Higher S-matrices

Theo Johnson-Freyd
Dalhousie University & Perimeter Institute

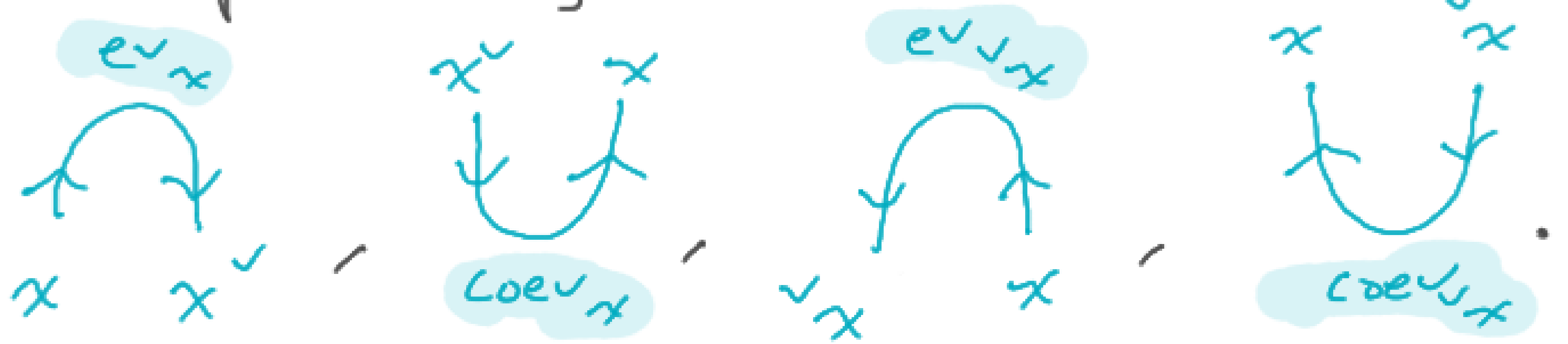
Joint work in progress with David Reutter

These slides available at categorified.net/S-matrices.pdf

1-categorical warm-up: string diagrams

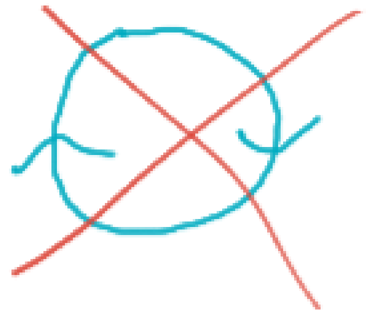
all objects have right + left duals

Let $\mathcal{C}$ a rigid monoidal 1-category. It defines a 2D "skein theory" aka graphical calculus. Given $x \in \mathcal{C}$, can interpret e.g.:



a priori, ${}^\vee x \ncong x^\vee$.

But cannot interpret, e.g.:



This very common string-diagram notation is bad, because it does not distinguish left and right duals. It should record how many times the string twists around.

1-categorical warm-up: Embedded Cobordism Hypothesis

You can place $x \in \mathcal{C}$ along an embedded 1-manifold $S \hookrightarrow \mathbb{R}^2$ if S is equipped with:

- tangential framing $\tau : T_S \simeq \underline{\mathbb{R}}$
  - tangent bundle
  - trivial bundle
  - [since $\dim S = 1$, equiv to orientation.
- normal framing $\nu : N_S \simeq \underline{\mathbb{R}}$
  - normal bundle
  - [since $\operatorname{codim} S = 1$, equiv to coorientation.
- [null homotopy η:

records the number of twists

$$\begin{array}{ccc} T_S \oplus N_S & \xrightarrow[\sim]{\tau \oplus \nu} & \mathbb{R}^2 \\ \wr\wr & \eta & \\ T_{\mathbb{R}^2}|_S & \nearrow_{\sim} & \end{array}$$

ambient tangent bundle restricted to S

blackboard framing

Same rule lets you draw skeins on any framed 2-manifold.

If $S = S^1 \hookrightarrow \mathbb{R}^2$, η DNE.

