfaces

Joint Work with M. Mendes Lopes and R. Pardini

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Plan of the talk

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Section 1
The classification problem

The world of projective varieties

Projective variety = closed, irreducible subset $X \subseteq \mathbb{P}^n$ with the Zarsiki topology.

The dream

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Classification up to what?

A morphism $f : X \to Y$ of projective varieties is a map that locally around any point p can be written as

$$\left(\frac{g_1(x_1, \dots, x_n)}{h_1(x_1, \dots, x_n)}, \dots, \frac{g_N(x_1, \dots, x_n)}{h_N(x_1, \dots, x_n)} \right)$$

with g_i and h_i homogeneous polynomials of the same degree, with h_i non vanishing around p. An isomorphism is a bijective morphism whose inverse is a morphism

We say that two varieties are birationally equivalent if they contain isomorphic non empty open sets

The invariants

On a smooth projective variety X of dimension n we can define the vector bundle of regular differential forms:

$$\Omega_X$$

It has rank n so its n-th exterior power is a line bundle, called canonical line bundle and denoted by

$$\omega_X := \bigwedge^n \Omega_X,$$

By taking cohomology we get important invariants:

- $q(X) := h^0(X, \Omega_X)$ is the irregularity of X
- $P_m(X) := h^0(X, \omega_X^{\otimes m})$ is the m-th plurigenus of X.

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These are birational invariants!!!

The Kodaira dimension

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Suppose that L is a line bundle such that $L^{\otimes m}$ has sections for m sufficiently large and divisible. Then $L^{\otimes m}$ induces a rational map (a morphism defined on a open set)

$$\varphi_{L^{\otimes m}} : X \dashrightarrow \mathbb{P}^N$$

The Kodaira dimension

Stockholm University

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The dimension of the image of $\varphi_{L^{\otimes m}}$ stabilizes. We call this stable value the Iitaka dimension of L, and we denote it by

$$\kappa(X, L),$$

If $h^0(X, L^{\otimes m}) = 0$ for every $m \geq 0$ we say that

$$\kappa(X, L) = -\infty,$$

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Special varieties

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- $\mathbb{P}^n$. We have that $q(\mathbb{P}^n) = 0$, $P_m(\mathbb{P}^n) = 0$ if $m > 0$ and $\kappa(\mathbb{P}^n) = -\infty$.
  A variety that is birational equivalent to $\mathbb{P}^n$ is rational.

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If the base field is $\mathbb{C}$ they are quotient of $\mathbb{C}^n/\Lambda$ with Λ a finetely generated free abelian group of maximal rank.

Constructed by Abel ⇝ to solve elliptic integrals

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In particular

$$q(A) = \dim A, \quad P_m(A) = 1 \text{ for every } m, \quad \text{and} \quad \kappa(A) = 0$$

Characterization

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Castelnuovo criterion

A smooth complex surface S is rational if, and only if,

$$q(S) = P_2(S) = 0,$$

Kawamata 1981

A smooth complex projective variety X is birationally equivalent to an abelian variety if, and only if

$$\kappa(X) = 0, \quad \text{and} \quad q(X) = \dim X,$$

Chen–Hacon 2001

A smooth complex projective variety X is birationally equivalent to an abelian variety if, and only if

$$P_1(X) = P_2(X) = 1, \quad \text{and} \quad q(X) = \dim X,$$

If $\boxed{k(X) = 0}$ $\iff$

$P_m(X) = 1$ for $m >> 0$ & enough divisible

Type 1 bielliptic surface

$h^0(X, \omega_X) = 0$ $\quad P_{2m}(X) = 1$

$h^0(X, \omega_X^{\otimes 2}) = 1$ $\quad P_{2m+1}(X) = 0$

Section 2
Iitaka's philosophy

A tale of two worlds

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Projective World

X smooth **complex** projective variety of dimension n

A tale of two worlds

Projective World

X smooth **complex** projective variety of dimension n

Quasi-Projective World

V smooth quasi-projective variety of dimension n

$\exists X$ smooth proj such that $V \simeq$ open set of X

snc "Every irreducible cp is smooth
and components meet eachothe
tranversally"

