Link invariants from finite categorical groups and a lifting of the Eisermann invariant

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This talk is based on work done in collaboration with

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and student projects with

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Overview

Introduction: the knot group

Consider *knots*, i.e. smooth embeddings $k : S^1 \to S^3$, or *long knots*, i.e. smooth embeddings $k : \mathbb{R} \to \mathbb{R}^3$, such that $k(t) = (t, 0, 0)$ outside a compact interval.

The complement of a knot C_K is the complement in S^3 of a tubular neighbourhood of the image K of k. It is a 3-manifold with boundary the 2-torus.

The *knot group* is the fundamental group π_1 of the knot complement, and is an invariant of knots (but not a complete invariant, since inequivalent knots may have isomorphic knot groups, e.g. the trefoil and its mirror image).

Introduction: the peripheral system

However, if we include two generators of π_1 of the boundary torus,

the *meridian* m (goes the short way round) and

the *longitude* ℓ (goes the long way round and has winding number zero with the knot itself),

then this so-called *peripheral system* of (knotgroup, meridian, longitude) is a complete invariant.

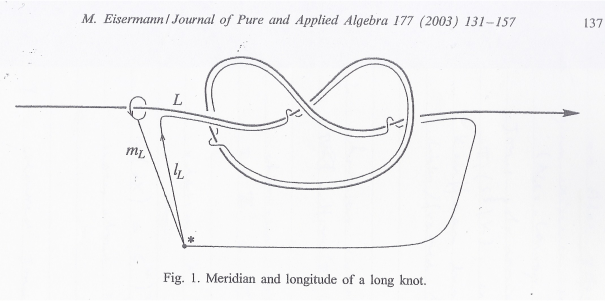


Fig. 1. Meridian and longitude of a long knot.

Introduction: the Eisermann invariant

The Eisermann invariant of knots (to be described soon) captures information about the peripheral system, and in particular, is able to distinguish a knot and its mirror image (m goes to m^{-1}), or a change in the orientation of the knot (both ℓ and m go to their inverses), amongst many other impressive feats.

Loosely speaking it does this by colouring the arcs of a knot diagram with elements of a finite group G, following certain rules which depend on a choice of $x \in G$ associated with the meridian, and then registering the values associated with the longitude. More precisely we have the following formula for $E(K)$:

$$E(K) = \sum_{\left\{ \substack{f\colon \pi_1(C_K) \to G \\ f(m) = x} \right\}} f(\ell),$$

taking values in the group algebra $\mathbb{Z}[G]$ of G.

Motivation for this research

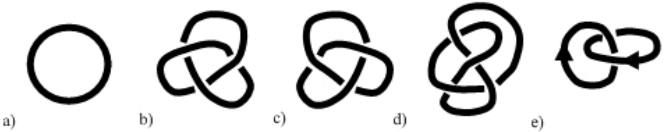
There are deeper ways of probing the knot complement C_K, using not just the loops of the fundamental group, but *surfaces* as well.

More precisely, we will be introducing the *fundamental 2-group*, or *fundamental crossed module*, Π_2, of a natural topological pair associated to a knot diagram.

The main point of this talk is to describe this construction, leading, in particular, to a *lifting* of the Eisermann invariant.

Knot diagrams

A knot diagram is a (regular) 2D representation of the image of the knot in 3D:

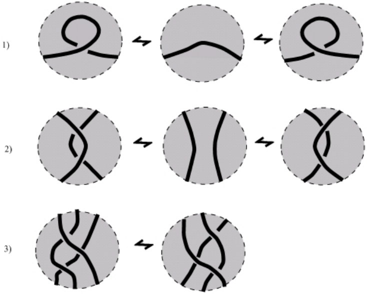


a) unknot, b) trefoil, c) mirror trefoil, d) Figure 8, e) Hopf link

The knot diagram consists of arcs and crossings.