1-categorical warm-up: extra fractional dualizability

Now suppose that $\mathcal{C}$ is fusion over field $\mathbb{k}$.

semisimple & $|\pi_0 \mathcal{C}| < \infty$ & 1 is simple

useful notation:

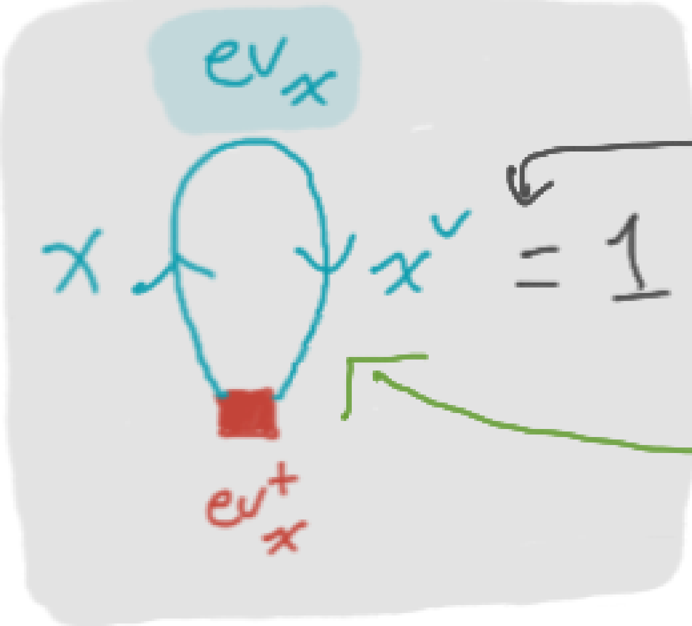
$\pi_0 \mathcal{C} :=$ simples / iso

$x \in \mathcal{C}$ is simple iff $x \otimes x^\vee \cong 1 \oplus \dots$

i.e. 1 appears with multiplicity one.

Then the projection map ev_x splits uniquely:

$$\begin{tikzcd} x x^\vee \arrow[r, two heads, "ev_x"'] & 1 \arrow[l, bend right, "\exists!\ ev_x^+"'] \end{tikzcd}, \text{ i.e.}$$



Think of this as value of x on D^2.

Think of as $\partial D^2 = S^1_{b}$ "S^1 with bounding 2-framing"

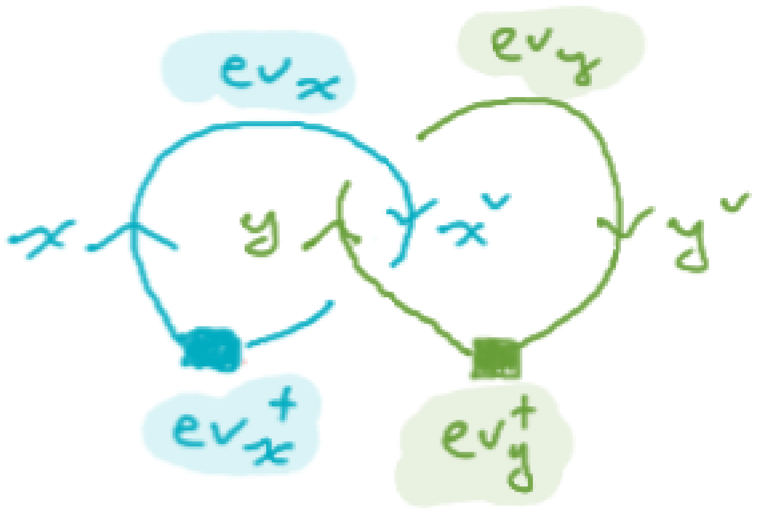
Corollary: $\forall x \in \mathcal{C}$, $x^\vee \cong {}^\vee x$.

Warning: Iso is canonical but usually non-monoidal.

A choice of monoidal iso is a pivotal structure.

1-categorical warm-up: framed S-matrix in braided fusion cat.

Now suppose $\mathcal{C}$ is braided. The framed S-matrix is the $\pi_0\mathcal{C} \times \pi_0\mathcal{C}$ matrix

$$\tilde{S}_{x,y} :=$$ 

Fundamental theorem of braided fusion 1-categories: [EGNO, FT, BJSS, ...]