A tale of two worlds

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Projective World

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Quasi-Projective World

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$\Updownarrow$

(X, D) with D a *snc* divisor

$V \hookrightarrow X$ proj

$X \setminus V$ pure codim 1

"divisor"

A tale of two worlds

Projective World	**Quasi-Projective World**
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Ω_X	
ω_X	

A tale of two worlds

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(X, D) with D a *snc* divisor

$\Omega_X(\log D)$

the sheaf of logarithmic 1-forms

$\Omega_X(\log D)_p$

$x_1 \ldots x_n$ local system of coordinates near p

st $D = \{x_1 \ldots x_s = 0\}$

$$\Omega_X(\log D)_p = \sum_{i=0}^{s} \mathcal{O}_{X,p} \frac{dx_i}{x_i} + \sum_{i=s+1}^{n} \mathcal{O}_{X,p} \frac{dx_i}{x_i}$$

"Logarithmic poles along D"

A tale of two worlds

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Projective World

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Ω_X

ω_X

Quasi-Projective World

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$\Updownarrow$

(X, D) with D a *snc* divisor

$\Omega_X(\log D)$ vector bundle of rank n

the sheaf of logarithmic 1-forms

$\bigwedge^n \Omega_X(\log D) \simeq \mathcal{O}_X(K_X + D)$
the log-canonical sheaf

Logarithmic invariants

Projective World	Quasi-Projective World
X smooth projective	V smooth quasi-projective $\Leftrightarrow$ (X, D)

Logarithmic invariants

Projective World	Quasi-Projective World
X smooth projective	V smooth quasi-projective $\Leftrightarrow$ (X, D)
$q(X)$	
$P_m(X)$	
$\kappa(X)$	

Logarithmic invariants

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Projective World	Quasi-Projective World
X smooth projective	V smooth quasi-projective $\Leftrightarrow$ (X, D)
$q(X)$	$\overline{q}(V) := h^0(X, \Omega_X(\log D))$ the log-irregularity
$P_m(X)$	
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$\kappa(X)$	$\overline{\kappa}(V)$ the log-Kodaira dimension

$\hookrightarrow \kappa(X, \mathcal{O}_X(K_X+D))$

Logarithmic invariants

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$\kappa(X)$	$\overline{\kappa}(V)$ the log-Kodaira dimension

- The logarithmic invariants do not depend on the compactification X.
- They are not birational invariants.

$\mathbb{P}^1 \setminus \{3\,pt\} \sim_{bir} \mathbb{P}^1$

$\overline{P}_1(V) = 2$ $\overline{q}(V) = 2$ $\overline{\kappa}(V) = 1$

0 0 $-\infty$

Iitaka's Philosophy

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To any statement in the projective world that is dictated by the behavior of regular forms there should be a corresponding statement in the quasi-projective world dictated by the behavior of logarithmic forms.

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To any statement in the projective world that is dictated by the behavior of regular forms there should be a corresponding statement in the quasi-projective world dictated by the behavior of logarithmic forms.

Log-Castelnuovo Criterion (Zhu 2014)

The following statement are equivalent for a smooth quasi-projective surface V:

1. V is log-rationally connected;
2. $h^0(X, \Omega_X(\log D)^{\otimes m}) = 0$ for any $m \geq 1$;
3. $\overline{\kappa}(V) = -\infty$ and $h^0(X, S^{12}\Omega_X(\log D)) = 0$,

Log-rationally connected = every 2 points lie on a curve of log-genus 0, that is either a $\mathbb{P}^1$ or an $\mathbb{A}^1$

Today's topic

We look for a quasi-projective analogue of the following statement:

Theorem (Enriques 1905, Chen–Hacon 2001)

Let S be a smooth complex projective surface such that

$$P_1(S) = P_2(S) = 1, \quad q(S) = 2.$$

4 (Enriques)

Then S is birationally equivalent to an abelian surface.

