Geometry of Localized Effective Theories and Algebraic Index

Si Li

YMSC, Tsinghua University

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infinite dim geometry $\Longrightarrow$ finite dim geometry

QFT *Math*

Typically, one starts with a path integral in quantum field theory

$$\int_{\mathcal{E}} e^{iS/\hbar}$$

In good situations (e.g. when supersymmetry exists), the ill-defined path integral is localized to a well-defined integral

$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_{\mathcal{M}} (-)$$

over a finite dim $\mathcal{M} \subset \mathcal{E}$. $\mathcal{M}$ is some interesting moduli space.

Example: Topological quantum mechanics (TQM)

TQM leads to a path integral on the loop space

$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \quad \overset{\hbar\to 0}{\Longrightarrow} \quad \int_X (\text{curvatures})$$

Topological nature implies the exact semi-classical limit $\hbar \to 0$, which localizes the path integral to constant loops.

- LHS= the analytic index expressed in physics
- RHS= the topological index.

This is the physics "derivation" of Atiyah-Singer Index Theorem.

Example: Witten's "Index Theorem" on loop space

Replace S^1 by an elliptic curve E:

$$\int_{\mathrm{Map}(E,X)} e^{-S/\hbar} \quad \overset{\hbar \to 0}{\Longrightarrow}$$

Intuitively, if we view

$$\mathrm{Map}(E, X) = \mathrm{Map}(S^1, LX)$$

as defining a quantum mechanics on LX, then this leads to **Witten**'s proposal for index of dirac operators on loop space.

Example: Mirror symmetry

Mirror symmetry is about a duality between

$$\boxed{\text{symplectic geometry}}\ (\text{A-model}) \Longleftrightarrow \boxed{\text{complex geometry}}\ (\text{B-model})$$

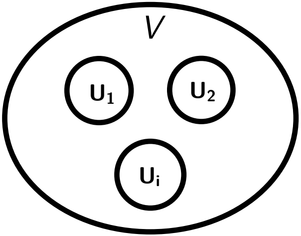
$$\begin{tikzcd}
\int_{\mathrm{Map}(\Sigma_g, X)} (\text{A-model}) \arrow[r, equal, "\text{Fourier transform}"] \arrow[d, "\text{localize}"] & \int_{\mathrm{Map}(\Sigma_g, X')} (\text{B-model}) \arrow[d, "\text{localize}"'] \\
\int_{\text{Holomorphic maps}(\Sigma_g, X)} \arrow[r, leftrightarrow, dashed] \arrow[d, Rightarrow] & \int_{\text{Constant maps}(\Sigma_g, X')} \text{???} \arrow[d] \\
\text{Gromov-Witten Theory} & \text{Hodge theory}
\end{tikzcd}$$

The B-model can be viewed as a suitable mysterious way to "count constant surfaces", which will be related to the variation of Hodge structures and its quantization.

Observable algebras

"Observables = functions on fields".

The topology of $X \Longrightarrow$ factorization algebra.



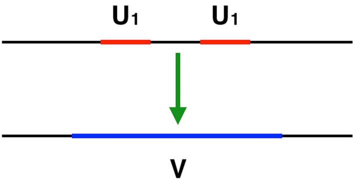
factorization product

$\bigotimes_i Obs(U_i) \to Obs(V)$

- **Beilinson-Drinfeld**: Factorization algebra in 2d CFT.
- **Costello-Gwilliam**: Factorization algebras from (perturbative) quantum field theory in BV formalism.

Example: $\dim X = 1$ (topological quantum mechanics)

QFT in $\dim = 1$ is quantum mechanics.



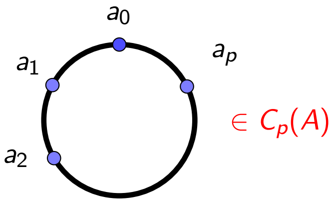
In the topological case, for any contractible open U, $Obs(U) = A$.
The factorization product doesn't depend on the location and size:

$$A \otimes A \to A.$$

We find an (homotopy) associative algebra.