Reidemeister moves

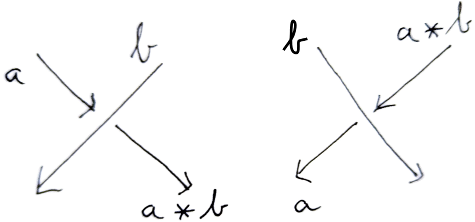
Of course, there are many different diagrams for the same knot. They are related by the famous Reidemeister moves:



i.e. three types of local moves which change a detail of the knot diagram as shown.

Quandles

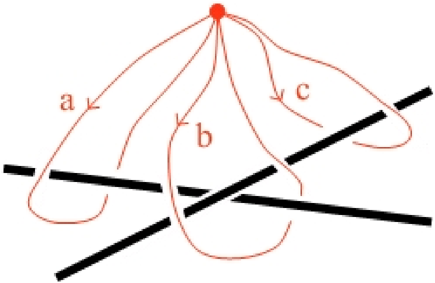
A quandle is a set Q with a binary operation $*$, and axioms such that arc colourings obeying



are in one-to-one correspondence for Reidemeister equivalent diagrams.

Knot diagrams and presentations of the knot group

Each knot diagram gives rise to a presentation of the knot group, with one generator for each arc and one relation for each crossing:



Relation for this crossing: $b = c^{-1}ac$

This is called the Wirtinger presentation of the knot group.

A knot invariant from a finite group G

Pick a finite group G and colour the arcs of a diagram with elements $X, Y, Z, \dots \in G$ such that at each crossing the relation

$$Y = Z^{-1} X Z$$

holds (same conventions as for the Wirtinger presentation).

Count the number of such colourings. This number is a knot invariant, since it counts the number of group homomorphisms from the, intrinsically defined, knot group $\pi_1(C_K)$ to G.

Our construction is somewhat similar, but involves 2-groups, also known as crossed modules of groups or categorical groups.

II Crossed modules of groups

Definition

A crossed module (of groups) is given by

$$\partial \colon E \to G \quad \text{(homomorphism of groups)}$$

$$\triangleright : G \times E \to E \quad \text{(left action of } G \text{ on } E \text{ by automorphisms}^*\text{)}$$

such that

1. $\partial(X \triangleright e) = X\partial(e)X^{-1}$ for each $X \in G, e \in E$,
2. $\partial(e) \triangleright f = efe^{-1}$ for each $e, f \in E$.

* i.e. $g \triangleright (e_1 e_2) = (g \triangleright e_1)(g \triangleright e_2)$ and $g \triangleright 1 = 1$

Examples of crossed modules

Crossed module recap: $\partial\colon E \to G$ and $\triangleright : G \times E \to E$
such that $\partial(X \triangleright e) = X\partial(e)X^{-1}$ and $\partial(e) \triangleright f = efe^{-1}$

1. (obvious example) $E = G$, $\partial = \mathrm{id}$, $X \triangleright Y = XYX^{-1}$
2. (just G) $\quad 1 \xrightarrow{\partial} G, \quad \triangleright$ trivial
3. (just abelian E) $\quad E \xrightarrow{\partial} 1, \quad 1 \triangleright e = e$

   Note that $\ker \partial$ is contained in the centre of E, hence is abelian (for $e \in \ker \partial$: $\quad efe^{-1} = \partial(e) \triangleright f = 1 \triangleright f = f$).
4. ("independent" G and abelian E) $\quad \partial(E) = 1, \quad g \triangleright e = e$.

We will also look at some examples with $E \neq G$ and "interacting".

Central extensions

A central extension of groups is an exact sequence

$$1 \to A \to E \to G \to 1$$

with the image of A central in E.

