ON THE STATISTICAL INTERPRETATION IN SCHWINGER'S PICTURE OF QUANTUM MECHANICS

March, 19th 2021

Work in progress in collaboration with

F. M. Ciaglia, A. Ibort, G. Marmo, L. Schiavone, A. Zampini

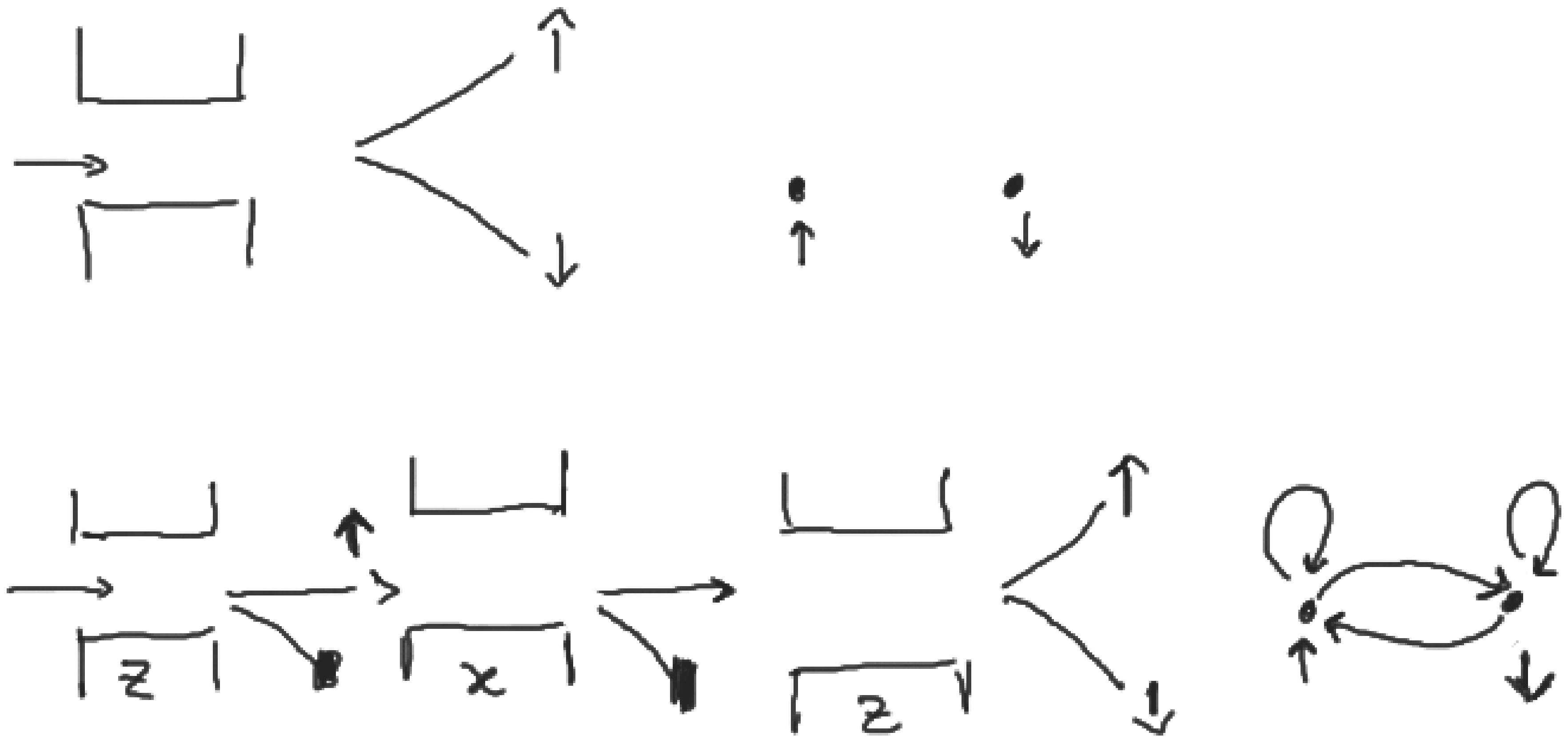
PLAN OF THE TALK

- Introduction
- Basic assumptions
- Groupoid-algebras revisited (Connes' non commutative theory of integration)
- The statistical interpretation
  - States on the groupoid-algebra
  - Quantum measures on the groupoid
- Conclusions and future developments

INTRODUCTION

Motivations:

SG experiments



Atomic Transitions

$$\nu_{n,p} = \nu_{n,m} + \nu_{m,p}$$

Schwinger's approach to Quantum Mechanics: "The laws of atomic physics must be expressed in a non-classical mathematical language that constitutes a symbolic expession of the properties of microscopic measurements" $\Rightarrow$ SELECTIVE MEASUREMENTS

SELECTIVE MEASUREMENTS $\longrightarrow$ TRANSITIONS OF GROUPOIDS

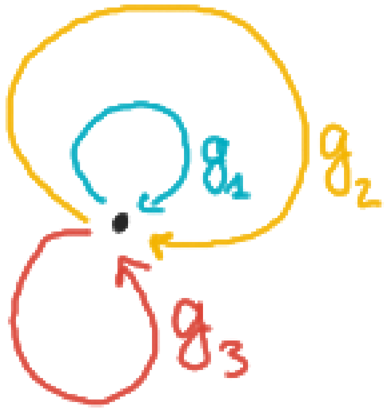
GROUPOID $\quad G \overset{s}{\underset{t}{\rightrightarrows}} \Omega$ (Algebraic properties)