$\tilde{S}$ is invertible $\Leftrightarrow$ $Z_{(2)}(\mathcal{C})$ is trivial $\Leftrightarrow$ $\mathcal{C}$ is Morita-invertible

0-categorical calculation

Müger aka syllepticentre, i.e. 1-cat of transparent objects.

in 4-cat of BFCs.

N.B.: If $\mathcal{C}$ is ribbon, then $\tilde{S}_{x,y} = \frac{1}{\dim x} \frac{1}{\dim y} S_{x,y}$. ← usual S-matrix

depends on ribbon structure

n-categorical version: higher fusion categories

fusion = multifusion & 1 is simple

slogan: $\dfrac{\text{multifusion } n\text{-categories}}{\text{multifusion } (n-1)\text{-categories}} = \dfrac{\text{multifusion 1-categories}}{\text{f.d. semisimple algebras}}$

Spelled out: A $\mathbb{k}$-linear monoidal n-category $\mathcal{C}$ is multifusion if it is:

- additive and Karoubi complete
- rigid: all objects have duals
- locally multifusion: $\forall x \in \mathcal{C}$, $\mathrm{End}_{\mathcal{C}}(x)$ is a multifusion $(n-1)$-category
- finite: $|\pi_0 \mathcal{C}| < \infty$

$\Rightarrow$ by induction, all k-morphisms are fully dualizable

$\Rightarrow |\pi_k \mathcal{C}| < \infty \ \forall k.$

$\pi_k \mathcal{C} = \pi_0 \Omega^k \mathcal{C}$

$\Omega := \mathrm{End}(1)$

Warning: $\pi_0 \mathcal{C} := \text{simples}/\text{nonzero morphisms} \neq \text{simples}/\text{iso}.$

The 1-categorical Schur's Lemma fails: $\exists$ nonzero non-iso morphisms between simples.

n-categorical version: Karoubi completion and components

A **condensation** $X \twoheadrightarrow Y$ is an n-categorical split surjection:

$$\begin{matrix} X \\ \Downarrow \\ Y \end{matrix} \; := $$ 

spelled out: $f: X \to Y$, $g: Y \to X$,

and a condensation $fg \twoheadrightarrow \mathrm{id}_Y$.

in the $(n-1)$-cat $\mathrm{End}(Y)$, hence defined by induction.

A **separated monad** $X \circlearrowleft p$ is an n-categorical idempotent:

a condensation $p^2 \twoheadrightarrow p$ which is assoc, coassoc, and frobenius.

An n-category is **Karoubi complete** if every separated monad factors through a condensation.

<u>n-categorical Schur's lemma:</u> In a s.s. n-category, every nonzero morphism with simple target extends to a condensation.

<u>Corollary:</u> $\pi_0 \mathcal{C} := \dfrac{\text{simples}}{\text{nonzero morphism}}$ is well defined. $\longleftarrow$ i.e. this is an equiv relation.

n-categorical version: embedded cobordism hypothesis

Let $\mathcal{C}^q$ be an E_p-monoidal q-category. A fully dualizable object can be inserted* along an embedded q-dim manifold $M^q \hookrightarrow \mathbb{R}^{p+q}$ provided it is equipped with:

- tangential framing $\tau : T_M \simeq \underline{\mathbb{R}^q}$
- normal framing $\nu : N_M \simeq \underline{\mathbb{R}^p}$
- null homotopy

trivial bundles

$$\begin{array}{ccc} T_M \oplus N_M & \xrightarrow{\tau\oplus\nu} & \\ \shortparallel & \cong \eta & \underline{\mathbb{R}^{p+q}} \\ T_{\mathbb{R}^{p+q}}|_M & \xrightarrow{id} & \end{array}$$

more generally, can replace $\mathbb{R}^{p+q}$ with any framed $(p+q)$-manifold.

*Notation: "$\int_M x$"

For us, $\mathcal{C} = \mathcal{C}^n$ is fusion, $q = n-k$, $\mathcal{C}^q := \Omega^k \mathcal{C} = \{\text{endo-}k\text{-morphisms of } 1\}$.

BTW: $\Omega := End(1)$ has a left adjoint $\Sigma := \{\text{dualizable modules}\}$.

n-categorical version: extra fractional dualizability.

Suppose $\mathcal{C}$ is a fusion n-category.