There are several algebraic structures for observables in TQM.
Let $\overline{A} := A/\mathbb{C} \cdot 1$. Define

$$C_p(A) := A \otimes \overline{A}^{\otimes p}, \quad \text{cyclic } p\text{-chains}.$$



It carries two natural differentials

$$\begin{cases} \text{Hochschild differential} & b : C_p(A) \to C_{p-1}(A) \\ \text{Connes operator} & B : C_p(A) \to C_{p+1}(A). \end{cases}$$

The periodic cyclic complex is defined by

$$CC_\bullet^{per}(A) := (C_\bullet(A)[u, u^{-1}], b + uB).$$

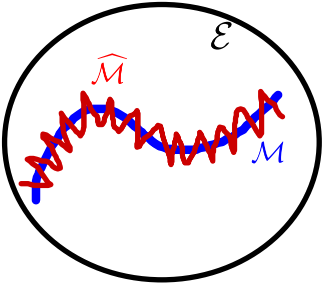
This can be viewed as S^1-equivariant observables on the cycle.

We will be mainly interested in σ-models about the mapping space

$$\mathcal{E} = \{\varphi : \Sigma \to X\}$$

and consider the case when it is localized to constant maps

$$\mathcal{M} = \{\text{const map} : \Sigma \to X\}\,.$$



Let $\widehat{\mathcal{M}}$ be the formal neighborhood of $\mathcal{M}$ inside $\mathcal{E}$. Then intuitively

$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_{\widehat{\mathcal{M}}} e^{iS^{\mathit{eff}}/\hbar} = \int_{\mathcal{M}} (-)\,.$$

The pair $(\widehat{\mathcal{M}}, S^{\mathit{eff}})$ will be called the localized effective theory.

$$\mathcal{E} = \{\varphi : \Sigma \to X\}$$

The idea is to formulate the effective geometry of σ-models as a sheaf on X of quantum algebras (observables) living on Σ

$$\begin{tikzcd}
\mathrm{Obs}^{\hbar}(\Sigma) \arrow[rr, "\langle - \rangle"] \arrow[dr] & & \mathcal{A}[[\hbar]] \arrow[dl] \\
& X &
\end{tikzcd}$$

- $\mathcal{A}$ is a BV algebra (IR theory of zero modes), equipped with

$$\int_{BV} : \mathcal{A} \to \mathbb{C}.$$

- $\langle - \rangle$ is a chain map (quantum master equation).
- Trace map $\mathrm{Tr}(-) = \int_{BV} \langle - \rangle : \mathrm{Obs}^{\hbar} \to \mathbb{C}((\hbar))$. Then

$$\text{Index} = \mathrm{Tr}(1).$$

Application: counting const loops $\Longrightarrow$ algebraic index

Let (X, ω) be a symplectic manifold, $(C^\infty(X)[\![\hbar]\!], \star)$ be a deformation quantization. There exists a unique linear map

$$\mathrm{Tr} : C^\infty(X)[\![\hbar]\!] \to \mathbb{R}(\!(\hbar)\!)$$

satisfying

- Trace property: $\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f)$
- Normalization:

$$\mathrm{Tr}(f) = \frac{1}{\hbar^n} \int_X \frac{\omega^n}{n!}(f + O(\hbar)) \quad n = \dim X/2.$$

The Algebraic Index Theorem [Fedosov, Nest-Tsygan] says that

$$\mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar} \hat{A}(X).$$

I will explain how to use localized effective theory to prove such index theorem, making the physics argument into math realization.

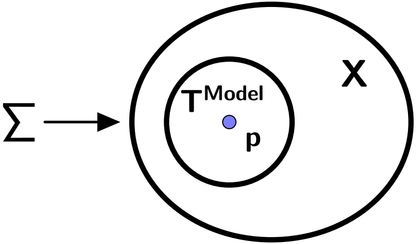
The framework of localized effective theory

The localized effective theory contains three main steps

1. Local model
2. Gluing and descent
3. Exact semi-classical approximation.

1. Local model

Assume that X is locally modeled by a flat geometry T^{Model} such that X is built up from gluing pieces of T^{Model}.



Quantum fluctuations around a point $p \in X$ is locally modeled by

$$\Sigma \to T^{Model}.$$

Since T^{Model} is flat, this is usually a free quantum field theory.

Even the theory is free, it carries a nontrivial factorization algebra of quantum observables $\mathrm{Obs}^{\hbar}$ living on Σ.

We are interested in an evaluation map

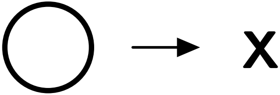
$$\mathrm{Tr} : \mathrm{Obs}^{\hbar} \to \mathbb{C}[\![\hbar]\!], \quad \delta \, \mathrm{Tr} = 0.$$

δ is some natural differential (BRST operator).