- G and Ω are sets, G is the collection of TRANSITIONS, Ω of the OBJECTS
- $s: G \to \Omega$ is the SOURCE MAP, $t: G \to \Omega$, the TARGET MAP
- Composition rule $\circ : G^{(2)} \subset G \times G \to G$ $\quad ((\alpha, \beta) \longrightarrow \alpha \circ \beta, \ s(\alpha) = t(\beta))$
  - associative $\quad (\alpha \circ \gamma) \circ \beta = \alpha \circ (\gamma \circ \beta)$
  - units, $\quad \alpha : x \to y, \quad \alpha \circ 1_x = 1_y \circ \alpha = \alpha$
  - inverses, $\quad \alpha^{-1} : y \to x$ such that $\alpha^{-1} \circ \alpha = 1_{s(\alpha)}, \ \alpha \circ \alpha^{-1} = 1_{t(\alpha)}$.

EXAMPLES

G is a group

$\Omega = \{e\}$



$\circ : G \times G \to G$

$(g_1, g_2) \to g_1 \circ g_2 = g_1 g_2$

- associative
- unit
- inverses

$s : G \to \Omega$
$\quad g \to g^{-1} g$

$t : G \to \Omega$
$\quad g \to g g^{-1}$

$S_N = \{\sigma_j\}$ is a finite set with N-elements

TRIVIAL GROUPOID

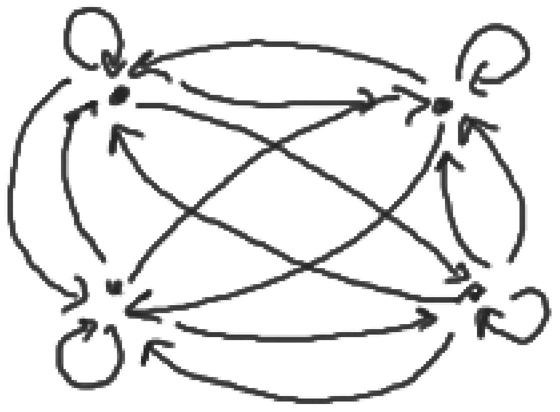
$G = S_N \rightrightarrows S_N = \Omega$

$s : S_N \to S_N$
$\quad \sigma_j \to \sigma_j$

$t : S_N \to S_N$
$\quad \sigma_j \to \sigma_j$

Only units

PAIR GROUPOID



$G = S_N \times S_N \rightrightarrows S_N = \Omega$

$s : (\sigma_j, \sigma_k) \to \sigma_k \,;\; t : (\sigma_j, \sigma_k) \to \sigma_j$

$(\sigma_\ell, \sigma_j) \circ (\sigma_j, \sigma_k) = (\sigma_\ell, \sigma_k)$

- associative
- units $\quad 1_{\sigma_j} = (\sigma_j, \sigma_j)$
- inverses $\quad (\sigma_j, \sigma_k)^{-1} = (\sigma_k, \sigma_j)$

CONVOLUTION ALGEBRAS & TRANSVERSE FUNCTIONS

Some basic ingredients from Connes' non commutative theory of integration:

- Let us consider two Borel spaces $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ and the spaces $\overline{\mathcal{F}}^+(X), \overline{\mathcal{F}}^+(Y)$ of Borel non-negative functions with values in $\overline{\mathbb{R}}^+ = [0, +\infty]$. A KERNEL λ from Y to X is a map $\lambda : \overline{\mathcal{F}}^+(Y) \to \overline{\mathcal{F}}^+(X)$, denoted $\lambda : Y \nrightarrow X$, such that:
  - $\lambda(af + bg) = a\lambda(f) + b\lambda(g)$, $f, g \in \overline{\mathcal{F}}^+(Y)$, $a, b \in \mathbb{R}^+$
  - normal: if $f_n \nearrow f$ is a monotonous increasing sequence of Borel functions converging pointwise to f, then $\lambda(f_n) \nearrow \lambda(f)$

Alternatively, a KERNEL $\lambda : Y \nrightarrow X$ defines a map $\lambda : X \to \mathcal{M}^+(Y)$, where $\mathcal{M}^+(Y)$ is the space of positive measures on Y, as follows:

$$x \to \lambda^x(\Delta) := \lambda(1_\Delta)(x) \quad , \text{ where } \Delta \in \mathcal{B}_Y, x \in X.$$

For any $\Delta \in \mathcal{B}_Y$, the map $\lambda_\Delta : X \to \overline{\mathbb{R}}^+$ is a measurable map.