E.g. $A = \mathbb{Z}_2$, $E = SU(2)$, $G = SO(3)$

Crossed modules from central extensions

Given any central extension:

$$1 \to A \to E \xrightarrow{\partial} G \to 1$$

we get a crossed module of groups

$$E \xrightarrow{\partial} G,$$

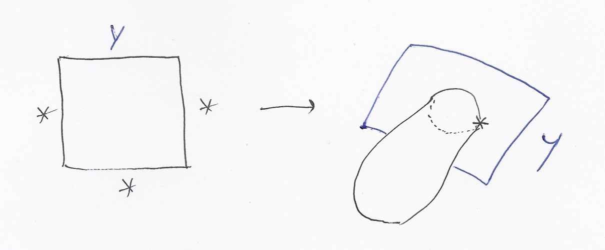
with lifted action

$$g \triangleright e = \tilde{e}\, e\, \tilde{e}^{-1}$$

where $\tilde{e} \in E$ is any element such that $\partial(\tilde{e}) = g$ (this is well-defined since $A = \ker \partial$ is central in E).

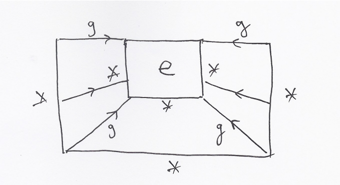
The fundamental crossed module of a topological pair 1

Given a pair of path-connected topological spaces (X, Y), with $Y \subset X$, and a basepoint $* \in Y$, we consider homotopy classes of maps from the square $[0,1]^2$ into X such that the image of the top edge is contained in Y and the image of the remaining 3 edges is $*$.



The fundamental crossed module of a topological pair 2

There is a well-defined horizontal multiplication of such squares, up to homotopy, like in the second homotopy group, giving rise to a group $\pi_2(X, Y)$.
We have an obvious group homomorphism ∂ from $\pi_2(X, Y)$ to $\pi_1(Y)$ given by restricting the map to the upper edge of the square, and $\pi_1(Y)$ acts on $\pi_2(X, Y)$ as follows:

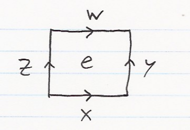


This gives the *fundamental crossed module* of the pair (X, Y):

$$\Pi_2(X, Y) = (\partial\colon \pi_2(X, Y) \to \pi_1(Y), \triangleright),$$

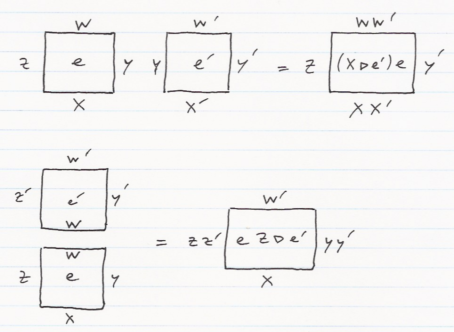
Crossed modules and multiplication of squares

Consider squares of the form



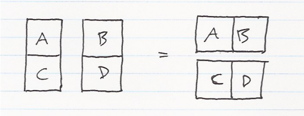
where $X, Y, Z, W \in G$ and $e \in E$, such that $\partial(e) = XYW^{-1}Z^{-1}$.

Define horizontal and vertical multiplication of squares:



Interchange law

These multiplications satisfy the interchange law:



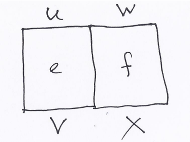
so that we can evaluate consistently the product of a 2D array of squares.

Categorical groups

2-groups can also be thought of as *categorical groups*, i.e. categories where both objects and morphisms are groups in a compatible way. The product of morphisms $\otimes$ is given by:

$$\begin{array}{c} U \\ {\scriptstyle (U,e)}\Big\downarrow \\ V \end{array} \otimes \begin{array}{c} W \\ {\scriptstyle (W,f)}\Big\downarrow \\ X \end{array} = \begin{array}{c} UW \\ \Big\downarrow {\scriptstyle \big(UW,(V \triangleright f)\, e\big)} \\ VX \end{array}$$

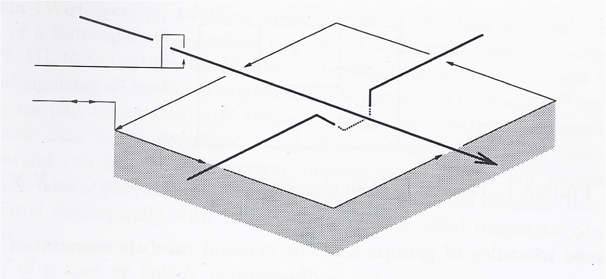
and can be thought of as horizontal multiplication of squares with 1_G on the left and right edges:



The composition of morphisms corresponds to vertical multiplication of such squares.