Example: Topological quantum mechanics

One way to formulate TQM is to consider the mapping space

$$\varphi : S^1_{dR} \to (X, \omega).$$



Here (X, ω) is a symplectic manifold. S^1_{dR} is the supermanifold

$$S^1_{dR} = (S^1, \Omega^\bullet_{S^1})$$

with underlying topology S^1 and the structure ring the sheaf of de Rham complex $\Omega^\bullet_{S^1}$.

Local Model

We first study the local model: $S^1_{dR} \to (\mathbb{R}^{2n}, \omega)$

$$\varphi \in \mathrm{Map}\left(S^1_{dR}, \mathbb{R}^{2n}\right) = \Omega^\bullet_{S^1} \otimes \mathbb{R}^{2n}.$$

The action is the free one

$$S_{free}[\varphi] = \int_{S^1} \omega(\varphi, d\varphi).$$

Constant maps are presented by

$$\text{Constant maps} = H^\bullet(S^1) \otimes R^{2n}.$$

Free correlation function

$$\mathrm{Obs}^{\hbar} = \text{functions on } \Omega^{\bullet}_{S^1} \otimes \mathbb{R}^{2n}.$$

$$\Omega^{\bullet}_{2n} = \text{functions on } H^{\bullet}(S^1) \otimes \mathbb{R}^{2n}, \quad \Delta = \mathcal{L}_{\Pi}.$$

Then we can use Feynman diagrams to define

$$\langle - \rangle_{free} := \int [D\varphi] e^{-S_{free}[\varphi]/\hbar}(-) : \boxed{\mathrm{Obs}^{\hbar} \to \Omega^{\bullet}_{2n}[[\hbar]]}$$

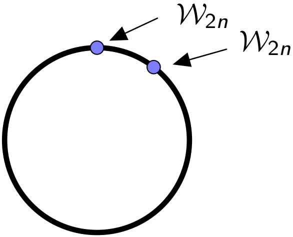
The correlation functions play the role of integrating out informations from normal neighborhood of constant maps.

Local observables on S^1 form the Weyl algebra

$$\mathcal{W}_{2n} = \left(\mathbb{C}[[p_i, q^i]][[\hbar]], \star\right)$$

where $\star$ is the Moyal-Weyl product

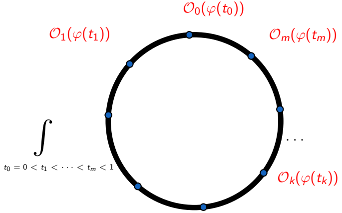
$$(f \star g)(p,q) := f(p,q) e^{\hbar\left(\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i} - \frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}\right)} g(p,q).$$



We consider the following types of observables

$$CC_\bullet^{per}(\mathcal{W}_{2n}) \to \mathrm{Obs}^{\hbar}$$

by sending $\mathcal{O}_0 \otimes \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_m$ ($\mathcal{O}_i \in \mathcal{W}_{2n}$) to the functional



Composing with the free correlation function

$$\boxed{\langle - \rangle_{free} : CC_\bullet^{per}(\mathcal{W}_{2n}) \to \Omega_{2n}^\bullet((\hbar))}.$$

This intertwines the Hochschild differential with BV operator

$$(b + \hbar\Delta)\langle - \rangle_{free} = 0.$$

The trace map

$$\begin{tikzcd} CC^{per}_\bullet(\mathcal{W}_{2n}) \arrow[r, "\langle - \rangle_{free}"] \arrow[dr, "\mathrm{Tr}"'] & \Omega^\bullet_{2n}((\hbar))[u,u^{-1}] \arrow[d, "\int_{\text{const maps}}:\quad \alpha \to e^{\hbar\iota_\pi/u}\alpha|_0"] \\ & \mathbb{R}((\hbar))[u,u^{-1}]. \end{tikzcd}$$

We end up with the following trace map

$$\mathrm{Tr} : CC^{per}_\bullet(\mathcal{W}_{2n}) \to \mathbb{R}((\hbar))[u,u^{-1}].$$

It can be checked that it satisfies the cocycle condition

$$\boxed{(b + uB)\,\mathrm{Tr} = 0}.$$

$b =$ the Hochschild differential and $B =$ the Connes' operator.