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Then S is birationally equivalent to an abelian surface.

↳ via the Albanese map constructed by integrating 1-forms over 1 cycle.

Questions:

- Why is this a good candidate for Iitaka's philosophy? ✓
- What is the right notion of equivalence to consider? WWPB equivalence
- What is the analogue of an abelian variety?

Section 3
Quasi-abelian varieties

The tale continues

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Projective World

Quasi-Projective World

A abelian variety
$A \simeq \mathbb{C}^n/\Lambda$ with Λ a free abelian group of rank $2n$

$$\Omega_A \simeq \mathcal{O}_A^{\oplus n}$$

$$\omega_A \simeq \mathcal{O}_A$$

$\kappa(A) = 0$
$P_m(A) = 1$ for every $m \geq 0$
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The tale continues

Stockholm University

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Quasi-Projective World

algebraic group
G quasi-abelian variety

$$0 \to \mathbb{G}_m^r \to G \to A \to 0$$

$$\mathbb{G}_m^r \hookrightarrow \mathbb{P}^r, \quad G \hookrightarrow Z, \quad Z \to A \text{ Abelian}$$

$$\mathbb{P}^r \to Z$$

$$\Delta = Z \setminus G \quad \text{snc}$$

The tale continues

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Quasi-Projective World

G quasi-abelian variety
$G \simeq \mathbb{C}^n/L$ with Λ a free abelian group of rank $\leq 2n$

$L = \pi_1(G)$

The tale continues

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$$\Omega_Z(\log \Delta) \simeq \mathcal{O}_Z^{\oplus n}$$

The tale continues

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The quasi-Albanese variety

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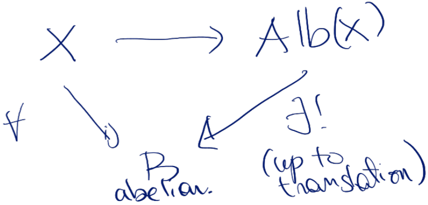
Projective World

Given a smooth projective variety X, there is an abelian variety $\mathrm{Alb}(X)$, and a morphism $a_X : X \to \mathrm{Alb}(X)$ satisfying the obvious universal property. We call the pair $(\mathrm{Alb}(X), a_X)$ the Albanese variety of X.



Quasi-Projective World

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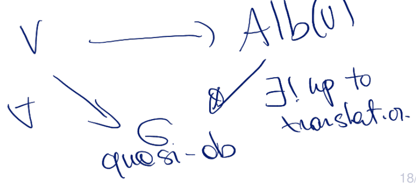

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Remark

In both cases it is constructed by integrating 1-forms over 1-cycles. It still fit Iitaka's Philosophy

Section 4
Characterization of quasi-abelian varieties

Iitaka's characterization

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Theorem (Iitaka 1979)

Let V be a smooth complex quasi-projective surface satisfying

$$\overline{\kappa}(V) = 0, \quad \overline{q}(V) = 2.$$

Then $a_V : V \to \mathrm{Alb}(V)$ is birational. Furthermore there are finitely many points $p_1, \ldots p_k$ in $\mathrm{Alb}(V)$, and an open set $V_0 \subseteq V$ such that $a_{V|_{V_0}} : V_0 \to \mathrm{Alb}(V) \setminus \{p_1, \ldots, p_k\}$ is proper.

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- In the language of Iitaka, a_V is a WWPB equivalence.

Weakly Weak Proper Birational.

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- In the language of Iitaka, a_V is a WWPB equivalence.
  WWPB= "Weakly Weak Proper Birational"
- WWPB equivalences preserve the logarithmic invariants.
- WWPB equivalences between affine varieties are isomorphisms.