III A tangle invariant from finite crossed modules

A topological pair from a knot diagram D



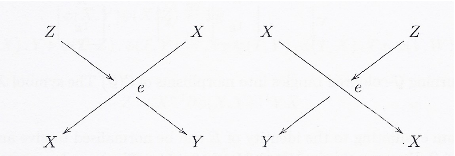
Put the arcs all at the same height and above the "water level", so that only the lowest parts of the undercrossings are below water. X_D is the knot complement, and $Y_D \subset X_D$ is the part above water. The square loop, which is homotopic to a product of generators of $\pi_1(Y)$ of the form $b^{-1}c^{-1}ac$, is trivial in X_D, but not in Y_D, since to contract it one must go under water. We have a generator of $\pi_2(X_D, Y_D)$ given by the dark surface, partly under water.

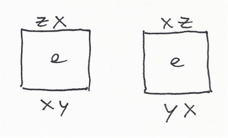
A knot invariant from a finite crossed module 1

The basic idea is to use a finite 2-group and colour the arcs of the diagram with G-elements, *and* the crossings of the diagram with E-elements to be compatible with the relations:

$$X_+ : \quad \partial(e) = XYX^{-1}Z^{-1} \qquad (1)$$

$$X_- : \quad \partial(e) = YXZ^{-1}X^{-1} \qquad (2)$$





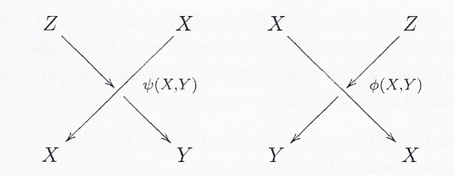
Each colouring describes a homomorphism from the fundamental crossed module $\Pi_2(X_D, Y_D)$ to the finite crossed module.
But counting these only gives an invariant depending on the homotopy type of the knot complement [FM].

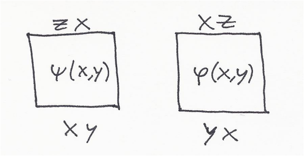
A knot invariant from a finite crossed module 2

We get a much more refined invariant if we restrict the colourings by making the colourings of two of the arcs determine the colouring of the crossing. To this end we introduce two functions:

$$\psi : G \times G \to E, \qquad \phi : G \times G \to E,$$

which determine the E-colouring of the two types of crossing, and hence the colouring of the remaining arc.

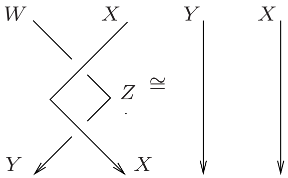


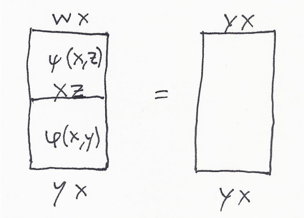


e.g. $\partial\psi(X, Y) = XY(ZX)^{-1}$, hence $Z = \partial\psi(X, Y)^{-1}XYX^{-1}$.

Invariance conditions on ψ and ϕ - Reidemeister 2

We study the conditions on ψ and ϕ for Reidemeister invariance, starting with Reidemeister 2:

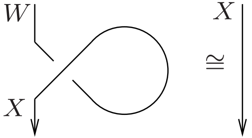


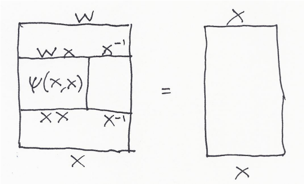


$$\phi(X, Y)\psi(X, Z) \overset{R2}{=} 1$$

(where Z is given as a function of X and Y, via ϕ).