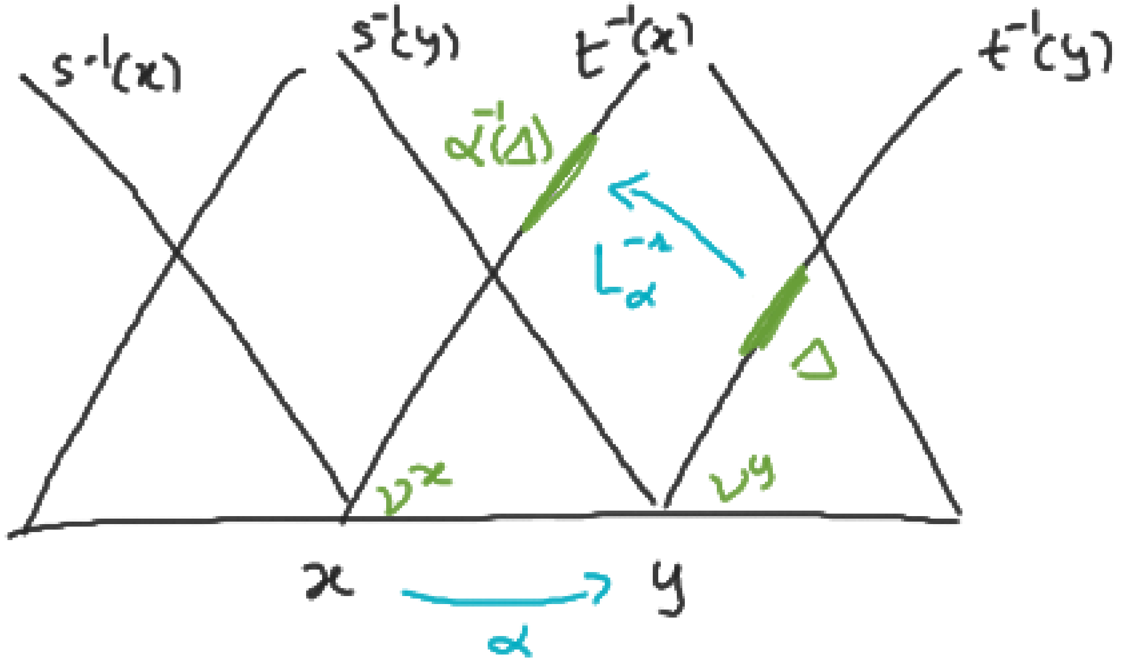
A Kernel is σ-FINITE if there is a family of disjoint Borel sets Δ_n such that $\bigcup_n \Delta_n = Y$ and $\lambda(\Delta_n)$ is finite $\forall n$. It is PROPER if there is a sequence of Borel sets $E_1 \subset E_2 \subset \ldots \subset E_n$, such that $\bigcup_n E_n = Y$ and $\lambda(1_{E_n})$ are bounded for any n.

Let $G \overset{s}{\underset{t}{\rightrightarrows}} \Omega$ be a Borel groupoid, then a G-Kernel is a Kernel $\lambda : G \overset{t}{\rightsquigarrow} \Omega$, fibered by $t : G \to \Omega$, i.e., $\mathrm{supp}(\lambda^x) \subset t^{-1}(x)$, $\forall x \in \Omega$ such that $\lambda^x \neq 0$ (if $\lambda^x \neq 0$ $\forall x \in \Omega$, then the Kernel λ is FAITHFUL).

A TRANSVERSE FUNCTION $\nu : G \overset{t}{\rightsquigarrow} \Omega$ is a G-Kernel such that

$$\alpha \circ \nu^x = \nu^y \qquad \text{(EQUIVARIANCE CONDITION)}$$

$\alpha : x \to y$ is a transition in G and $(\alpha \circ \nu^x)(\Delta) = \nu^x(L_\alpha^{-1}\Delta)$ $\forall \Delta \subset t^{-1}(y) = G^y$.



A MODULAR FUNCTION $\Delta : G \to \mathbb{R}^+$ is a homomorphism of groupoids, i.e.,

$$\Delta(\alpha \circ \beta) = \Delta(\alpha) \cdot \Delta(\beta)$$

Given a modular function $\Delta : G \to \mathbb{R}^+$ and a transverse function $\nu : G \overset{t}{\to} \Omega$ we introduce the CONVOLUTION ALGEBRA $C_\nu(G)$, of functions $f : G \to \mathbb{C}$:

- INVOLUTION $+ : C_\nu(G) \to C_\nu(G)$ $\qquad f^+(\gamma) = \Delta^{-1}(\gamma)\, \overline{f}(\gamma^{-1})$
- $\|f\|_I = \max\left(\sup_{x \in \Omega} \nu(|f|),\ \sup_{x\in\Omega} \nu(|f^+|) \right) < \infty$
- CONVOLUTION PRODUCT $\star : C_\nu(G) \times C_\nu(G) \to C_\nu(G)$ $\qquad f \star g(\gamma) = \int_{G^{t(\gamma)}} f(\alpha)\, g(\alpha^{-1} \circ \gamma)\, d\nu^{t(\gamma)}(\alpha)$

This is an involutive Banach algebra. Real elements will be interpreted as observables of the system described by G.

GROUP : ν^e is the left Haar measure. The modular function of G is a homomorphism of the group.

In this case $C_\nu(G) = L^1(G, \nu^e)$ equipped with:

- INVOLUTION $\qquad f^+(g) = \overline{f}(g^{-1})$
- CONVOLUTION PRODUCT $\qquad f_1 \star f_2(g) = \int_G f_1(h)\, f_2(h^{-1}g)\, d\nu(h)$

For instance $G = (\mathbb{R}, +) \Rightarrow C_\nu(G) = L^1(\mathbb{R}, dx)$

$$f_1 \star f_2(t) = \int_{\mathbb{R}} f_1(x) f_2(t-x)\, dx$$

PAIR GROUPOID $M \times M \rightrightarrows M$ $(M, \mathcal{B}_M)$ a Borel space.