Main Result

Theorem (Mendes Lopes, Pardini,)

Let V be a smooth complex quasi-projective surface with $\overline{q}(V) = 2$. Assume that either one of the following hold:

1. $\overline{P}_1(V) = \overline{P}_2(V) = 1$, and $q(X) > 0$;
2. $\overline{P}_1(V) = \overline{P}_3(V) = 1$, and $q(X) = 0$.

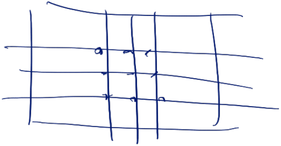
Then $a_V : V \to \mathrm{Alb}(V)$ is a WWPB equivalence.

Corollary

If V is an affine surface with $\overline{P}_1(V) = \overline{P}_3(V) = 1$ and $\overline{q}(V) = 2$, then V is isomorphic to $\mathbb{G}_m^2$

$X \simeq \mathbb{P}^2$

Example



Let X' be a $\mathbb{Z}/3$-cyclic cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on 3 fibers of each fibration.

Then a (minimal) smooth model of X' is an elliptic $K3$ surface such that the fibration $X \to \mathbb{P}^1$ has 3 fibers of type IV*, F_1, F_2, and F_3.

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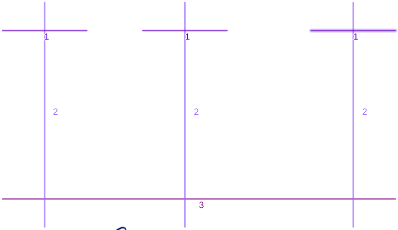
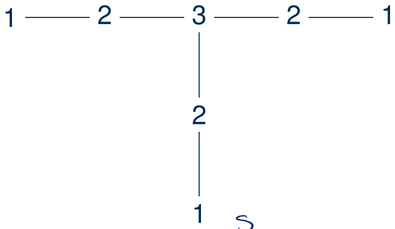
How they look in nature



Dual Graph



Rmk F_i^s is the support of F_i

$2F_i^s \not\geq F_i$

$3F_i^s \geq F_i$

Example

Denote by F_i^s the support of the fiber F_i. Let

$$V := X \backslash \{F_1^s,\ F_2^s,\ F_3^s\}.$$

We have that

- $\overline{P}_1(V) = \overline{P}_2(V) = 1$,
- $\overline{q}(V) = 2$,
- $q(X) = 0$, and...

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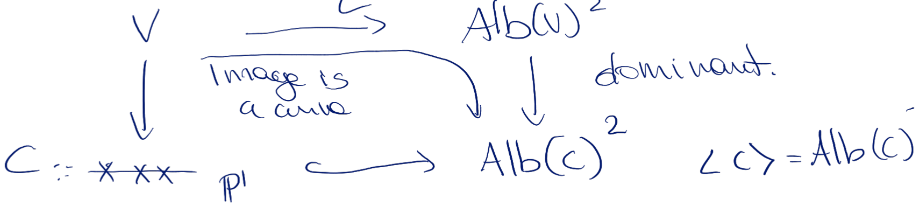
...the quasi-Albanese morphism is not dominant!!!



Higher dimension?

Higher dimension?

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Theorem (Kawamata 1981)

Let V be a quasi-projective variety with

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Effective versions of this result using the first 2 plurigenera were given by Chen–Hacon in 2001 (when V is projective) and Pareschi–Popa–Schnell in 2014 (when V is compact Kähler).

Conjectures

(Very^n Bold) Conjecture

Let V be a quasi-projective variety with

$$\overline{P}_1(V) = \overline{P}_3(V) = 1, \quad \overline{q}(V) = \dim V.$$

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With n as big as you want

Okay Conjecture

Let V be a quasi-projective variety. Then there exist k a positive integer independent of the dimension of V such that, if

$$\overline{P}_1(V) = \overline{P}_k(V) = 1, \quad \overline{q}(V) = \dim V,$$

the quasi-Albanese morphism $a_V : V \to \mathrm{Alb}(V)$ is a WWPB equivalence.