Then $x \in \Omega^k \mathcal{C}$ is simple iff

$$\int_{S_b^{n-k-1} := \partial D^{n-k}} x \in \Omega^{n-1}\mathcal{C} \quad \text{a fusion 1-category}$$

with "boundary $(n-k)$-framing": maximum dimension CH allows.

$$= 1 \oplus \ldots$$

i.e. 1 appears with multiplicity 1.

$\int_{D^{n-k}} x$ is an "evaluation" morphism

S^{n-1-k}

If so, then the projection $\int_{D^{n-k}} x : \int_{S_b^{n-k-1}} x \twoheadrightarrow 1$ splits uniquely.

Corollary: There is a canonical (but not monoidal!) way to insert simple x along S_b^{n-k}. ← "boundary $(n-k+1)$-framing"

Cannot do this w/o semisimplicity.

n-categorical version: framed S-matrix

$\tilde{S}$ is a version of the "Whitehead bracket"

Let $x \in \Omega^k \mathcal{C}$ and $y \in \Omega^{n-k}\mathcal{C}$ be simple. Define:

$\tilde{S}_{x,y} :=$ 

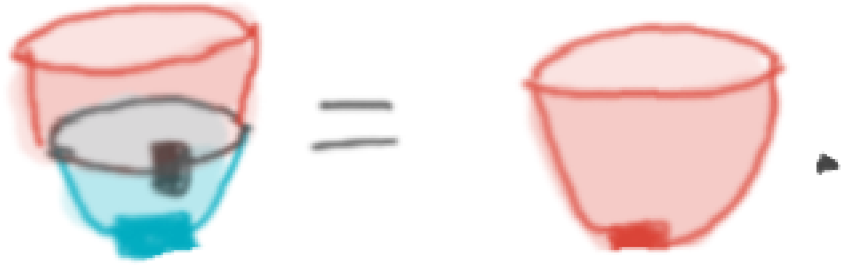
in picture, $k=1$ and $n=3$.

Theorem: $\tilde{S} : \pi_k \mathcal{C} \times \pi_{n-k}\mathcal{C} \to \mathbb{k}$.

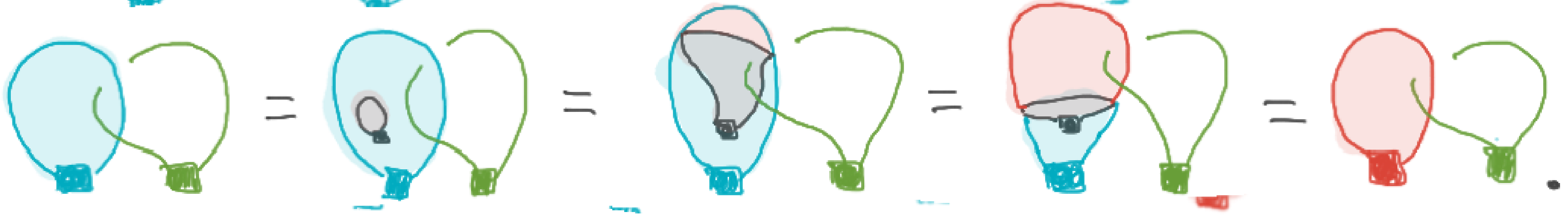
Pf: Suppose x, x' in same component. Choose simple $(k+1)$-morphism $f : x \to x'$.

(a) $1 =$ 

(b) By uniqueness of splittings,



(c) $(n-k-1)+k < n$ so f, y cannot link:



TQFT interpretation: context

If $\mathcal{C}$ is Morita-invertible, this is an anomalous TQFT aka topological order.

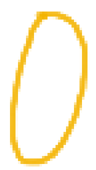
multifusion n-category $\mathcal{C}$ = relative $(n+1)$D framed TQFT.

= $(n+2)$D TQFT w/ $(n+1)$D boundary cond.

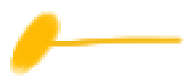
Morita class of $\mathcal{C}$ — $\mathcal{C}$ itself

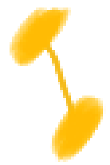
This is a thing which assigns values to m-manifolds, $m \leq n+2$, with two types of "boundaries": open boundaries where cobordisms glue, and closed boundaries where you place the boundary condition.