Tr has an explicit formula in terms of Feynman diagrams (**Feigin-Felder-Shoikhet** formula).

2. Gluing and descent

Once we have a local theory modeled on T^{Model}, the next step is to glue them to form a sheaf on X. The gluing symmetry is typically described by a Harish-Chandra pair

$$(\mathfrak{g}, K), \quad \mathrm{Lie}(K) \subset \mathfrak{g}.$$

- K is the linearized transformations.
- $\mathfrak{g}$ contains all nonlinear transformations.

Gluing geometry of X

The geometry of X will be represented by a principal K-bundle

$$\begin{array}{ccc} K & \longrightarrow & P \\ & & \downarrow \\ & & X \end{array}$$

together with a flat $(\mathfrak{g}, K)$-connection γ, i.e. $\gamma \in \Omega^1(P, \mathfrak{g})$

- $\iota_\xi \gamma = \xi, \forall \xi \in \mathrm{Lie}(K)$;
- γ is K-equivariant;
- $d\gamma + \frac{1}{2}[\gamma, \gamma] = 0$.

It induces a flat connection on the associated vector bundle $P \times_K V$ for any $(\mathfrak{g}, K)$-module V.

Gelfand-Fuks descent

Let

$$C^\bullet(\mathfrak{g}, K; V) := \mathrm{Hom}_K(\wedge^\bullet(\mathfrak{g}/\mathrm{Lie}(K)), V), \quad \partial_{\mathrm{Lie}}$$

be the Lie algebra cochain complex valued in V. Let

$$\Omega^\bullet(P, V)_{basic} \simeq \Omega^\bullet(X, P \times_K V), \quad d + \gamma$$

be the de Rham complex of the associated flat bundle $P \times_K V$.

There is a natural cochain map of Gelfand-Fuks descent

$$\begin{aligned} \mathrm{desc} : C^\bullet(\mathfrak{g}, K; V) &\to \Omega^\bullet(P, V)_{basic} \\ \alpha &\to \alpha(\gamma, \gamma, \cdots, \gamma). \end{aligned}$$

This allows us to descent a local coupling to a global object on X

$$\mathrm{desc} : \underset{\text{Lie theoretic}}{H^\bullet(\mathfrak{g}, K; V)} \to \underset{\text{geometric}}{H^\bullet(X, P \times_K V)}$$

This set-up follows closely the philosophy of **Gelfand-Kazhdan** formal geometry. Several examples are established along this line

- **Kontsevich** and **Cattaneo-Felder**: Poisson σ-model
- **Costello**: holomorphic CS theory on an elliptic curve E
- **Grady-Gwilliam**: TQM on $X = T^*M$
- **Grady-Li-L**: TQM on a symplectic manifold X
- **Gorbounov-Gwilliam-Williams**: $\beta - \gamma$-system.
- $\cdots$

Example: Topological quantum mechanics

The relevant Harish-Chandra pair $(\mathfrak{g}, K)$ is

$$\mathfrak{g} = \mathcal{W}_{2n}, \quad K = Sp_{2n}$$

The principal bundle is the frame bundle

$$\begin{tikzcd} Sp_{2n} \arrow[r] & Fr(T_X) \arrow[d] \\ & X \end{tikzcd}$$

Fedosov shows that there exists flat $(\mathfrak{g}, K)$-connection

$$\gamma \in \Omega^1(Fr(T_X), \mathfrak{g}).$$

γ will be called the Fedosov connection.

γ induces a flat connection on the associated Weyl bundle

$$\mathcal{W}(X) := Fr(X) \times_{Sp_{2n}} \mathcal{W}_{2n}.$$

Let

$$\mathcal{W}_D = \{\text{flat sections of } \mathcal{W}(X)\}.$$

- $\mathcal{W}_D$ defines a deformation quantization of $C^\infty(X)$
- $\mathcal{W}_D$ can be viewed as quantum observables glued on X.