Section 5
Strategy

Outline of the proof

There are three main steps:

1. the quasi-Albanese morphism is dominant;
2. the quasi-Albanese morphism is birational;
3. the quasi-Albanese morphism is a WWBP equivalence.

$2 = \overline{q}(V) \geq q(X) \geq 0$

- $q(X) = 2$
- $q(X) = 1$
- $q(X) = 0$

Outline of the proof

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Step 1	😎		
Step 2	😎		
Step 3			

ok in higher dimens.

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Case $q(X) = 2$

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- The first two points follows (almost) directly from Chen–Hacon, with the help of generic vanishing (and recent extensions to pairs).
- It remains to show that the Albanese morphism of X contracts the boundary D.

Case $q(X) = 1, 0$

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- The argument for the dominance of the quasi-Albanese morphism proceeds smoothly thanks to generic vanishing when $q(X) = 1$.

Case $q(X) = 1, 0$

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Case $q(X) = 1, 0$

- The argument for the dominance of the quasi-Albanese morphism proceeds smoothly thanks to generic vanishing when $q(X) = 1$.
- You do not really want to know the argument in the case $q(X) = 0$...
  A great tool is the following formula

$2 \leq$ 2 0,1 0 0

$$h^0(X, K_X + D) = p_a(D) + p_g(X) - q(X) + h^1(X, -D).$$

- Once one knows that the quasi-Albanese morphism is dominant (and hence generically finite), we can use the logarithmic ramification formula