The homomorphisms $\Delta : M \times M \to \mathbb{R}^+$ are written as $\Delta(y,x) = e^{V(y)} e^{-V(x)}$

The transverse functions $\nu : M \times M \xrightarrow{t} M$ are written in terms of regular Radon measures $\tilde{\nu}$ on M as follows $\nu^x = \delta_x \otimes \tilde{\nu}$.

The convolution product is $(f \star g)(y,x) = \int_M f(y,w)\, g((w,y)\circ(y,x))\, \delta_y \otimes d\tilde{\nu}(w)$

$$= \int_M f(y,w)\, g(w,x)\, d\tilde{\nu}(w)$$

The involution is $f^{+}(y,x) = e^{V(x)} e^{-V(y)} \overline{f(x,y)}$

The norm is the max between

$$\left(\sup_{y \in M} \int_M |f(y,x)|\, d\tilde{\nu}(x) \;,\; \sup_{y \in M} e^{-V(y)} \int_M |f^{+}(y,x)|\, e^{V(x)} d\tilde{\nu}(x) \right)$$

TRIVIAL GROUPOID $M \rightrightarrows M$ $\nu^x = K \delta_x$ and the convolution algebra is made up of complex-valued functions $f : M \to \mathbb{C}$ with the product

$$(f \star g)(x) = f(x) \cdot g(x)\, K$$

$$\| f \|_\infty = \sup_{x \in M} |f|$$

TRANSVERSE MEASURES & REPRESENTATIONS OF $C_\nu(G)$

A TRANSVERSE MEASURE Λ with module $\Delta : G \to \mathbb{R}^+$ is a map

$$\Lambda : \mathcal{E}^+(G) \longrightarrow \overline{\mathbb{R}^+} \quad (\mathcal{E}^+(G) \text{ the space of faithful transverse functions}):$$

- LINEAR: $\Lambda(a\nu + b\mu) = a\Lambda(\nu) + b\Lambda(\mu)$, $a, b \in \mathbb{R}^+$, $\nu, \mu \in \mathcal{E}^+(G)$
- NORMAL: If $\nu_m \nearrow \nu$, then $\Lambda(\nu_m) \nearrow \Lambda(\nu)$
- Δ-SIMMETRIC: For any $\lambda : G \overset{t}{\to} \Omega$, such that $\lambda^x(G^x) = 1 \ \forall x \in \Omega$,

$$\Lambda(\nu) = \Lambda(\nu \star \Delta\lambda), \text{ where } (\nu \star \Delta\lambda)^x(f) = \int f(\alpha \circ \beta)\, d\lambda^{s(\alpha)}(\beta)\, d\nu^x(\alpha)$$

For a given faithful and proper $\nu \in \mathcal{E}^+(G)$, there is a bijection $\Lambda \longrightarrow \Lambda_\nu$ between transverse measures Λ and positive measures Λ_ν on Ω such that

$$\int_G f(\gamma^{-1})\, \Delta^{-1}(\gamma)\, d\nu^x(\gamma)\, d\Lambda_\nu(x) = \int_G f(\gamma)\, d\nu^x(\gamma)\, d\Lambda_\nu(x).$$

Therefore $\mu = \Lambda_\nu \circ \nu$ is a measure on G and $\Delta^{-1} = \dfrac{d\mu^{-1}}{d\mu}$

Given the measure $\mu = \Lambda_\nu \circ \nu$ we can construct the Hilbert space $\mathcal{L}^2(G,\mu)$.
The dense subset $C_0(G) \subset \mathcal{L}^2(G,\mu) \cap C_\nu(G)$ is closed under the convolution product $\star$ and the involution $+$. $C_0(G) \ni f \to T_f \in B(\mathcal{L}^2(G,\mu))$, $T_f\, g = f \star g$ is a bounded representation of $(C_0(G), \star)$ and we call

$$\mathcal{V}(G) = \{T_f \,,\, f \in C_0(G)\}''$$

the groupoid von-Neumann algebra of G. $\mathcal{V}(G)$ possesses a canonical weight ω_μ defined on its positive elements as

$$\omega_\mu(T_f) = \begin{cases} \Lambda_\nu(\nu|g|^2) & \text{if } g = \sqrt{T_f} \\ +\infty & \text{otherwise} \end{cases} \qquad \Lambda_\nu(\nu|g|^2) = \int_G d\mu(x)\, d\nu^x(\gamma)\, |g|^2(\gamma)$$

Using the same ingredients we can define a positive linear functional $\omega_\Lambda : C_\nu(G) \to \mathbb{R}^+$