Coupling with gluing symmetry

We will glue our trace map

$$\mathrm{Tr} \in \mathrm{Hom}\left(CC^{per}_\bullet(\mathcal{W}_{2n}), \mathbb{C}(\!(\hbar)\!)[u, u^{-1}]\right)$$

on X by lifting it to

$$\mathrm{Tr}^{\mathbb{L}} \in C^\bullet(\mathfrak{g}, K; \mathrm{Hom}\left(CC^{per}_\bullet(\mathcal{W}_{2n}), \mathbb{C}(\!(\hbar)\!)[u, u^{-1}]\right))$$

satisfying the coupled equation

$$(\partial_{\mathrm{Lie}} + (b + uB))\,\mathrm{Tr}^{\mathbb{L}} = 0.$$

Connection v.s. Interaction

There is a very natural way to realize this, by turning on an interaction by a universal connection. In our context, we consider

$$\hat{\Theta} \in C^1(\mathfrak{g}; \mathcal{W}_{2n})$$

which represents the natural embedding $\mathfrak{g} \hookrightarrow \mathcal{W}_{2n}$.

Introduce the following interacting action

$$\int_{S^1} \hat{\Theta}(\varphi)$$

which is a $C^1(\mathfrak{g}; \mathbb{C}) = \mathfrak{g}^*$-valued local functional on S^1. Define

$$\mathrm{Tr}^{\mathbb{L}}(-) := \mathrm{Tr}((-)e^{\frac{1}{\hbar}\int_{S^1}\hat{\Theta}}) \in C^\bullet(\mathfrak{g}, K; -)$$

Proposition (Gui-L-Kai)

$\mathrm{Tr}^{\mathbb{L}}$ satisfies the coupled equation

$$(\partial_{\mathrm{Lie}} + (b + uB))\,\mathrm{Tr}^{\mathbb{L}} = 0.$$

Descent

Now we can use Fedosov connection to descent $\mathrm{Tr}^{\mathbb{L}}$ to give

$$\mathrm{desc}(\mathrm{Tr}^{\mathbb{L}}) : CC_{\bullet}^{per}(\mathcal{W}_D) \to \Omega^{2n-\bullet}(X)((\hbar))[u, u^{-1}].$$
$$b + uB \to d$$

This gives our glued correlation functions on quantum observables.

The index is given by the partition function

$$\mathrm{Index} = \int_X \mathrm{desc}(\mathrm{Tr}^{\mathbb{L}})(1).$$

3. Exact semi-classical approximation

Now we explain the strategy to understand the physics idea of exact semi-classical approximation. Suppose

$$\mathsf{Tr} : \mathsf{Obs}^{\hbar} \to \mathbb{C}((\hbar))$$

which is suitably coupled with gluing symmetry by

$$\mathsf{Tr}^{\mathbb{L}} = \mathsf{Tr} + \cdots \in C^{\bullet}(\mathfrak{g}, K; \mathsf{Hom}(\mathsf{Obs}^{\hbar}, \mathbb{C}((\hbar))))$$

In particular, the partition function (the index) is given by $\mathsf{Tr}(1)$.

We need to figure out the Gauss-Manin connection ∇ along $\hbar$-variation. Then exact semi-classical approximation of the index is

$$\boxed{(\nabla_{\hbar\partial_{\hbar}})\,\mathsf{Tr}(1) = \partial_{\mathsf{Lie}} - \text{exact.}}$$

This will allow us to compute the index via one-loop Feynman diagrams, the same as what physicists would do.

Example: Topological quantum mechanics

The $\hbar$-variation is computed by Getzler's Gauss-Manin connection ∇ on periodic cyclic homologies. We have

$$\nabla_{\hbar\partial_\hbar} \text{ acts on } CC_\bullet^{per}(\mathcal{W}_{2n}).$$

The calculation of index consists of the following steps

1. Feynman diagram computation implies:

$$\mathrm{Tr}^{\mathbb{L}}(1) = e^{\omega_\hbar/\hbar}(\hat{A} + O(\hbar)) \in C^\bullet(g, K; \mathbb{R}((\hbar))[u, u^{-1}]).$$

   Here $\omega_\hbar$ is the char. class of the deformation quantization.

2. Computation of Gauss-Manin connection implies

$$\nabla_{\hbar\partial_\hbar}\left(e^{-\omega_\hbar/\hbar}\,\mathrm{Tr}^{\mathbb{L}}(1)\right) = \partial_{\mathrm{Lie}} - \text{exact}.$$

3. Combining the above two computations, we find

$$[\mathrm{Tr}^{\mathbb{L}}(1)] = e^{\omega_\hbar/\hbar}\hat{A} \in H^\bullet(g, K; \mathbb{R}((\hbar))[u, u^{-1}]).$$

This is the algebraic index at the Lie algebraic level.