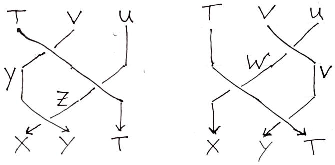
Invariance conditions on ψ and ϕ - Reidemeister 1

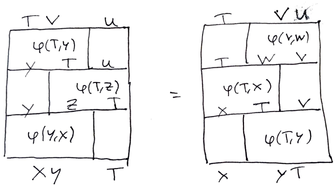




$$\psi(X,X) \overset{R1}{=} 1$$

Invariance conditions on ψ and ϕ - Reidemeister 3





$$\phi(Y,X)\,.\,Y \triangleright \phi(T,Z)\,.\,\phi(T,Y) \overset{R3}{=}$$

$$X \triangleright \phi(T,Y)\,.\,\phi(T,X)\,.\,T \triangleright \phi(V,W)$$

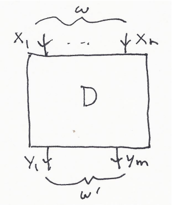
Note that the action $\triangleright$ of G on E enters the equation!

A tangle invariant

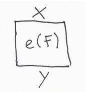
Theorem (Faria Martins + P)

Given a finite crossed module $\mathcal{G}$ and a pair of functions $\Phi = (\psi, \phi)$ satisfying (R1-3), let D be a tangle diagram, and fix a G-colouring of the top and bottom strands, given by words ω, ω'.

Let $C_\Phi(D, \omega, \omega')$ denote the set of $\mathcal{G}$-colourings of the arcs and crossings of D, using Φ, and compatible with ω, ω'. Then we have a tangle invariant given by

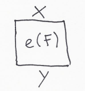


$$I_\Phi(D, \omega, \omega') \doteq \sum_{F \in C_\Phi(D,\omega,\omega')}$$



where $X = X_1 \ldots X_n$, $Y = Y_1 \ldots Y_m$, and $e(F) \in E$ is the evaluation of the corresponding array of squares.

A tangle invariant 2

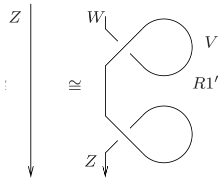
Regarding 

as a morphism in the categorical group $\mathsf{C}(\mathcal{G})$, we see that the invariant takes values in $\mathbb{N}\big[\mathrm{Hom}_{\mathsf{C}(\mathcal{G})}\big(X, Y\big)\big]$.

The proof involves checking invariance under a fairly long list of tangle moves, similar to the ones that gave rise to equations (R1-3).

A tangle invariant 3

We call ψ and ϕ an *unframed Reidemeister pair*, and there is an analogous theorem for *framed Reidemeister pairs* with a modified Reidemeister 1 move:



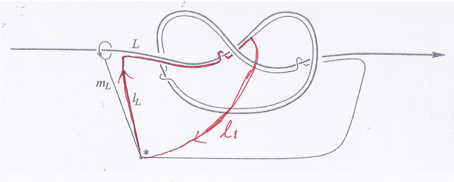
This invariant includes, as special cases, all rack and quandle cohomology invariants.

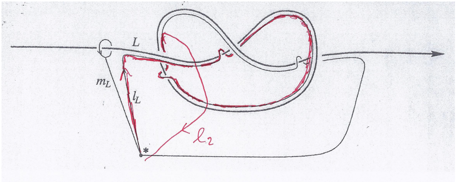
It also includes the Eisermann invariant, and a lifted version thereof, as we shall see.

IV The Eisermann invariant and its lifting

The Eisermann invariant and partial longitudes

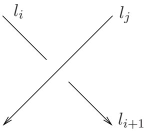
Following Eisermann we introduce partial longitudes ℓ_i which follow the knot only beyond the ith undercrossing:



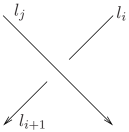


The Eisermann quandle and its Reidemeister pair

The relation between successive partial longitudes can be expressed using the arc generators x_i, where $x_0 = x$, the meridian [Eis2]:



$l_{i+1} = x^{-1}\, l_i\, l_j^{-1}\, x\, l_j$
or
$l_{i+1} = l_i\, {x_i}^{-1}\, x_j$