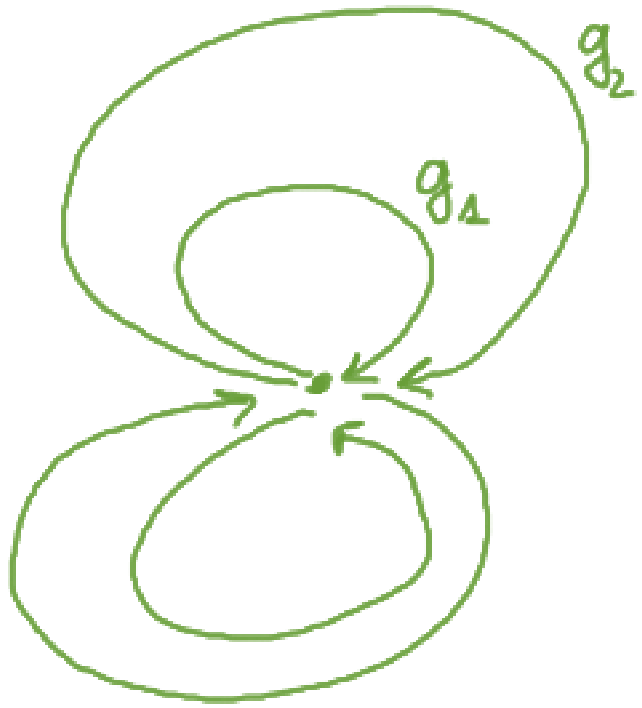
$$\omega_\Lambda(f^+ \star f) = \Lambda_\nu(\nu(f^+ \star f)) = \int_\Omega d\mu(x)\, |\nu^x(f)|^2 \geq 0 \,, \quad \text{where} \quad \nu^x(f) = \int_{G^x} f(\gamma)\, d\nu^x(\gamma)$$

The GNS construction produces the Hilbert space $\mathcal{L}^2(\Omega, \Lambda_\nu)$ and the elements in $C_\nu(G)$ act as bounded operator $T_f^\Lambda \Psi_g = \Psi_{f\star g}$, where $\Psi_g(x) = \nu^x(g) = \int_{G^x} f(\gamma)\, d\nu^x(\gamma)$. The von-Neumann algebra $\mathcal{V}_\Lambda(G) = \{T_f^\Lambda\}^{\overline{w}}$ is the von-Neumann groupoid algebra in the reduced form.

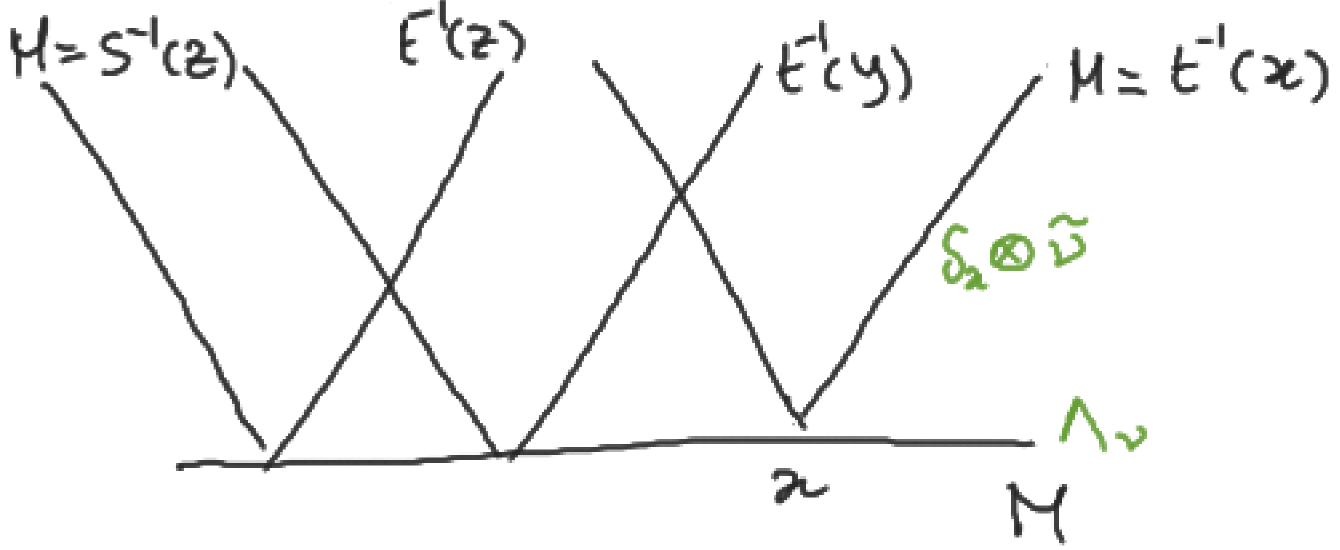
GROUPS : Given the Haar measure ν on G and Δ the modular function there is, up to a constant, a unique transverse measure Λ such that $\Lambda_\nu \circ \nu = \nu$. The von-Neumann groupoid-algebra corresponds to the group-algebra obtained by completion of the algebra of continuous function acting on $\mathcal{L}^2(G,\nu)$ via convolution, w.r.t. the $\mathcal{L}^2$-norm.

$$\omega_\Lambda (f^+ \star f) = \nu (f^+ \star f) = \int_{G \times G} \Delta^{-1}(h) \overline{f}(h^{-1}) f(h^{-1}g) \, d\nu(h) d\nu(g) = |\nu(f)|^2$$

The equivalence class of functions $[f]$ is associated to a complex number k_f and there is a one dimensional support space $\mathcal{L}^2(G, \Lambda_\nu) = \mathbb{C}$.



PAIR GROUPOID : $M \times M \rightrightarrows M$



$\nu^x = \delta_x \otimes \tilde{\nu}$, $\Delta(y,x) = e^{V(y)} e^{-V(x)}$

For every ν and Δ there is a unique, apart from a constant factor, transverse measure which is Δ-symmetric and we can disintegrate it as follows:

$\Lambda_\nu \circ \nu = K\,(e^{V}\tilde{\nu}) \otimes \tilde{\nu}$. In the case $\Delta(y,x) = 1$ the von-Neumann groupoid algebra is $B(\mathcal{L}^2(G,\tilde{\nu}))$.