Algebraic Index Theorem

After geometric descent, we obtain

$$\text{Index} = \int_X \text{desc}(\text{Tr}^{\mathbb{L}})(1) = \int_X e^{\omega_\hbar/\hbar}\hat{A}(X).$$

This is the simplest version of algebraic index theorem which was first formulated by Fedosov and Nest-Tsygan as the algebraic analogue of Atiyah-Singer index theorem.

Coupling with bundle

This construction has a natural generalization by coupling with a rank r vector bundle E on X. The Harish-Chandra pair $(\mathfrak{g}, K)$ is

$$\mathfrak{g} = \mathcal{W}_{2n}\mathrm{Id}_n + \hbar\mathfrak{gl}_r(\mathcal{W}_{2n}), \quad K = \mathrm{Sp}_{2n} \times Gl_r.$$

The algebraic index can be computed in a similar way by

$$\mathrm{Index} = \int_X \mathrm{desc}(\mathrm{Tr}^{\mathbb{L}})(1) = \int_X e^{\omega_\hbar/\hbar}\hat{A}(X)\mathrm{Ch}(E).$$

This shows that

$$\boxed{\text{counting constant loops} \Longrightarrow \text{index theorem!}}$$

Remarks

1. If we consider 2d worldsheet, then the effective field theory of a chiral conformal field theory for $\mathbb{C} \to X$ gives

$$\begin{array}{c} \text{VOA} \\ \big\downarrow \\ X \end{array}$$

together with a flat connection (**L-2016**).

2. In 1d TQM case, integrals over $\mathrm{Conf}(S^1)$ intertwines the Hochschild differential with the BV operator

$$(b + \hbar\Delta) \langle - \rangle_{1d} = 0.$$

In 2d chiral CFT, the regularized integral (as defined in **L-Zhou 2020**) over $\mathrm{Conf}(\Sigma)$ intertwines the chiral differential

$$(d_{\mathrm{ch}} + \hbar\Delta) \langle - \rangle_{2d} = 0.$$

Work in progress with **Zhengping Gui**.

Thank you!

Sketch of Feynman diagram computation

The index (partition function) has a Feynman diagram expansion

$$\mathrm{Tr}^{\mathbb{L}}(1) = u^n e^{\left(\frac{1}{\hbar}\sum_{\Gamma_0}^{\text{con. tree}} \frac{W_{\Gamma_0}}{|\mathrm{Aut}(\Gamma_0)|}\right)} \left(\sum_{\Gamma_1}^{\text{1-loop}} \frac{W_{\Gamma_1}}{|\mathrm{Aut}(\Gamma_1)|} + O(\hbar)\right).$$

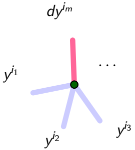


Figure: vertex

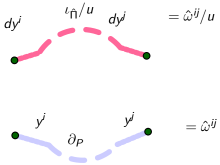


Figure: propagator

Here the vertex is valued in $\mathcal{W}_{2n} = \mathbb{C}[[y^i]][[\hbar]]$.

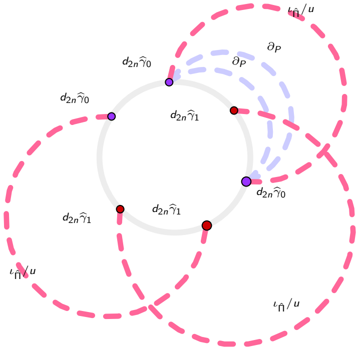


Figure: An example of $W_{\Gamma\chi}$.

The tree and one-loop Feynman diagram computation gives

$$\mathrm{desc}(\mathrm{Tr}^{\mathbb{L}})(1) = e^{-\omega_\hbar/\hbar}\left(\hat{A}(M)\mathrm{Ch}(E) + O(\hbar)\right).$$

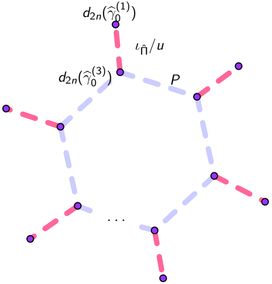


Figure: $\hat{A}(M)$

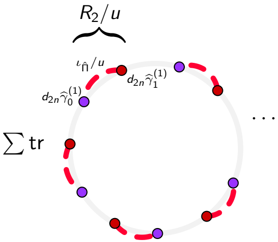


Figure: $\mathrm{Ch}(E)$