$$K_X + D \sim \overline{R}_g$$

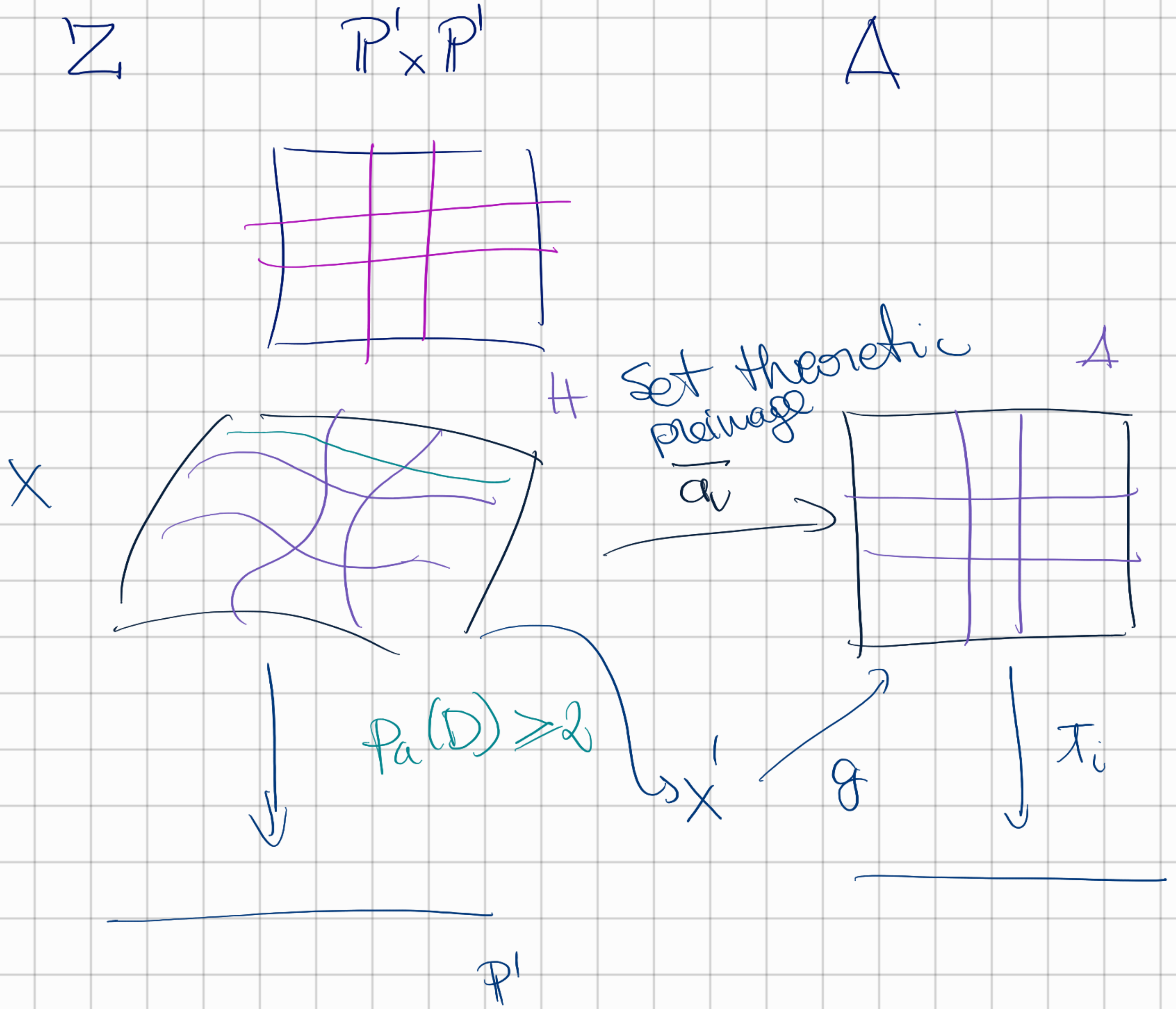
$q(X) = 0$ we already know that

a_V is dominant.

$Alb(V)$ is such that $q(Z) = 0$

$$0 \to \mathbb{G}_m^r \to Alb(V) \to Alb(X) \to 0$$

$$Alb(V) \simeq \mathbb{G}_m^2$$



If Γ is a component of D not in H then Γ is contracted by $\bar{a}_V$

g is étale on $Alb(V)$

$X' \setminus g^{-1}(\Delta)$ is a quasi-abelian variety

$\rightsquigarrow$ Universal property

$X' \setminus g^{-1}(\Delta) \simeq Alb(V)$

$\Rightarrow$ $\deg g = 1$

$\bar{a}_V$ is birational

a_V is birational

To show WWPB same argument the cp in $D + R_{\bar{a}_V}$ which are not in H are contracted

$\bar{P}_2(X) = 1$ $\quad 2(K_X + D) = K_X + D + R_g$

Thank You!!!

Section 6
WWPB

WPB

Definition

A birational map $\varphi : V \dashrightarrow W$ is a WPB-equivalence if it can be written as a composition of

- proper birational morphism and their inverse;
- open immersions $V \subset V'$ such that $V' \setminus V$ has codimension at least 2, and their inverses.

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WPB-equivalent varieties have the same logarithmic plurigenera and irregularity. Therefore, it would seem that WPB-equivalence is the right equivalence to consider when studying the birational geometry of open varieties.

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Problem

the set of WPB-maps is not saturated.

WWPB

$$\mathcal{W} := \{f : V \dashrightarrow W \mid \text{there is } U \text{ and } g \text{ or } h \text{ such that} \\ \text{either } f \circ g \text{ or } h \circ f \text{ are WPB}\},$$

Definition

A rational map $f : V \dashrightarrow W$ between algebraic varieties is a WWPB equivalence, if we can write $f = f_1 \circ \cdots \circ f_k$ with f_i birational maps such that either f_i or f_i^{-1} is in $\mathcal{W}$.

It can be proven that WWPB-equivalent varieties have the same plurigenera and the same irregularity.

Thank You!!!