On the other hand $\omega_\Lambda(f^{+} \star f) = \int_M d\tilde{\nu}(x)\,|\nu^x(f)|^2 = \int_M d\tilde{\nu}(x) \int_M d\nu^x(\gamma)\,\bar{f}(\gamma) \int_M d\nu^x(\beta)\,f(\beta)$

and $\int_M d\nu^x(\gamma)\,f(\gamma) = \int_M d\tilde{\nu}(y)\,f(x,y) = \Psi_f(x)$

The action $T_h \Psi_f = \Psi_{h \star f}$ and $\Psi_{h\star f}(x) = \nu^x(h \star f) = \int d\tilde{\nu}(y)\,d\tilde{\nu}(w)\,h(x,w)\,f(w,y)$

TRIVIAL GROUPOID $M \rightrightarrows M$, Λ_ν are positive measures $\tilde{\nu}$ on M such that $\Lambda_\nu \circ \nu = p\,\tilde{\nu}$. $\omega_\Lambda(f^{+} \star f) = \int d\tilde{\nu}\,p(x)\,|f|^2(x)$

STATES AND QUANTUM MEASURES ON A GROUPOID

Given (Λ, ν) we have defined the weight

$$\omega_\Lambda(f^\dagger \star f) = \Lambda_\nu(\nu(f^\dagger \star f)).$$

Chosen a positive linear function φ on $G \rightrightarrows \Omega$, i.e.,

$$\int d\mu(\alpha)\, \varphi(\alpha)\, (f^\dagger \star f)(\alpha) \geq 0$$

we define a family of states $\omega_\Lambda^\varphi(f^\dagger \star f) = \Lambda_\nu(\nu(\varphi(f^\dagger \star f)))$ if the normalization condition is satisfied. Via these states we have the standard statistical interpretation in terms of expectation values on the algebra of observables. In particular, Dirac-Feynman states are defined via

$$\varphi(\alpha) = \sqrt{p(s(\alpha))\, p(t(\alpha))}\; e^{iS(\alpha)}, \text{ where } S(\alpha \circ \beta) = S(\alpha) + S(\beta)$$

where $S(\alpha)$ is called the action functional.

QUANTUM MEASURES

In classical probability theory we assocuates a subset A (event) of some configuration space M a real positive number $0 \le \mu(A) \le 1$ obeying the sum rule

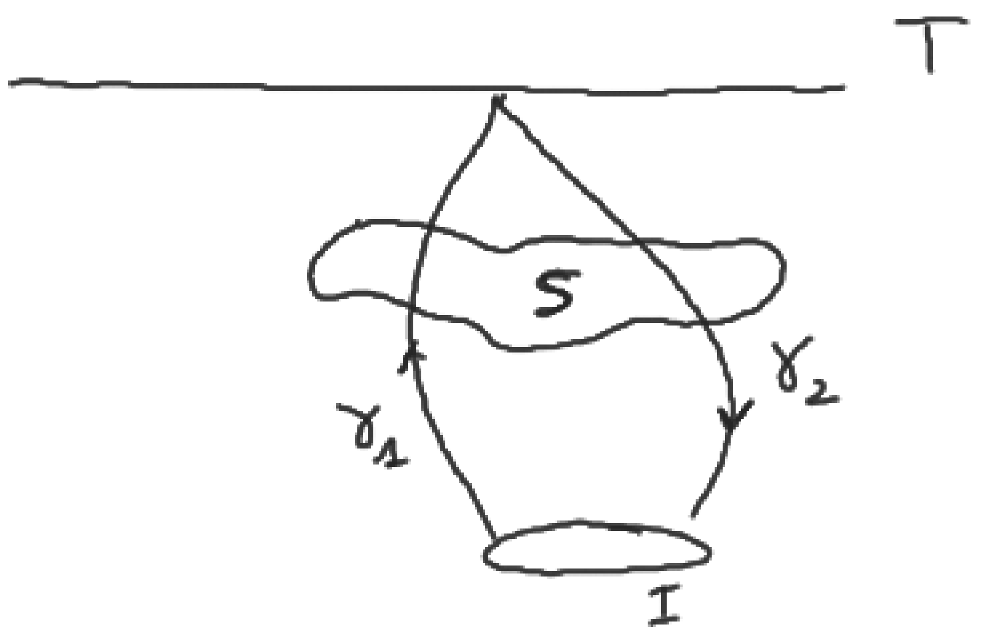
$$I_2(A,B) := \mu(A \sqcup B) - \mu(A) - \mu(B) = 0 \qquad \forall A,B \text{ s.t. } A \cap B = \emptyset .$$

According to Sorkin's description of QM, classical measure theory has to be generalized to what is called a quantum measure $\mu(A) \geq 0$ satisfying

$$I_2(A,B) \neq 0$$

$$I_3(A,B,C) := \mu(A \sqcup B \sqcup C) - \mu(A \sqcup B) - \mu(A \sqcup C) - \mu(B \sqcup C) + \mu(A) + \mu(B) + \mu(C) = 0$$ (GRADE-2 MEASURE)

Statistical interpretation : PRECLUSION OF EVENTS



Sorkin's quantum measure $\quad \mu(S) = \sum_{\gamma_1,\gamma_2 \in S_T} I_2(\gamma_1,\gamma_2)$