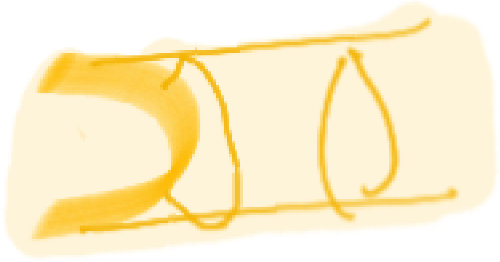
pt = . $\longmapsto \mathrm{Mod}(\mathcal{C}) \in (n+1)\text{-}\mathrm{Cat}$

 $\longmapsto \underset{\mathcal{C}\otimes\mathcal{C}^{op}}{\mathcal{C}\otimes\mathcal{C}} =: HH_0\mathcal{C}$

$\in n\text{-}\mathrm{Cat}$


(half open) $\longmapsto \mathcal{C}_{\mathcal{C}} \in \mathrm{Mod}\,\mathcal{C}$

 $\longmapsto \mathcal{C} \in n\text{-}\mathrm{Cat}$

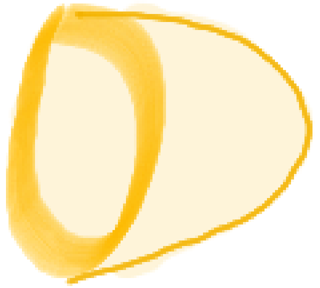
 $\longmapsto \left(\mathcal{C} \xrightarrow{tr} HH_0\mathcal{C}\right)$

 $\longmapsto \left(id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}\right)$

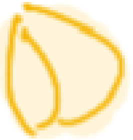
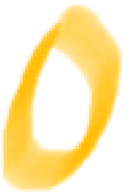
etc.

TQFT interpretation: loop categories

The higher-categorical state-operator correspondence is:

 $\longmapsto \Omega^k \mathcal{C} \in (n-k)\text{-Cat}$

D^{k+1} with only closed boundary. N.B: $\partial D^{k+1} = S^k_b$.

Why the name? If $\mathcal{C} = \mathrm{End}(\mathcal{X})$ is Morita triv, then our "relative TQFT" is an ordinary absolute $(n+1)$D TQFT, $\mathrm{pt} \mapsto \mathcal{X}$, and $\mathcal{C} = \{\text{extended operators}\}$.  $\mapsto$ triv, so  $=$  $= S^k_b$.

The usual state-operator correspondence is the statement that $\{\text{States on a sphere}\} = \{\text{operators that can be inserted in that sphere}\}$.

Cor: $M \mapsto$ value of $(\underbrace{D^{k+1}}_{\text{bulk}} \times M, \underbrace{S^k_b}_{\text{closed boundary}} \times M)$ is the tqft s.t. $\mathrm{pt} \mapsto \Omega^k \mathcal{C}$

TQFT interpretation: π_k, K^0, surgery, and invertibility

finite s.s. m-cat $\mathcal{X}$ $=$ absolute $(m+1)$D tqft s.t. $pt \mapsto \mathcal{X}$.

In this tqft, $S^m_b \longmapsto$ v-space with basis $\pi_0 \mathcal{X}$ $=$ "$K^0(\mathcal{X})$" Grothendieck gp.

Pf: "absolute" means $(m+2)$D bulk is trivial. $S^m_b = (D^{m+1}, S^m) \mapsto \Omega^m \mathrm{End}(\mathcal{X})$.

Cor: $\pi_k \mathcal{C}$ is a basis for value of $(D^{k+1} \times S^{n-k}_b, S^k_b \times S^{n-k}_b)$.

$\tilde{S}$ matrix = "surgery" bordism $D^{k+1} \times D^{n-k+1}$.
The "closed" boundary is $S^k_b \times S^{n-k}_b \times [0,1]$. This is invertible!
So $\tilde{S}$ is invertible $\Leftrightarrow$ "bulk" of sugery bordism is.

Theorem: $\mathcal{C}$ is Morita invertible $\Longrightarrow$ $\tilde{S}$ is invertible

$\Uparrow$ $\Swarrow$

$Z(\mathcal{C})$ is trivial

$\blacksquare$