$l_{i+1} = x\, l_i\, l_j^{-1}\, x^{-1}\, l_j$
or
$l_{i+1} = l_i\, x_i\, {x_j}^{-1}$

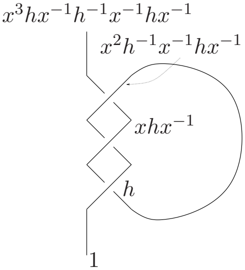
The corresponding unframed Reidemeister pair is given by [FMP]:

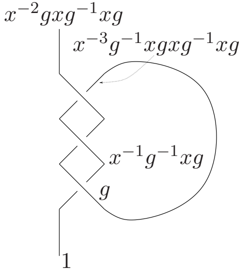
$$\begin{aligned} \phi^x(g, h) &= [hx^{-1}, gx^{-1}] \\ \psi^x(g, h) &= [g, h][hg^{-1}, x] \end{aligned}$$

where $[g, h]$ denotes the group commutator $ghg^{-1}h^{-1}$, and the crossed module here has $E = G$ and $\partial = \mathrm{id}$.

Example of the Eisermann invariant

Using the Eisermann quandle and the zero'th partial longitude $\ell_0 = 1$, we obtain an expression for the final partial longitude, i.e. the longitude, ℓ, e.g. for the trefoil and its mirror:





Representing this in $G = S_5$, with $x = (12345)$, the longitude values for valid colourings distinguish the two knots [Eis1]:

$$\mathrm{id}_{S_5} + 5(15432) \qquad \text{versus} \qquad \mathrm{id}_{S_5} + 5(12345)$$

The lifted Eisermann invariant

For a crossed module obtained from a central extension of groups,

$$1 \to A \to E \xrightarrow{\partial} G \to 1$$

there is a natural construction of an unframed Reidemeister pair, lifting the previous one for the Eisermann invariant:

$$\begin{aligned} \hat{\phi}^x(g,h) &= \{hx^{-1}, gx^{-1}\} \\ \hat{\psi}^x(g,h) &= \{g,h\}\{hg^{-1}, x\} \end{aligned}$$

where $\{\,,\} : G \times G \to E$ is defined by

$$\{g,h\} = [s(g), s(h)]$$

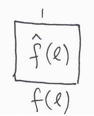
for an arbitrary section $s : G \to E$.

More generally, use the Peiffer lifting $\{\,,\}$ of a 2-crossed module.

The lifted Eisermann invariant 2

Since $\partial\{g,h\} = [g,h]$, the relation between the unlifted and lifted Eisermann invariants is:

$$E(K) = \sum_{\left\{\substack{f\,:\,\pi_1(C_K)\to G \\ f(m)=x}\right\}} f(\ell),$$

$$\hat{E}(K) = \sum_{\left\{\substack{f\,:\,\pi_1(C_K)\to G \\ f(m)=x}\right\}}$$ 

where $\hat{f}(\ell)$ is the evaluation of the array of squares using the lifted Reidemeister pair $(\hat{\psi}^x, \hat{\psi}^x)$.

The lifted Eisermann invariant 3

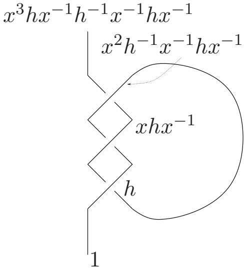
Let $\mathrm{GL}(n,k)$ denote the invertible n by n matrices with entries in $\mathbb{Z}_k$, and $\mathrm{PGL}(n,k)$ the corresponding projective linear group.

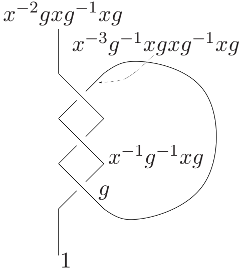
In [FMP] we showed for the choice $\mathrm{GL}(2,5) \xrightarrow{\partial} \mathrm{PGL}(2,5) \cong S_5$ that the lifted invariant distinguishes the trefoil and its mirror for two different choices of x (as opposed to just a single choice).