DECOHERENCE FUNCTIONALS & POSITIVE DEFINITE FUNCTIONS

A quantum measure can be derived as the quadratic function associated with a biadditive set function called DECOHERENCE FUNCTIONAL

$D : \mathcal{B}_M \times \mathcal{B}_M \to \mathbb{C}$: ($\mathcal{B}_M$ denotes the σ-algebra of measurable sets in M)

- σ-additivity $D(\cdot, A)$ is a complex measure for any $A \in \mathcal{B}_M$.
- POSITIVITY $\forall m \in \mathbb{N}$ and family $\{A_i\}_{i=1\dots m}$, $A_i \in \mathcal{B}_M$, then $\sum_{i,j} \bar{\xi}_i \xi_j D(A_i, A_j) \geq 0$, $\xi \in \mathbb{C}^m$.

A quantum measure is obtained from D according to:

$$\mu(A) = D(A, A)$$

Given a groupoid $G \rightrightarrows \Omega$ and a pair (Λ, ν) we define

$$D(A,B) = \Lambda_\nu(\nu(\chi_A \star \chi_B^+))$$

$$\mu(A) = \Lambda_\nu(\nu(\chi_A \star \chi_A^+))$$

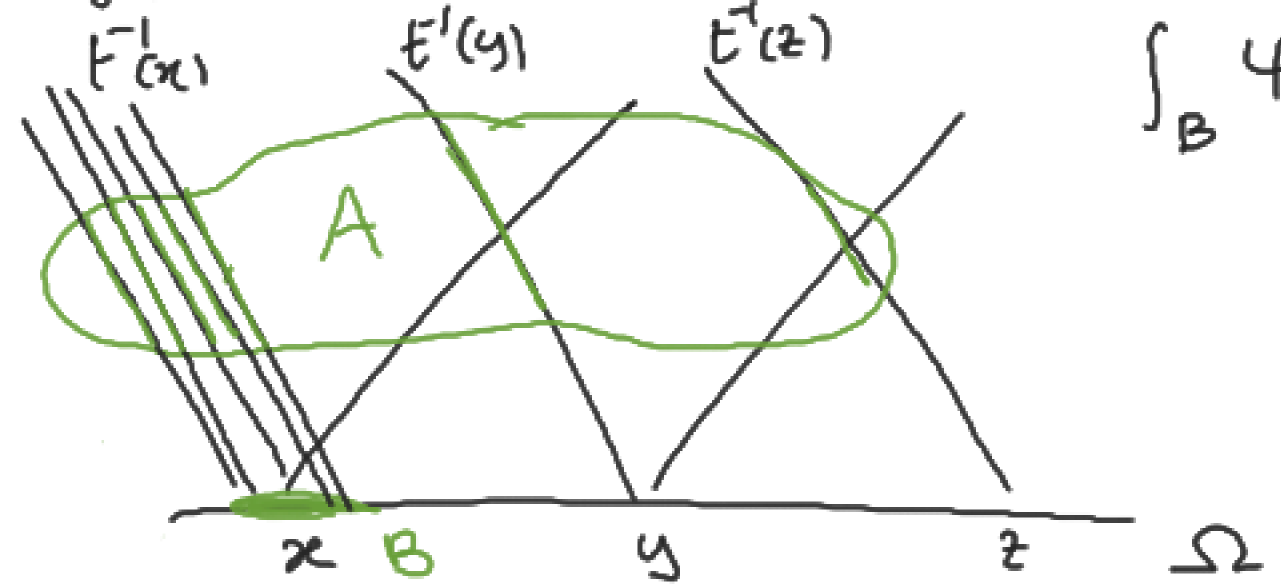
Using a positive definite function $\varphi : G \to \mathbb{C}$ one can define a family of decoherence functionals and quantum measures

$$D^{\varphi}(A,B) = \Lambda_\nu(\nu(\varphi \chi_A * \chi_B^{+})) \to \mu^{\varphi}(A) = \Lambda_\nu(\nu(\varphi \chi_A * \chi_A^{+}))$$

In particular, for a Dirac-Feynmann state we obtain the measure

$$\mu^{S}(A) = \int_{\Omega} d\Lambda_\nu(x) \int_{G^x \times G^x} d\nu^x(\alpha)\, d\nu^x(\beta)\, K(\alpha^{-1} \circ \beta)\, e^{iS(\beta)} e^{-iS(\alpha)} \chi_A(\alpha) \chi_A(\beta)$$

Subsets of G are events and their measure is interpreted in terms of precision. Moreover, if $\varphi(\alpha) = 1$ $\forall \alpha \in G$, the GNS construction associates a vector $\psi_A \in \mathcal{L}^2(\Omega, \Lambda_\nu)$ with any equivalence class of sets $[A]$ having the same measure, with values



$$\int_B \psi_A(x)\, d\Lambda_\nu(x) = \int_B \nu^x(A)\, d\Lambda_\nu(x) \quad , \forall B \in \mathcal{B}_\Omega .$$

Moreover, given a representation of the groupoid Π and a vector ξ of this representation we can define the positive definite function

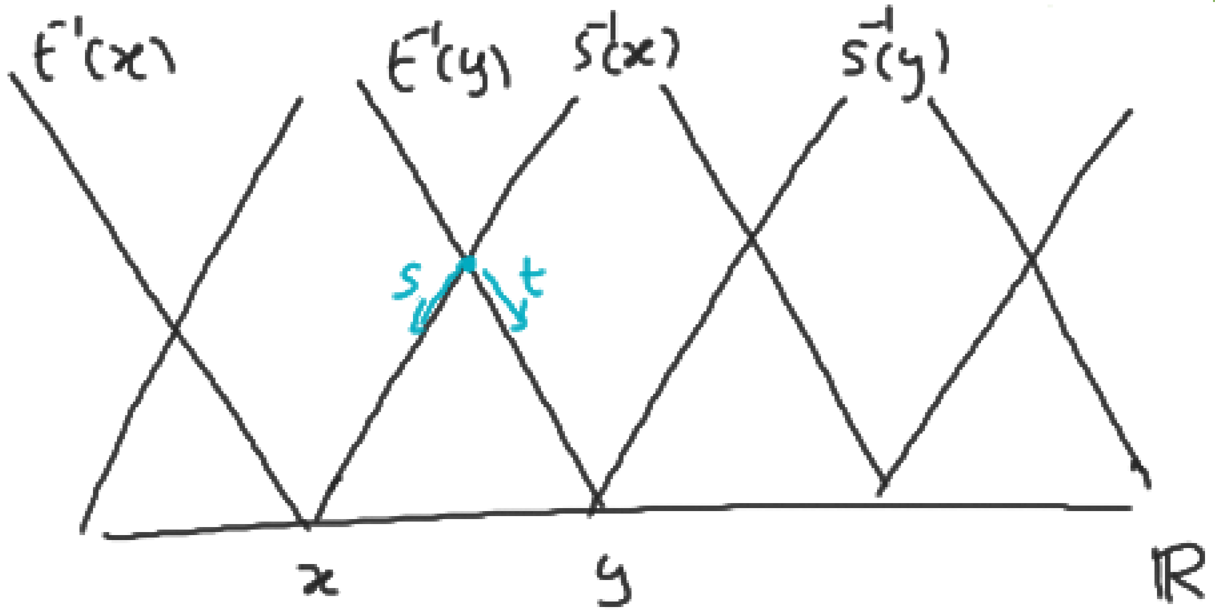
$$\varphi_\xi : G \to \mathbb{C} \quad , \quad \varphi_\xi(\alpha) = \langle \xi | \Pi(\alpha) | \xi \rangle$$

Then, if I construct the state ω_Λ^φ, the associated GNS construction produces the Hilbert space $\mathcal{L}^2(\Omega, \Lambda_\nu^\varphi)$ ω_Λ^φ has an associated density matrix $\hat{\rho}_\varphi$. If $T_\beta = T_{\delta_\beta}$ is the operator associated with the Kernel δ_β, the quantity

$$\varphi(\beta) = \mathrm{Tr}(\hat{\rho}_\varphi T_\beta)$$

can be interpreted as the probability amplitude of the transition β.

EXAMPLE : The pair groupoid $\mathbb{R}\times\mathbb{R} \rightrightarrows \mathbb{R}$



$$\left.\begin{array}{l} \nu = \delta_x \otimes \tilde{\nu} \\ \Delta(x,y) = 1 \end{array}\right\} \Rightarrow \Lambda_\nu \circ \nu = \tilde{\nu} \otimes \tilde{\nu}$$

The von Neumann groupoid algebra is $B(\mathcal{L}^2(\mathbb{R},\tilde{\nu}))$, as well as its reduced form.

A Dirac-Feynman state can be written as $\varphi(y,x) = \sqrt{p(y)p(x)}\, e^{i(\sigma(y)-\sigma(x))}$

$$\omega^\varphi_\Lambda(f^* \star f) = \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} d\tilde{\nu}(y)\, d\tilde{\nu}(x)\, d\tilde{\nu}(w) \sqrt{p(x)p(y)}\, e^{i(\sigma(y)-\sigma(x))}\, \overline{f}(y,w) f(w,x) =$$

$$\int d\tilde{\nu}(w) \left(d\tilde{\nu}(y) \sqrt{p(y)}\, e^{i\sigma(y)} \overline{f}(y,w) \right) \left(d\tilde{\nu}(x) \sqrt{p(x)}\, e^{-i\sigma(x)} f(w,x) \right) =$$

$$= \int d\tilde{\nu}(x) \left| \tilde{\nu}^* \left(\sqrt{p}\, e^{-i\sigma} f \right) \right|^2 = \int d\tilde{\nu}(x)\, |\Psi_f(x)|^2$$

$T_\beta = T_{\delta_\beta}$ with $\beta : x \to y$. $(\delta_\beta \star f)(\alpha) = \int f(\gamma^{-1}\circ\alpha)\, d\lambda_\beta^{t(\alpha)}(\gamma) = f(\beta^{-1}\circ\alpha)$, $\alpha \in t^{-1}(y)$

$$\Psi_{\delta_\beta \star f}(y) = \int d\nu^y(\alpha)\, f(\beta^{-1}\circ\alpha) = \int d\nu^y(\alpha)\, f(\alpha) = \Psi_f(y)$$

$$\overline{\Psi}_f(x)\, \Psi_{\delta_\beta \star f}(y) = \int d\tilde{\nu}(w)\, d\tilde{\nu}(z) \sqrt{p(w)p(z)}\, e^{iS(w,z)}\, \overline{f}(y,w) f(z,z)$$

THANKS FOR
YOUR ATTENTION