In a student project, Sofia Brito showed that the lifted invariant for the choice $\mathrm{GL}(2,3) \xrightarrow{\partial} \mathrm{PGL}(2,3)$ distinguished the trefoil and its mirror for various choices of x, whereas the ordinary Eisermann invariant doesn't distinguish for any choice of x.

Recent work: longitude expressions

Consider again the colourings of the trefoil and its mirror using the Eisermann quandle.



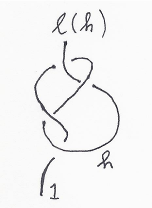


Instead of choosing a specific group, one could look at the expression obtained for the longitude in terms of x (the meridian) and one or more auxiliary partial longitudes, subject to constraint(s).

Recent work: longitude expressions 2

This is based on student projects with Maria Madrugo and João Tavares.

As a simpler example

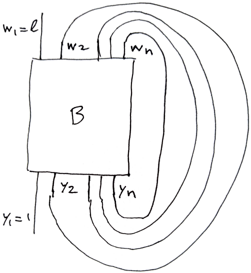
for the figure eight knot:  we get:

$$\begin{aligned} \ell(h) &= hx^{-1}h^{-1}xhxh^{-1}x^{-1}h \quad &\text{(expression)} \\ 1 &= h^{-1}x^{-1}hxh^{-1}xhx^{-1}h^{-1} \quad &\text{(constraint)} \end{aligned}$$

Via a simple substitution, $h \mapsto x^{-1}gx$, we get a new expression $\ell(g)$ and constraint for the mirror figure eight.

Recent work: longitude expressions 3

João Tavares and I are trying a systematic study of expressions and constraints using *long braids*.



Presentation in terms of x, $y_2, \ldots, y_n$, with expressions $w_1, \ldots, w_n$ coming fom using the Eisermann quandle, and constraints $w_i y_i^{-1} = 1$, $i = 2, \ldots, n$.

Recent work: longitude expressions 4

Theorem (Sofia Lambropoulou, using L-equivalence) Long knots are isotopic iff any corresponding long braids differ by

- conjugation by $\sigma_2^{(-1)}, \ldots \sigma_{n-1}^{(-1)}$,
- conjugation by σ_1^2 or σ_1^{-2},
- Markov stabilization $B \leftrightarrow B\sigma_n^{(-1)}$

The second move keeps the long strand on the left hand side of the braid.

Using programmes developed by João Tavares we are looking for stable patterns under these moves in the expressions for the longitude $w_1 = \ell$ and the partial longitudes w_j, $j = 2, \ldots, n$, emerging at the top of the braid.

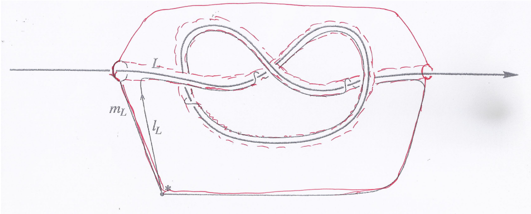
Final considerations

Returning to a generic crossed module and the lifted Eisermann invariant, it would be interesting to identify an intrinsic global surface associated with the knot (i.e. a 3D viewpoint) . . .

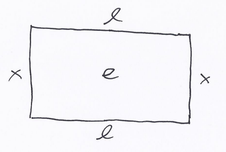
. . . as opposed to the locally defined mini-surfaces under each crossing, which collectively produce the invariant from a particular long knot diagram (i.e. a 2D diagrammatic viewpoint).

Final considerations 2

The following torus should play a significant role:



Cutting this torus along x and ℓ gives a square:



Since ℓ and x commute, e belongs to $\ker \partial$, and should, in some sense, capture the extra abelian information in the lifting.

References

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[Eis2] M. Eisermann, *Homological characterization of the unknot*, J. Pure Appl. Algebra 177 (2003), no. 2, 131–157.

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[FMP] João Faria Martins and Roger Picken, *Link invariants from finite categorical groups*, Homology, Homotopy and Applications, 17(2) (2015), 205–233;
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THANKS FOR LISTENING